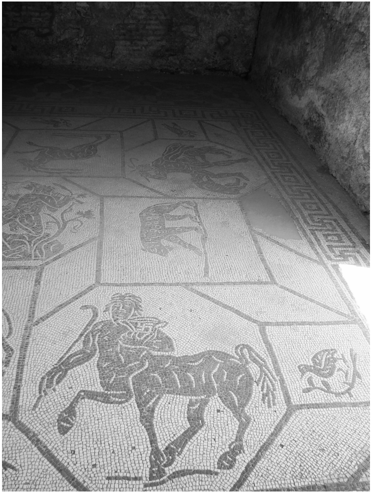
On Centauric Subshifts

Andrei Romashchenko
joint work with Bruno Durand

CIRM, 22.06.2016

Centauric tilings?

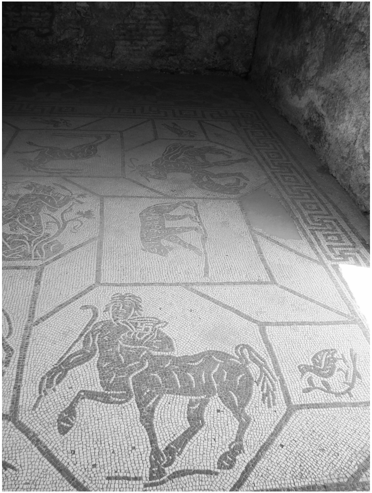
Centauric tilings?



Casa della Fortuna Annonaria, Ostia.

flickr photo by F. Tronchin

Centauric tilings?



Casa della Fortuna Annonaria, Ostia.

flickr photo by F. Tronchin

We mean tilings with *seemingly* mutually exclusive properties.

Shifts of finite type / Wang tilings

The idea: Simple **local rules** imply the global properties of an **infinite structure**.

Shifts of finite type / Wang tilings

The idea: Simple **local rules** imply the global properties of an **infinite structure**.

More specifically: in an **SFT** we have a finite set of *forbidden finite patterns*

Shifts of finite type / Wang tilings

The idea: Simple **local rules** imply the global properties of an **infinite structure**.

More specifically: in a **tiling** we have the matching rules for neighboring tiles

Shifts of finite type / Wang tilings

The idea: Simple **local rules** imply the global properties of an **infinite structure**.

More specifically: in a **tiling** we have the matching rules for neighboring tiles

Motivation: dynamical systems, computability, mathematical logic, quasi-crystals, . . .

Shifts of finite type / Wang tilings

The idea: Simple **local rules** imply the global properties of an **infinite structure**.

More specifically: in a **tiling** we have the matching rules for of neighboring tiles

Motivation: dynamical systems, computability, mathematical logic, quasi-crystals, . . .

Wang tiles

Formal definitions:

Color: an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$

Wang tiles

Formal definitions:

Color: an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$

Tile: a unit square with colored sides,

Wang tiles

Formal definitions:

Color: an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$

Tile: a unit square with colored sides,

i.e, element of C^4 

Wang tiles

Formal definitions:

Color: an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$

Tile: a unit square with colored sides,

i.e, element of C^4 

Tile set: a set $\tau \subset C^4$

Wang tiles

Formal definitions:

Color: an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$

Tile: a unit square with colored sides,
i.e, element of C^4 

Tile set: a set $\tau \subset C^4$

Tiling: a mapping $f : \mathbb{Z}^2 \to \tau$
that respects the matching rules

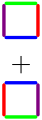
Tiling: a mapping $f : \mathbb{Z}^2 \rightarrow \tau$ such that

$$f(i,j).\text{right} \quad = \quad f(i+1,j).\text{left},$$

e.g.,

$$f(i,j).\text{top} \quad = \quad f(i,j+1).\text{bottom},$$

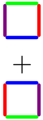
e.g.,



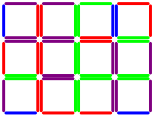
Tiling: a mapping $f : \mathbb{Z}^2 \to \tau$ such that

$$f(i,j).\text{right} \quad = \quad f(i+1,j).\text{left},$$ e.g.,

$$f(i,j).\text{top} \quad = \quad f(i,j+1).\text{bottom},$$ e.g.,



Example. A finite pattern from a valid tiling:



