Ultralimits and computability

Noah Schweber (with Uri Andrews, Mingzhong Cai, David Diamondstone)

Computability, Randomness, and Applications, 23 June 2016

Ultrafilter jumps

Definition

For $X \subseteq \omega$ — viewed as an array $X = (X_i)_{i \in \omega}$ — and $\mathcal{U}$ an ultrafilter, let

$$\lim_{\mathcal{U}}(X) = \{j : \{i : \langle i, j \rangle \in X\} \in \mathcal{U}\}.$$

Ultrafilter jumps

Definition

For $X \subseteq \omega$ — viewed as an array $X = (X_i)_{i \in \omega}$ — and $\mathcal{U}$ an ultrafilter, let

$$\lim_{\mathcal{U}}(X) = \{j : \{i : \langle i, j \rangle \in X\} \in \mathcal{U}\}.$$

For a Turing degree $\mathbf{a}$, let

$$\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}(X) : X \leq_T \mathbf{a}\}.$$

Ultrafilter jumps

Definition

For $X \subseteq \omega$ — viewed as an array $X = (X_i)_{i \in \omega}$ — and $\mathcal{U}$ an ultrafilter, let

$$\lim_{\mathcal{U}}(X) = \{j : \{i : \langle i, j \rangle \in X\} \in \mathcal{U}\}.$$

For a Turing degree $\mathbf{a}$, let

$$\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}(X) : X \leq_T \mathbf{a}\}.$$

Remark

Can also define $\delta_{\mathcal{U}}(\mathcal{S})$ for arbitrary families of sets $\mathcal{S}$.

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ a Turing degree. Then:

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ a Turing degree. Then:

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ a Turing degree. Then:

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal.*

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ *a Turing degree. Then:*

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal.*
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a countable Scott set.*

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ *a Turing degree. Then:*

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal.*
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a countable Scott set.*

This gives a new proof of a well-known result:

Corollary

WKL_0 is strictly weaker than ACA_0.

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ a Turing degree. Then:

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal.*
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a countable Scott set.*

This gives a new proof of a well-known result:

Corollary

WKL_0 is strictly weaker than ACA_0.

Standard proof: iterate the Low Basis Theorem.

Closure properties

Perhaps surprisingly, ultralimits of Turing degrees have meaningful closure properties!

Theorem

Let $\mathcal{U}$ be a nonprincipal ultrafilter, $\mathbf{a}$ a Turing degree. Then:

- $\delta_{\mathcal{U}}(\mathbf{a})$ *contains* $\Delta^0_2(\mathbf{a})$.
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal.*
- $\delta_{\mathcal{U}}(\mathbf{a})$ *is a countable Scott set.*

This gives a new proof of a well-known result:

Corollary

WKL_0 is strictly weaker than ACA_0.

Standard proof: iterate the Low Basis Theorem.

Combinatorial proof: Let $\mathcal{U}$ be a nonprincipal ultrafilter such that $\{e : W_e \in \mathcal{U}\}$ is (say) Δ^0_{17}. Then $\delta_{\mathcal{U}}(REC)$ is a subset of Δ^0_{17}.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, I/II

Let $\mathbf{a}$ be a degree, $\mathcal{U}$ an ultrafilter.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, I/II

Let $\mathbf{a}$ be a degree, $\mathcal{U}$ an ultrafilter.

$\Delta^0_2(\mathbf{a}) \subseteq \delta_{\mathcal{U}}(\mathbf{a})$: by limit lemma.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, I/II

Let $\mathbf{a}$ be a degree, $\mathcal{U}$ an ultrafilter.

$\Delta^0_2(\mathbf{a}) \subseteq \delta_{\mathcal{U}}(\mathbf{a})$: by limit lemma.

