Gambling Against Some Odds

Jake Pardo

The Pennsylvania State University

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Restricted Value Martingales

Definition

1. A *martingale* is a function $M : \{0,1\}^{<\mathbb{N}} \to \mathbb{R}$ such that

$$M(\sigma) = \frac{M(\sigma 0) + M(\sigma 1)}{2}$$

for every string $\sigma \in \{0,1\}^{<\mathbb{N}}$.

2. Let $S \subseteq \mathbb{R}^+$; an *S-martingale* is any martingale M that satisfies

$$|M(\sigma 1) - M(\sigma)| \in S$$

for all $\sigma \in \{0,1\}^{<\mathbb{N}}$.

3. The *success set* of an S-martingale M is

$$succ(M) = \{X \in \{0,1\}^{\mathbb{N}} : \limsup_n M(X \upharpoonright n) = \infty\}.$$

Anticipation and Evasion

Henceforth let $A, B \subseteq \mathbb{R}^+$, and assume $0 \in A \cap B$.

Definition

- B *singly anticipates* A if for every A-martingale X, there exists a B-martingale Y such that

$$succ(X) \subseteq succ(Y).$$

  Otherwise we say A *singly evades* B. If A and B singly anticipate each other then we say A and B are *strongly equivalent*.

- B *(countably) anticipates* A if for every A-martingale X, there exists a countable set of B-martingales $\{Y_1, Y_2, \ldots\}$ such that

$$succ(X) \subseteq \bigcup_{i \in \mathbb{N}} succ(Y_i).$$

  Otherwise we say A *(countably) evades* B.

Results

Definition

A scales into B if there exists $r > 0$ such that $rA \subseteq \overline{B}$.

Lemma (Bavly/Peretz)

If A scales into B then B singly anticipates A.

Lemma (Bavly/Peretz)

Every subset of $\mathbb{R}^+$ is strongly equivalent to its closure.

Bigger Results

Theorem (Bavly/Peretz)

If $\sup A < \infty$ *and* $0 \notin \overline{B \setminus \{0\}}$, *then* B *anticipates* A *if and only if* A *scales into* B.

Theorem (Bavly/Peretz)

If B *is well-ordered, i.e.* $\forall x \in \mathbb{R}^+ x \notin \overline{B \setminus [0,x]}$, *then* B *anticipates* A *if and only if* A *scales into* B.

Theorem (Bavly/Peretz)

Suppose there is a non-increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

1. $\overline{B} \supset f(x) \cdot (A \cap [0,x])$ *for every* $x \in \mathbb{R}^+$, *and*
2. $\int_0^\infty f(x)dx = \infty$.

Then B *singly anticipates* A.

A Few Examples

- $\{2n : n \in \mathbb{N}\}$ evades and singly anticipates $\{2n + 1 : n \in \mathbb{N}\}$.
- $\{1, \pi\}$ evades $\mathbb{N}$.
- $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$ evades $\mathbb{N}$.
- $\{\frac{1}{n} : n \in \mathbb{N}\}$ evades $\{0\} \cup [1, \infty)$.
- $\mathbb{R}^+$ and $[0, 1]$ are strongly equivalent.
- $A = \{2^{-n} : n \in \mathbb{Z}\}$ and $B = \{2^{-n} : n \in \mathbb{N}\}$ are strongly equivalent.
- If $B \cap [0, \epsilon]$ is dense in $[0, \epsilon]$ for any $\epsilon > 0$, then B singly anticipates A for any A.

Open Questions

1. Are there any other kinds of sets that singly anticipate everything?
2. Are single anticipation and countable anticipation different? That is, are there sets $A, B \subseteq \mathbb{R}^+$ such that B countably anticipates A but does not singly anticipate A?
3. What can be said about sets that include 0 as an accumulation point, e.g. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ or $\left\{\frac{1}{x^n} : n \in \mathbb{N}\right\}$ for some $x \in \mathbb{R}^+$?

A Partial Answer

Theorem (P.)

Let $A = \mathbb{N}$ *and* $B_x = \{x^{-n} : n \in \mathbb{N}\} \cup \{0\}$ *for* $x \in \mathbb{R}^+ \setminus \left\{ \sqrt[m]{\varphi} : m \in \mathbb{N}, m \geq 1 \right\}$ *where* φ *is the golden ratio,* $\frac{1+\sqrt{5}}{2}$*. Then* A *singly evades* B_x*.*

Intuitions and Assumptions

We can think in terms of a casino game between Anne (making wagers from A) and Bob (making wagers from B). In order to prove our result we must show that Anne can make infinite money in the long run while Bob does not make infinite money. We may assume that:

- First Anne bets, then Bob bets, and then the casino flips a coin. Each player who bet correctly on the result of the coin flip gains their wager, and each player who bet incorrectly loses their wager.
- The casino rigs the results of each coin flip in Anne's favor.
- Anne always bets on heads, and Bob always bets on the same outcome as Anne.

The Process at Stage i

Fix x as above with $x > 1$ (the $x \leq 1$ case follows from above); we develop a strategy that guarantees Anne has at least $2x^{i-1} + 5$ at the beginning of stage i and that gains her at least $2x^i$ during stage i – all while limiting how much Bob wins.
At the beginning of stage i (we call this START), Anne bets 1, and then the casino acts based on what Bob bets:

- If Bob bets an amount greater than or equal to $\frac{1}{x^{i-1}}$, the casino makes both players lose and gambling proceeds along the *Punishing Path*.
- If Bob bets an amount less than or equal to $\frac{1}{x^i}$, the casino makes both players win and gambling proceeds along the *Rewarding Route*.