One point of view:
local rules can enforce high algorithmic complexity

- There exists a tile set τ such that:

One point of view:
local rules can enforce high algorithmic complexity

- There exists a tile set τ such that:
  - all τ-tilings are aperiodic [Berger, 1966]

One point of view:
local rules can enforce high algorithmic complexity

- There exists a tile set τ such that:
  - all τ-tilings are aperiodic [Berger, 1966]
  - no computable τ-tiling [Hanf, Myers, 1974]

One point of view:
local rules can enforce high algorithmic complexity

- There exists a tile set τ such that:
  - all τ-tilings are aperiodic [Berger, 1966]
  - no computable τ-tiling [Hanf, Myers, 1974]
  - high information density: each $N \times N$-square in a τ-tiling has high Komogorov complexity [Durand, Levin, Shen, 2001]

One point of view:
local rules can enforce high algorithmic complexity

- There exists a tile set τ such that:
  - all τ-tilings are aperiodic [Berger, 1966]
  - no computable τ-tiling [Hanf, Myers, 1974]
  - high information density: each $N \times N$-square in a τ-tiling has high Komogorov complexity [Durand, Levin, Shen, 2001]
- Every **effectively closed shift** in $1D$ can be *simulated* by vertical columns of a $2D$ tiling [Aubrun-Sablik, Durand-R.-Shen]

Another point of view:

local rules can enforce interesting dynamical properties

Another point of view:

local rules can enforce interesting dynamical properties

An example: all τ-tilings have exactly the same the set of *finite patterns*

Another point of view:
local rules can enforce interesting dynamical properties

An example: all τ-tilings have exactly the same the set of *finite patterns* (a minimal dynamical system)

Another point of view:
local rules can enforce interesting dynamical properties

An example: all τ-tilings have exactly the same the set of *finite patterns* (a minimal dynamical system)

A weaker version: Every τ-tiling must be quasiperiodic

Another point of view:
local rules can enforce interesting dynamical properties

An example: all τ-tilings have exactly the same the set of *finite patterns* (a minimal dynamical system)

A weaker version: Every τ-tiling must be quasiperiodic, i.e., each finite pattern either *never* appears or appears in *all large enough* squares.

Another point of view:
local rules can enforce interesting dynamical properties

An example: all τ-tilings have exactly the same the set of *finite patterns* (a minimal dynamical system)

A weaker version: Every τ-tiling must be quasiperiodic, i.e., each finite pattern either *never* appears or appears in *all large enough* squares.
(a uniformly recurrent dynamical system)

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Stereoscopic picture: merging together both views

Some restrictions:

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

- for every minimal SFT the set of finite patterns is computable

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

- for every minimal SFT the set of finite patterns is computable
- every minimal SFT contains a computable configuration

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

- for every minimal SFT the set of finite patterns is computable
- every minimal SFT contains a computable configuration
- for every quasiperiodic SFT the function of quasiperiodicity is computable [Ballier, Jeandel]

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

- for every minimal SFT the set of finite patterns is computable
- every minimal SFT contains a computable configuration
- for every quasiperiodic SFT the function of quasiperiodicity is computable [Ballier, Jeandel]
- Turing spectrum of quasiperiodic SFT must be upward close [Jeandel, Vanier]

Stereoscopic picture: merging together both views

Some restrictions: an irreducible SFT cannot be too complex,

- for every minimal SFT the set of finite patterns is computable
- every minimal SFT contains a computable configuration
- for every quasiperiodic SFT the function of quasiperiodicity is computable [Ballier, Jeandel]
- Turing spectrum of quasiperiodic SFT must be upward close [Jeandel, Vanier]
- after all, the standard constructions does not work!

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Theorem. There exists a tile set τ such that all tilings are aperiodic **and** quasiperiodic.

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Theorem. There exists a tile set τ such that all tilings are aperiodic **and** quasiperiodic.
Moreover, exactly the same finite patterns appear in all τ-tilings (minimality).

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Theorem. There exists a tile set τ such that all tilings are aperiodic **and** quasiperiodic.
Moreover, exactly the same finite patterns appear in all τ-tilings (minimality).