$\delta_{\mathcal{U}}(\mathbf{a})$ closed under join: $\lim_{\mathcal{U}}(\langle X_i \oplus Y_i \rangle) = \lim_{\mathcal{U}}(\langle X_i \rangle) \oplus \lim_{\mathcal{U}}(\langle Y_i \rangle)$.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, I/II

Let $\mathbf{a}$ be a degree, $\mathcal{U}$ an ultrafilter.

$\Delta^0_2(\mathbf{a}) \subseteq \delta_{\mathcal{U}}(\mathbf{a})$: by limit lemma.

$\delta_{\mathcal{U}}(\mathbf{a})$ closed under join: $\lim_{\mathcal{U}}(\langle X_i \oplus Y_i \rangle) = \lim_{\mathcal{U}}(\langle X_i \rangle) \oplus \lim_{\mathcal{U}}(\langle Y_i \rangle)$.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Turing ideal: let $S = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$, Φ^S_e total. Define Y as

$$Y_i(n) = \begin{cases} 1 & \text{if } \Phi^{X_i}_e(n)[i] \downarrow = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\Phi^S_e = \Phi^{\lim_{\mathcal{U}}(Y)}_e$:

- $\sigma \prec S$ implies $\sigma \prec X_i$ for $\mathcal{U}$-many σ
- Look at $\sigma \prec S$ such that $\Phi^{\sigma}_e(n)[|\sigma|] \downarrow$.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?
Suppose $T = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$ is an infinite tree.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?
Suppose $T = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$ is an infinite tree.
Conflate ω and $2^{<\omega}$, so each X_i is a subset of $2^{<\omega}$.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?
Suppose $T = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$ is an infinite tree.
Conflate ω and $2^{<\omega}$, so each X_i is a subset of $2^{<\omega}$.

Let Y_i be the "tree part" of X_i: $Y_i = \{\sigma \in 2^{<\omega} : \forall \tau \preccurlyeq \sigma(\tau \in X_i)\}$. Since T is a tree, $\lim_{\mathcal{U}}(Y) = T$.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?
Suppose $T = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$ is an infinite tree.
Conflate ω and $2^{<\omega}$, so each X_i is a subset of $2^{<\omega}$.

Let Y_i be the "tree part" of X_i: $Y_i = \{\sigma \in 2^{<\omega} : \forall \tau \preccurlyeq \sigma(\tau \in X_i)\}$. Since T is a tree, $\lim_{\mathcal{U}}(Y) = T$.
For $i \in \omega$, can effectively-in-**a** find a maximal finite path of length at most i through Y_i — call this p_i.

$\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set containing $\mathbf{a}'$, II/II

Why is $\delta_{\mathcal{U}}(\mathbf{a})$ a Scott set?
Suppose $T = \lim_{\mathcal{U}}(X) \in \delta_{\mathcal{U}}(\mathbf{a})$ is an infinite tree.
Conflate ω and $2^{<\omega}$, so each X_i is a subset of $2^{<\omega}$.

Let Y_i be the "tree part" of X_i: $Y_i = \{\sigma \in 2^{<\omega} : \forall \tau \preccurlyeq \sigma(\tau \in X_i)\}$. Since T is a tree, $\lim_{\mathcal{U}}(Y) = T$.
For $i \in \omega$, can effectively-in-**a** find a maximal finite path of length at most i through Y_i — call this p_i.

Then $\lim_{\mathcal{U}}(p_i)$ is a path through T.

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, I/II

We proved that $\delta_{\mathcal{U}}(\mathbf{a})$ is always a countable Scott set containing $\mathbf{a}'$.
What about the converse?

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, I/II

We proved that $\delta_{\mathcal{U}}(\mathbf{a})$ is always a countable Scott set containing $\mathbf{a}'$.
What about the converse?

Theorem

If $\mathbf{a}$ *is a Turing degree and* $\mathcal{J}$ *is a countable Scott set containing* $\mathbf{a}'$*, then* $\delta_{\mathcal{U}}(\mathbf{a}) = \mathcal{J}$ *for some* $\mathcal{U}$*.*

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, I/II

We proved that $\delta_{\mathcal{U}}(\mathbf{a})$ is always a countable Scott set containing $\mathbf{a}'$. What about the converse?

Theorem

If $\mathbf{a}$ is a Turing degree and $\mathcal{J}$ is a countable Scott set containing $\mathbf{a}'$, then $\delta_{\mathcal{U}}(\mathbf{a}) = \mathcal{J}$ for some $\mathcal{U}$.

Remark

More generally, if $\mathcal{I}$ is a countable Turing ideal and $\mathcal{J}$ is a countable Scott set containing the jump of every $\mathbf{a}' \in \mathcal{I}$, then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{J}$ for some $\mathcal{U}$.

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, I/II

We proved that $\delta_{\mathcal{U}}(\mathbf{a})$ is always a countable Scott set containing $\mathbf{a}'$. What about the converse?

Theorem

If $\mathbf{a}$ *is a Turing degree and* $\mathcal{J}$ *is a countable Scott set containing* $\mathbf{a}'$*, then* $\delta_{\mathcal{U}}(\mathbf{a}) = \mathcal{J}$ *for some* $\mathcal{U}$*.*

Remark

More generally, if $\mathcal{I}$ is a countable Turing ideal and $\mathcal{J}$ is a countable Scott set containing the jump of every $\mathbf{a}' \in \mathcal{I}$, then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{J}$ for some $\mathcal{U}$.

Given $\mathbf{a}$ and $\mathcal{J}$ appropriate, need to meet:

- Image requirements: each $Y \in \mathcal{J}$ is $\lim_{\mathcal{U}}(X)$ for some $X \leq_T \mathbf{a}$.
- Domain requirements: When we put sets into $\mathcal{U}$, we never force $\lim_{\mathcal{U}}(X)$ to be outside $\mathcal{J}$ for any $X \leq_T \mathbf{a}$.

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.
An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.

An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.

Equivalently: $\{(A_i, B_i) : i \in I\}$ is satisfiable if for all $F \subset I$ finite and $n \in \omega$,

$$[\bigcap_{j \in F, m < n, B_j(m)=1} (A_j)^m] \cap [\bigcap_{j \in F, m < n, B_j(m)=0} \overline{(A_j)^m}] \text{ is infinite.}$$

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.

An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.

Equivalently: $\{(A_i, B_i) : i \in I\}$ is satisfiable if for all $F \subset I$ finite and $n \in \omega$,

$$[\bigcap_{j \in F, m < n, B_j(m)=1} (A_j)^m] \cap [\bigcap_{j \in F, m < n, B_j(m)=0} \overline{(A_j)^m}] \text{ is infinite.}$$

Enumerate $\mathbf{a} = \{X_i : i \in \omega\}$, $\mathcal{I} = \{Y_i : i \in \omega\}$. We build set of axioms $\mathcal{C}$ in stages:

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.
An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.
Equivalently: $\{(A_i, B_i) : i \in I\}$ is satisfiable if for all $F \subset I$ finite and $n \in \omega$,

$$[\bigcap_{j \in F, m < n, B_j(m)=1} (A_j)^m] \cap [\bigcap_{j \in F, m < n, B_j(m)=0} \overline{(A_j)^m}] \text{ is infinite.}$$

Enumerate $\mathbf{a} = \{X_i : i \in \omega\}$, $\mathcal{I} = \{Y_i : i \in \omega\}$. We build set of axioms $\mathcal{C}$ in stages:

- $\mathcal{C}_0 = \emptyset$

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.
An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.
Equivalently: $\{(A_i, B_i) : i \in I\}$ is satisfiable if for all $F \subset I$ finite and $n \in \omega$,

$$[\bigcap_{j \in F, m < n, B_j(m)=1} (A_j)^m] \cap [\bigcap_{j \in F, m < n, B_j(m)=0} \overline{(A_j)^m}] \text{ is infinite.}$$

Enumerate $\mathbf{a} = \{X_i : i \in \omega\}$, $\mathcal{I} = \{Y_i : i \in \omega\}$. We build set of axioms $\mathcal{C}$ in stages:

- $\mathcal{C}_0 = \emptyset$
- For $\mathcal{C}_{2k+1}$, want to make sure Y_k gets mapped to: pick $X \in \mathbf{a}$ sufficiently free over already-used sets.