(Ballier and Ollinger [2009] did it with a version of Robinson's tile set)

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Theorem [Durand-R. 2015] There exists a tile set τ such that all tilings are *non computable* **and** *quasiperiodic*.

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Question: Can we enforce by local rules *non computability* **and** *minimality*?

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Question: Can we enforce by local rules *non computability* **and** *minimality*?
Answer: NO!

Stereoscopic picture: merging together both views

Can we enforce at the same time

(1) *high algorithmic complexity*
(aperiodicity, non-computability, etc.)

and

(2) *simple combinatorial structure*
(quasiperiodicity, minimality, etc.)?

Question: Can we enforce by local rules *non computability* **and** *minimality*?
Answer: NO! Every minimal SFT contains a computable point.

The message of this talk

Theorem 1. There exists a tile set τ such that all τ-tilings are *non computable* **and** *quasiperiodic*.

A stronger positive result

Theorem 2. There exists a tile set τ such that Kolmogorov complexity of every finite pattern is large **and** all tilings are quasiperiodic.

Another positive result (motivated by Pascal Vanier)

What about the Turing spectrum of quasiperiodic tilings?

Another positive result (motivated by Pascal Vanier)

What about the Turing spectrum of quasiperiodic tilings?

Preliminary remark 1: For every tile set τ, the set of τ-tilings is alway *effectively closed*.

Another positive result (motivated by Pascal Vanier)

What about the Turing spectrum of quasiperiodic tilings?

Preliminary remark 1: For every tile set τ, the set of τ-tilings is alway *effectively closed*.

Preliminary remark 2: For every quasiperiodic tile set *the Turing spectrum* of these tilings is alway *upward closed*.

Another positive result (motivated by Pascal Vanier)

What about the Turing spectrum of quasiperiodic tilings?

Preliminary remark 1: For every tile set τ, the set of τ-tilings is alway *effectively closed*.

Preliminary remark 2: For every quasiperiodic tile set *the Turing spectrum* of these tilings is alway *upward closed*. (Thanks, Pascal!)

Another positive result (motivated by Pascal Vanier)

What about the Turing spectrum of quasiperiodic tilings?

Preliminary remark 1: For every tile set τ, the set of τ-tilings is alway *effectively closed*.

Preliminary remark 2: For every quasiperiodic tile set *the Turing spectrum* of these tilings is alway *upward closed*. (Thanks, Pascal!)

Theorem 3. For every effectively closed set $\mathcal{A}$ there exists a tile set τ such that

- all τ-tilings are *quasiperiodic*,
- the Turing spectrum of all τ-tilings $=$ the *upper closure* of $\mathcal{A}$.

(*upper closure* $:=$ all degrees in $\mathcal{A}$ $+$ the degrees above them)

Another positive result (motivated by Emmanuel Jeandel)

Theorem 4. For every *minimal* 1D subshift $\mathcal{A}$ there exists a tile set τ such that

- the set of τ-tilings is *minimal*
- $\mathcal{A}$ is *simulated* by vertical columns of τ-tilings

Another positive result (motivated by Emmanuel Jeandel)

Theorem 4. For every *minimal* 1D subshift $\mathcal{A}$ (*minimal* $\Rightarrow$ *computable*) there exists a tile set τ such that

- the set of τ-tilings is *minimal*
- $\mathcal{A}$ is *simulated* by vertical columns of τ-tilings

Another positive result (motivated by Emmanuel Jeandel)

Theorem 4. For every *minimal* 1D subshift $\mathcal{A}$ (*minimal* $\Rightarrow$ *computable*) there exists a tile set τ such that

- the set of τ-tilings is *minimal*
- $\mathcal{A}$ is *simulated* by vertical columns of τ-tilings

cf.

Theorem [Aubrun-Sablik, Durand-R.-Shen 2013]
For every *effectively closed* 1D subshift $\mathcal{A}$ there exists a tile set τ such that $\mathcal{A}$ is *simulated* by vertical columns of τ-tilings.

Once again, the first nontrivial statement:

Theorem. There exists a tile set τ such that all τ-tilings are *aperiodic* **and** *quasiperiodic*.