Controlling $\delta_{\mathcal{U}}(\mathbf{a})$, II/II

Fix appropriate $\mathbf{a}$ and $\mathcal{I}$.
An **axiom** is a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ (meaning: "$\lim_{\mathcal{U}}(A) = B$"). Set of axioms is *consistent* if satisfied by some nonprincipal ultrafilter.
Equivalently: $\{(A_i, B_i) : i \in I\}$ is satisfiable if for all $F \subset I$ finite and $n \in \omega$,

$$[\bigcap_{j \in F, m < n, B_j(m)=1} (A_j)^m] \cap [\bigcap_{j \in F, m < n, B_j(m)=0} \overline{(A_j)^m}] \text{ is infinite.}$$

Enumerate $\mathbf{a} = \{X_i : i \in \omega\}$, $\mathcal{I} = \{Y_i : i \in \omega\}$. We build set of axioms $\mathcal{C}$ in stages:

- $\mathcal{C}_0 = \emptyset$
- For $\mathcal{C}_{2k+1}$, want to make sure Y_k gets mapped to: pick $X \in \mathbf{a}$ sufficiently free over already-used sets.
- For $\mathcal{C}_{2k+2}$, want to make sure X_k gets mapped inside $\mathcal{I}$: $\mathbf{a}'$-computable tree of consistent extensions.

Some questions

Question

Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed? Or $\delta_{\mathcal{U}}(\mathcal{J})$, for "sufficiently closed" Turing ideals $\mathcal{J}$?

Some questions

Question

Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed? Or $\delta_{\mathcal{U}}(\mathcal{J})$, for "sufficiently closed" Turing ideals $\mathcal{J}$?

Remark

By Theorem 2.5, if $\mathcal{I}$ is arithmetically closed then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{I}$ for some $\mathcal{U}$.

Some questions

Question

Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed? Or $\delta_{\mathcal{U}}(\mathcal{J})$, for "sufficiently closed" Turing ideals $\mathcal{J}$?

Remark

By Theorem 2.5, if $\mathcal{I}$ is arithmetically closed then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{I}$ for some $\mathcal{U}$.

Remark

However, no ultrafilter satisfies $\delta_{\mathcal{U}}(\mathbf{a}) = ARITH(\mathbf{a})$ for *every* $\mathbf{a}$:

Some questions

Question

Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed? Or $\delta_{\mathcal{U}}(\mathcal{J})$, for "sufficiently closed" Turing ideals $\mathcal{J}$?

Remark

By Theorem 2.5, if $\mathcal{I}$ is arithmetically closed then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{I}$ for some $\mathcal{U}$.

Remark

However, no ultrafilter satisfies $\delta_{\mathcal{U}}(\mathbf{a}) = ARITH(\mathbf{a})$ for *every* $\mathbf{a}$: $\mathbf{a}'$ can diagonalize away from $\mathbf{a}$.

Some questions

Question

Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed? Or $\delta_{\mathcal{U}}(\mathcal{J})$, for "sufficiently closed" Turing ideals $\mathcal{J}$?

Remark

By Theorem 2.5, if $\mathcal{I}$ is arithmetically closed then $\delta_{\mathcal{U}}(\mathcal{I}) = \mathcal{I}$ for some $\mathcal{U}$.

Remark

However, no ultrafilter satisfies $\delta_{\mathcal{U}}(\mathbf{a}) = ARITH(\mathbf{a})$ for *every* $\mathbf{a}$: $\mathbf{a}'$ can diagonalize away from $\mathbf{a}$.

Question

Are combinatorial properties of $\mathcal{U}$ (Ramsey, p-point, . . .) connected with closure properties of $\delta_{\mathcal{U}}(\mathbf{a})$?

Negative results

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.

Negative results

Question

For what **a** is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.
2. DNR_2-degrees.

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.
2. DNR_2-degrees.
3. High degrees.

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.
2. DNR_2-degrees.
3. High degrees.

Proof.

For $(1), (2)$: can compute A such that $\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$ for each $e \in Tot$ and $i \in \omega$.

Negative results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.
2. DNR_2-degrees.
3. High degrees.

Proof.

For $(1), (2)$: can compute A such that $\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$ for each $e \in Tot$ and $i \in \omega$.

For (3): any dominating function f computes an A such that $\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$ for each $e \in Tot$ and *cofinitely many* $i \in \omega$.

□

Negative results

Question

For what **a** is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Nonexamples:

1. $\geq_T \mathbf{0}'$.
2. DNR_2-degrees.
3. High degrees.

Proof.

For $(1), (2)$: can compute A such that $\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$ for each $e \in Tot$ and $i \in \omega$.

For (3): any dominating function f computes an A such that $\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$ for each $e \in Tot$ and *cofinitely many* $i \in \omega$.

□

Note that same proof shows that no $\delta_{\mathcal{U}}$ is $\mathbf{a} \mapsto ARITH(\mathbf{a})$.

Positive results

Question

For what **a** is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Positive results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Theorem

Degrees bounded by reals of the following types are ultrafilter-low:

Positive results

Question

For what **a** is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Theorem

Degrees bounded by reals of the following types are ultrafilter-low:

- *2-generic.*

Positive results

Question

For what $\mathbf{a}$ is there some $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$?

Call such degrees/reals "ultrafilter-low."

Theorem

Degrees bounded by reals of the following types are ultrafilter-low:

- *2-generic.*
- *Computably traceable.*

Some questions

In search of an exact characterization of ultrafilter-lowness:

Some questions

In search of an exact characterization of ultrafilter-lowness:

Question

Are measure-1 many reals ultrafilter-low? (Are random reals ultrafilter-low?)

Some questions

In search of an exact characterization of ultrafilter-lowness:

Question

Are measure-1 many reals ultrafilter-low? (Are random reals ultrafilter-low?)

Question

Are there Δ^0_2 ultrafilter-low reals?

Some questions

In search of an exact characterization of ultrafilter-lowness:

Question

Are measure-1 many reals ultrafilter-low? (Are random reals ultrafilter-low?)

Question

Are there Δ^0_2 ultrafilter-low reals?

Even the following is open:

Question

Is there a degree **a** such that $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(REC)$ for **every** $\mathcal{U}$?

By Theorem 2.5, such an **a** would have to be Δ^0_2 — in fact, low in the usual sense.

What sort of structure is induced on $\beta\mathbb{N}$?

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all degrees $\mathbf{a}$.

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all degrees $\mathbf{a}$.

Question

Are there $\mathcal{U}, \mathcal{V}$ with $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$ for all/cone many $\mathbf{a}$?

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_\mathcal{U}(\mathbf{a}) \subseteq \delta_\mathcal{V}(\mathbf{a})$ for a cone of degrees **a**.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_\mathcal{U}(\mathbf{a}) \subseteq \delta_\mathcal{V}(\mathbf{a})$ for all degrees **a**.

Question

Are there $\mathcal{U}, \mathcal{V}$ with $\delta_\mathcal{U}(\mathbf{a}) \subsetneq \delta_\mathcal{V}(\mathbf{a})$ for all/cone many **a**?

Remark

For *ideals*, answer is no: $...\delta_\mathcal{U} \circ \delta_\mathcal{V} \circ \delta_\mathcal{U} \circ \delta_\mathcal{V} \circ \delta_\mathcal{U} \circ \delta_\mathcal{V}(REC)$.