Sketch of the proof:

Sketch of the proof:

In what follows we explain how to guarantee *aperiodicity* + *quasiperiodicity* of a tiling.

Sketch of the proof:

In what follows we explain how to guarantee *aperiodicity* + *quasiperiodicity* of a tiling.

The plan:

- enforce self-similarity of a tiling

Sketch of the proof:

In what follows we explain how to guarantee *aperiodicity* + *quasiperiodicity* of a tiling.

The plan:

- enforce self-similarity of a tiling

  self-simulation: using ideas of S. Kleene, J. von Neumann, P. Gács

Sketch of the proof:

In what follows we explain how to guarantee *aperiodicity* + *quasiperiodicity* of a tiling.

The plan:

- enforce self-similarity of a tiling

  self-simulation: using ideas of S. Kleene, J. von Neumann, P. Gács

  (Remember Linda's talk!)

Sketch of the proof:

In what follows we explain how to guarantee *aperiodicity* + *quasiperiodicity* of a tiling.

The plan:

- enforce self-similarity of a tiling

  self-simulation: using ideas of S. Kleene, J. von Neumann, P. Gács

  (Remember Linda's talk!)
- enforce replication of all patterns that you *may* have in a tiling

Fix a tile set τ and an integer $N > 1$.

Fix a tile set τ and an integer $N > 1$.

Definition 1. A τ**-macro-tile**: an $N \times N$ square made of matching τ-tiles.

Fix a tile set τ and an integer $N > 1$.

Definition 1. A τ-**macro-tile**: an $N \times N$ square made of matching τ-tiles.

Definition 2. A tile set ρ is **simulated** by τ: there exists a family of τ-macro-tiles R *isomorphic* to ρ such that every τ-tiling can be *uniquely* split by an $N \times N$ grid into macro-tiles from R.

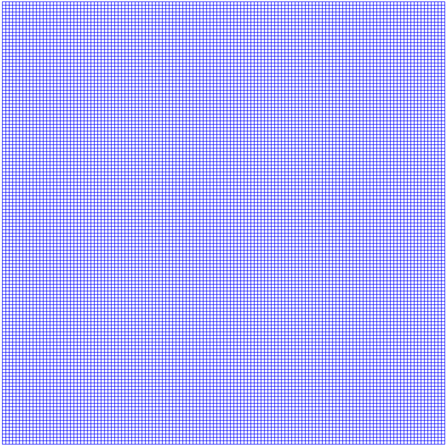
Self-similar tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

Self-similar tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

Theorem. Self-similar tile set is aperiodic.

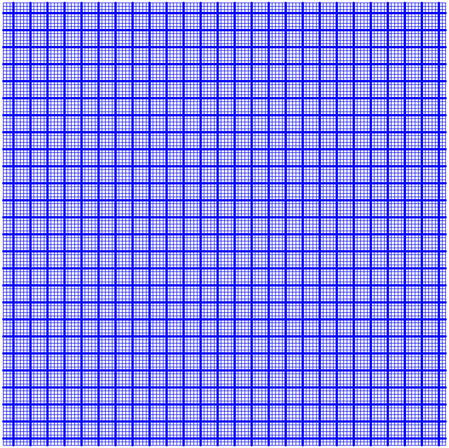
Self-similar tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

Theorem. Self-similar tile set is aperiodic.
Sketch of the proof:



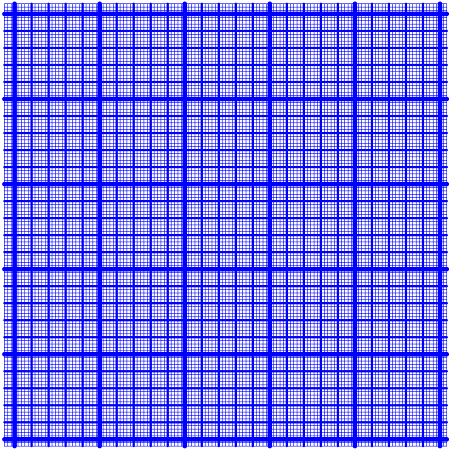
Self-similar tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

Theorem. Self-similar tile set is aperiodic.
Sketch of the proof:



Self-similar tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

Theorem. Self-similar tile set is aperiodic.
Sketch of the proof:



Simulating a given tile set ρ by macro-tiles.