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all degrees $\mathbf{a}$.

Question

Are there $\mathcal{U}, \mathcal{V}$ with $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$ for all/cone many $\mathbf{a}$?

Remark

For *ideals*, answer is no: $...\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} \circ \delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} \circ \delta_{\mathcal{U}} \circ \delta_{\mathcal{V}}(REC)$.

Composition: $\mathcal{U} * \mathcal{V} = \{X : \{b : \{a : \langle a, b \rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}\}$.

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees **a**.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all degrees **a**.

Question

Are there $\mathcal{U}, \mathcal{V}$ with $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$ for all/cone many **a**?

Remark

For *ideals*, answer is no: $...\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} \circ \delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} \circ \delta_{\mathcal{U}} \circ \delta_{\mathcal{V}}(REC)$.

Composition: $\mathcal{U} * \mathcal{V} = \{X : \{b : \{a : \langle a, b\rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}\}$.

Remark

While $\delta_{\mathcal{U}*\mathcal{V}} = \delta_{\mathcal{U}} \circ \delta_{\mathcal{V}}$, this is *not* true on the level of individual sets.

What sort of structure is induced on $\beta\mathbb{N}$?

Degree structure: Set $\mathcal{U} \leq_J \mathcal{V}$ if $\delta_\mathcal{U}(\mathbf{a}) \subseteq \delta_\mathcal{V}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.
Lightface version: $\mathcal{U} \leq_j \mathcal{V}$ if $\delta_\mathcal{U}(\mathbf{a}) \subseteq \delta_\mathcal{V}(\mathbf{a})$ for all degrees $\mathbf{a}$.

Question

Are there $\mathcal{U}, \mathcal{V}$ with $\delta_\mathcal{U}(\mathbf{a}) \subsetneq \delta_\mathcal{V}(\mathbf{a})$ for all/cone many $\mathbf{a}$?

Remark

For *ideals*, answer is no: $...\delta_\mathcal{U} \circ \delta_\mathcal{V} \circ \delta_\mathcal{U} \circ \delta_\mathcal{V} \circ \delta_\mathcal{U} \circ \delta_\mathcal{V}(REC)$.

Composition: $\mathcal{U} * \mathcal{V} = \{X : \{b : \{a : \langle a, b\rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}\}$.

Remark

While $\delta_{\mathcal{U}*\mathcal{V}} = \delta_\mathcal{U} \circ \delta_\mathcal{V}$, this is *not* true on the level of individual sets.

Remark

Unlike e.g. addition of ultrafilters, $*$ has no idempotents — we observed before that $\delta_\mathcal{U}(\mathbf{a}) \supsetneq \delta_\mathcal{U}(\mathbf{a}')$.

Basic properties of the boldface degree structure

Definition

$\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.

Basic properties of the boldface degree structure

Definition

$\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.

Proposition

Given $\{\mathcal{U}_\eta : \eta < \omega_1\}$, *there is an upper bound* $\mathcal{V} >_J \mathcal{U}_\eta$.

Basic properties of the boldface degree structure

Definition

$\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.

Proposition

Given $\{\mathcal{U}_\eta : \eta < \omega_1\}$, there is an upper bound $\mathcal{V} >_J \mathcal{U}_\eta$.

Proof.

Let $h : \mathbb{R} \to \omega_1 : r \mapsto \omega_1^r$. For $r \in \mathbb{R}$, let $\hat{r}$ be such that $\hat{r} \geq_T s$ for all $s \in \delta_{\mathcal{U}_\eta}(deg(r)), \eta < h(r)$. Can construct $\mathcal{V}$ with $\delta_{\mathcal{V}}(deg(r)) \ni \hat{r}$. □

Basic properties of the boldface degree structure

Definition

$\mathcal{U} \leq_J \mathcal{V}$ if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for a cone of degrees $\mathbf{a}$.