Simulating a given tile set ρ by macro-tiles.
Representation of the tile set ρ:

Simulating a given tile set ρ by macro-tiles.

Representation of the tile set ρ:

- colors of a tile set ρ $\Longrightarrow$ k-bits strings

Simulating a given tile set ρ by macro-tiles.

Representation of the tile set ρ:

- colors of a tile set ρ $\Longrightarrow$ k-bits strings
- a tile set ρ $\Longrightarrow$ a predicate $\mathcal{P}(x_1, x_2, x_3, x_4)$ on 4-tuples of colors

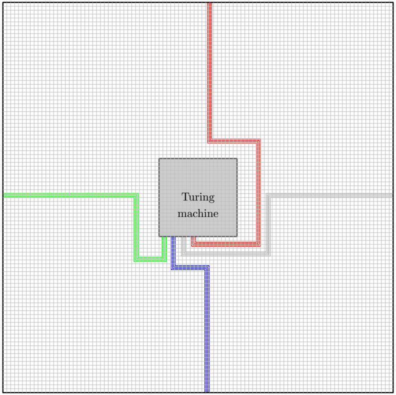
Simulating a given tile set ρ by macro-tiles.
Representation of the tile set ρ:

- colors of a tile set ρ $\Longrightarrow$ k-bits strings
- a tile set ρ $\Longrightarrow$ a predicate $\mathcal{P}(x_1, x_2, x_3, x_4)$ on 4-tuples of colors

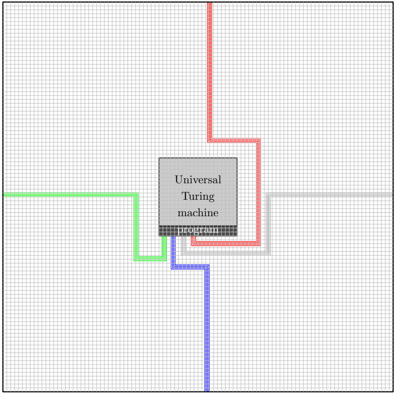
$\Downarrow$

a TM that accepts only 4-tuples of colors for the ρ-tiles

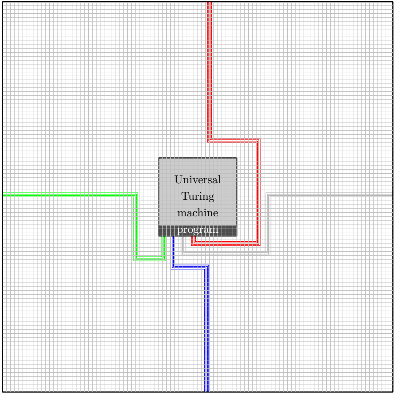
Implementation scheme:



A more generic construction:
universal TM + program



A more generic construction:
universal TM + program

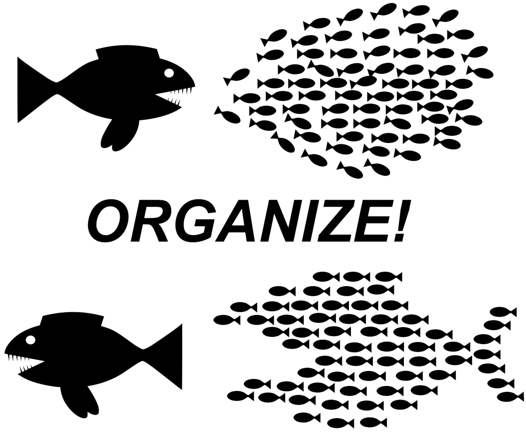


A fixed point: simulating tile set = simulated tile set

A similar metaphor in pop culture:

(*Picture by Worker, http://OpenClipArt.org/detail/102679/organize*)