Proposition

Given $\{\mathcal{U}_\eta : \eta < \omega_1\}$, there is an upper bound $\mathcal{V} >_J \mathcal{U}_\eta$.

Proof.

Let $h : \mathbb{R} \to \omega_1 : r \mapsto \omega_1^r$. For $r \in \mathbb{R}$, let $\hat{r}$ be such that $\hat{r} \geq_T s$ for all $s \in \delta_{\mathcal{U}_\eta}(deg(r)), \eta < h(r)$. Can construct $\mathcal{V}$ with $\delta_{\mathcal{V}}(deg(r)) \ni \hat{r}$. □

Proposition

Modulo $\equiv_J$, there are more than continuum-many ultrafilters.

RK-reductions

Definition

$\mathcal{U} \leq_{RK} \mathcal{V}$ if for some $f : \omega \to \omega$, $X \in \mathcal{U} \iff f^{-1}(X) \in \mathcal{V}$.

RK-reductions

Definition

$\mathcal{U} \leq_{RK} \mathcal{V}$ if for some $f : \omega \to \omega$, $X \in \mathcal{U} \iff f^{-1}(X) \in \mathcal{V}$. “Lightface” version $\leq_{rk}$: same but with computable f.

RK-reductions

Definition

$\mathcal{U} \leq_{RK} \mathcal{V}$ if for some $f : \omega \to \omega$, $X \in \mathcal{U} \iff f^{-1}(X) \in \mathcal{V}$. "Lightface" version $\leq_{rk}$: same but with computable f.

Proposition

$\leq_J, \leq_j$ refined by $\leq_{RK}, \leq_{rk}$ respectively.

RK-reductions

Definition

$\mathcal{U} \leq_{RK} \mathcal{V}$ if for some $f : \omega \to \omega$, $X \in \mathcal{U} \iff f^{-1}(X) \in \mathcal{V}$. "Lightface" version $\leq_{rk}$: same but with computable f.

Proposition

$\leq_J, \leq_j$ refined by $\leq_{RK}, \leq_{rk}$ respectively.

Proof.

Suppose $\mathcal{U} \leq_{RK} \mathcal{V}$ via $f \in deg(\mathbf{a})$. For $X \in \mathbf{a}$ let $Y_i = \{n : n \in X_{f(i)}\}$. Then $n \in \lim_{\mathcal{V}}(Y) \iff n \in \lim_{\mathcal{U}}(X)$. □

RK-reductions

Definition

$\mathcal{U} \leq_{RK} \mathcal{V}$ if for some $f : \omega \to \omega$, $X \in \mathcal{U} \iff f^{-1}(X) \in \mathcal{V}$. "Lightface" version $\leq_{rk}$: same but with computable f.

Proposition

$\leq_J, \leq_j$ refined by $\leq_{RK}, \leq_{rk}$ respectively.

Proof.

Suppose $\mathcal{U} \leq_{RK} \mathcal{V}$ via $f \in deg(\mathbf{a})$. For $X \in \mathbf{a}$ let $Y_i = \{n : n \in X_{f(i)}\}$. Then $n \in \lim_{\mathcal{V}}(Y) \iff n \in \lim_{\mathcal{U}}(X)$. □

Question

Do $\leq_J$ ($\leq_j$) and $\leq_{RK}$ ($\leq_{rk}$) coincide?

What about *uncountable* Turing ideals?

Recall the characterization theorem:

Theorem

For $\mathcal{I}, \mathcal{J}$ countable Turing ideals, the following are equivalent:

- $\mathcal{J}$ *contains* $\mathbf{a}'$ *for every* $\mathbf{a} \in \mathcal{I}$, *and is a Scott set.*
- $\mathcal{J} = \delta_{\mathcal{U}}(\mathcal{I})$ *for some nonprincipal ultrafilter* $\mathcal{U}$.

What about *uncountable* Turing ideals?

Recall the characterization theorem:

Theorem

For $\mathcal{I}, \mathcal{J}$ countable Turing ideals, the following are equivalent:

- $\mathcal{J}$ *contains* $\mathbf{a}'$ *for every* $\mathbf{a} \in \mathcal{I}$*, and is a Scott set.*
- $\mathcal{J} = \delta_{\mathcal{U}}(\mathcal{I})$ *for some nonprincipal ultrafilter* $\mathcal{U}$*.*

What if we take ultrafilter jumps of uncountable Turing ideals?

What about *uncountable* Turing ideals?

Recall the characterization theorem:

Theorem

For $\mathcal{I}, \mathcal{J}$ countable Turing ideals, the following are equivalent:

- *$\mathcal{J}$ contains $\mathbf{a}'$ for every $\mathbf{a} \in \mathcal{I}$, and is a Scott set.*
- *$\mathcal{J} = \delta_{\mathcal{U}}(\mathcal{I})$ for some nonprincipal ultrafilter $\mathcal{U}$.*

What if we take ultrafilter jumps of uncountable Turing ideals?

Theorem (S.)

Consistently with ZFC + PD, the theorem fails to generalize badly. Specifically: for $V \models ZFC + PD$, there is a forcing extension $V[G]$ and a Turing ideal $\mathcal{I} \in V[G]$ such that

- *$\mathcal{I}$ is an elementary submodel of $\mathcal{P}(\omega)$;*
- *but $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any ultrafilter $\mathcal{U} \in V[G]$.*

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Forcing is countably closed, so no new reals. Also, for any name ν for an ultrafilter, have dense set of ("**good**") conditions $p = (M, A)$ deciding $\nu(X)$ for each $X \in M$.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Forcing is countably closed, so no new reals. Also, for any name ν for an ultrafilter, have dense set of ("**good**") conditions $p = (M, A)$ deciding $\nu(X)$ for each $X \in M$.

By PD, given condition p can find good $q = (M, A) \leq p$ such that $\delta_\nu(M) \notin M$ — we just build a sufficiently generic nice condition below p.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Forcing is countably closed, so no new reals. Also, for any name ν for an ultrafilter, have dense set of ("**good**") conditions $p = (M, A)$ deciding $\nu(X)$ for each $X \in M$.

By PD, given condition p can find good $q = (M, A) \leq p$ such that $\delta_{\nu}(M) \notin M$ — we just build a sufficiently generic nice condition below p. Pass to $(M, A \cup \{\delta_{\nu}(M)\})$.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Forcing is countably closed, so no new reals. Also, for any name ν for an ultrafilter, have dense set of ("**good**") conditions $p = (M, A)$ deciding $\nu(X)$ for each $X \in M$.

By PD, given condition p can find good $q = (M, A) \leq p$ such that $\delta_\nu(M) \notin M$ — we just build a sufficiently generic nice condition below p. Pass to $(M, A \cup \{\delta_\nu(M)\})$. $\mathcal{I}$ is union of left coordinates of generic.

Building an "unfixed" ideal

We want to force a projectively closed $\mathcal{I}$ with $\delta_{\mathcal{U}}(\mathcal{I}) \neq \mathcal{I}$ for any $\mathcal{U}$. Conditions: pairs (M, A) with M a countable elementary submodel of $\mathcal{P}(\omega)$, and A a countable set of reals not in M. Ordered by extension in both coordinates.

Forcing is countably closed, so no new reals. Also, for any name ν for an ultrafilter, have dense set of ("**good**") conditions $p = (M, A)$ deciding $\nu(X)$ for each $X \in M$.

By PD, given condition p can find good $q = (M, A) \leq p$ such that $\delta_\nu(M) \notin M$ — we just build a sufficiently generic nice condition below p. Pass to $(M, A \cup \{\delta_\nu(M)\})$. $\mathcal{I}$ is union of left coordinates of generic.

Remark

In other direction, note that assuming $V = L$ gives opposite answer for sufficiently closed ideals (definable ultrafilters).

Thanks!