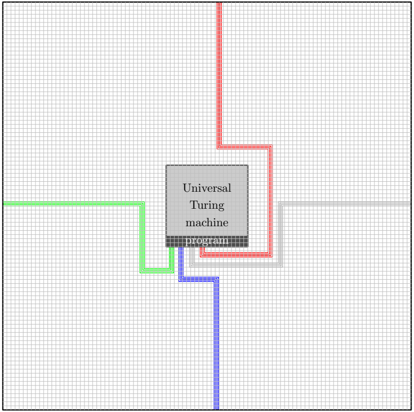
A similar metaphor in pop culture:



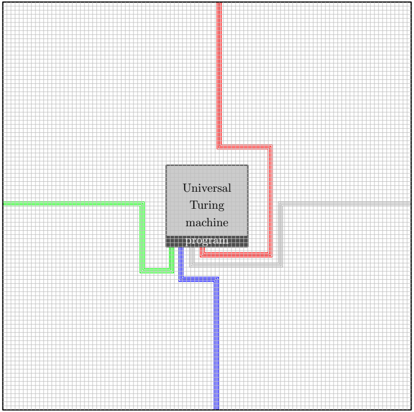
(*Picture by Worker, http://OpenClipArt.org/detail/102679/organize*)

...but we need (infinitely) many levels of self-simulation.

What about quasiperiodicity?

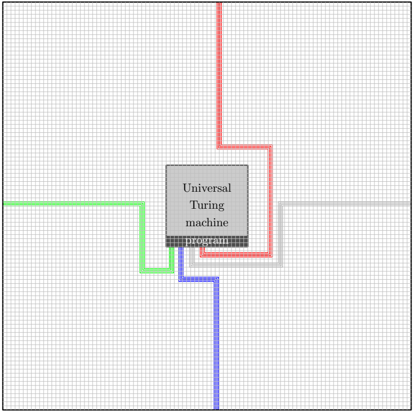


What about quasiperiodicity?



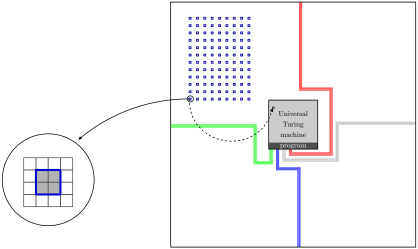
Good news: for self-similar tilings it is enough to prove that each 2×2-pattern in a tiling has "siblings" hereabouts.

What about quasiperiodicity?

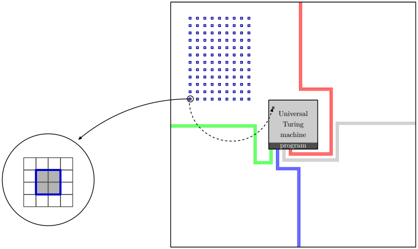


Bad news: the problematic parts are the *computation zone* and the *communication wires*.

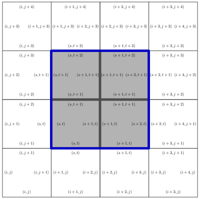
Replicate all 2 × 2 patterns that *may* appear in the computational zone!



Replicate all 2×2 patterns that *may* appear in the computational zone!



A slot for a 2×2 pattern from the comput. zone:



Theorem 1. There exists a tile set τ such that all tilings are *quasiperiodic* and *non computable*.

Theorem 1. There exists a tile set τ such that all tilings are *quasiperiodic* and *non computable*.

Proof: The same technique + variable zoom factor

Theorem 1. There exists a tile set τ such that all tilings are *quasiperiodic* and *non computable*.

Proof: The same technique + variable zoom factor + embed in a tiling an (infinite) verification of a separator for a pair of recursively non separable sets.

Theorem 1. There exists a tile set τ such that all tilings are *quasiperiodic* and *non computable*.

Proof: The same technique + variable zoom factor + embed in a tiling an (infinite) verification of a separator for a pair of recursively non separable sets.

Proofs of Theorems 2-4:
The same idea + more technical tricks.

Theorem 1. There exists a tile set τ such that all tilings are *quasiperiodic* and *non computable*.

Proof: The same technique + variable zoom factor + embed in a tiling an (infinite) verification of a separator for a pair of recursively non separable sets.

Proofs of Theorems 2-4:
The same idea + more technical tricks.

That's all!