The Punishing Path

Once the gamblers are on the "punishing path", Anne repeatedly bets 1, and then the casino acts based on what Bob bets:

- If Bob bets an amount greater than or equal to $\frac{1}{x^{i-1}}$, the casino makes both players lose.
- If Bob bets an amount less than or equal to $\frac{1}{x^{i}}$, the casino makes both players win.

The Punishing Path

This goes on until one of two things happens:

1. Anne has the same amount of money she had at START, while Bob has lost at least $\frac{1}{x^{i-1}} - \frac{1}{x^i}$ dollars. At this point we go back to START.
2. Anne has lost $\lceil x^{i-1} \rceil + 1$ dollars and Bob has lost at least $\frac{\lceil x^{i-1} \rceil + 1}{x^{i-1}}$ dollars. At this point Anne bets everything, Bob bets at most 1, and then the casino makes them both win. Anne now has more money than she had when the "punishing path" started, while Bob has lost at least $\frac{1}{x^{i-1}}$. At this point we again return to START.

No matter what happens we remain in stage i.

The Rewarding Route

Anne bets according to the Fibonacci sequence: she already bet \$1 getting onto the "rewarding route", then she bets \$1 again, then \$2, then \$3, then \$5, and so on as long as she keeps winning. If Anne ever loses, one of two things must have happened. If her most recent bet was either \$1 or \$2, then that means Anne now has the same amount of money she had at START, so in this instance the "rewarding route" ends and we go back to START. Otherwise, Anne goes back two steps in the Fibonacci sequence and continues betting from that point, e.g. if Anne loses when she bets \$13, she will next bet \$5, then \$8, and so on.

If Bob ever bets an amount that exceeds the sum of his "previous" two bets, then the casino makes both gamblers lose; otherwise, both players win.

The Rewarding Route

In order for Anne to gain the $2x^i$ dollars necessary to reach stage $i+1$, the "rewarding route" must continue until Anne is able to win a bet of F_k dollars for some $k \in \mathbb{N}$ satisfying

$$2x^i \leq \sum_{j=0}^{k} F_j.$$

Using several properties of the Fibonacci numbers one can show that it is sufficient to pick $k = \lceil i \log_{\varphi}(x) \rceil + 3$ to guarantee Anne meets her goal.

Bob's Total Winnings

To determine how much Bob can earn in total, we must consider several different cases for x:

1. $x \in (2, \infty)$,
2. $x \in (\varphi, 2]$,
3. $x \in (\sqrt[n+1]{\varphi}, \sqrt[n]{\varphi}) \cap (\sqrt[m+1]{2}, \sqrt[m]{2}]$ where $n, m \in \mathbb{N}$ and $m \geq n \geq 1$.

The Easy Case

Assume $x > 2$. Bob bet at most $\frac{1}{x^i}$ to get onto the "rewarding route", Bob's first bet on the "rewarding route" will always be at most $\frac{1}{x^i}$, and since $\frac{2}{x^i} < \frac{1}{x^{i-1}}$, Bob's next wager must also be at most $\frac{1}{x^i}$. Clearly this pattern repeats ad infinitum. Thus, if gambling proceeds for k rounds as above, the most Bob can earn during stage i is $\frac{k+1}{x^i}$ dollars. Thus, over the course of all stages $i \in \mathbb{N}$ Bob's total earnings are bounded above by

$$\sum_{i=0}^{\infty} \frac{\lceil i \log_{\varphi}(x) \rceil + 3}{x^i}$$

which is finite.

The Medium Case

Assume $x \in (\varphi, 2]$. We see that

$$\frac{x}{x^i} \leq \frac{2}{x^i} < \frac{x^2}{x^i}$$

so Bob's second bet on the "rewarding route" is at most $\frac{1}{x^{i-1}}$. Consider now Bob's third wager; since his last two wagers were $\frac{1}{x^i}$ and $\frac{1}{x^{i-1}}$, clearly he can bet at least $\frac{1}{x^{i-1}}$ – can he possibly bet $\frac{1}{x^{i-2}}$ though? In fact he cannot: since $x > \varphi$ we know that

$$x^2 - x - 1 > 0.$$

Rearranging the terms and dividing by x^i yields

$$\frac{1}{x^{i-2}} > \frac{1}{x^{i-1}} + \frac{1}{x^i}$$

so Bob's third bet must be at most $\frac{1}{x^{i-1}}$.

The Medium Case

Now that Bob's previous two wagers are $\frac{1}{x^{i-1}}$ and $\frac{1}{x^{i-1}}$, we see that this pattern continues: Bob's next two bets will be at most $\frac{1}{x^{i-2}}$, the two bets after that will be at most $\frac{1}{x^{i-3}}$, and so on. Thus, after k rounds on the "rewarding route", Bob's winnings are bounded above by

$$\sum_{j=0}^{\lceil \frac{k}{2} \rceil} \frac{2}{x^{i-j}} = \frac{2}{x^i} \cdot \frac{x^{\lceil \frac{k}{2} \rceil + 1} - 1}{x - 1} < \frac{2x^4}{x-1} \cdot x^{i\left(\frac{\log_\varphi(x)}{2} - 1\right)}.$$

Summing up over all stages $i \in \mathbb{N}$, we see that Bob's total earnings are bounded by

$$\frac{2x^4}{x-1} \sum_{i=0}^{\infty} x^{i\left(\frac{\log_\varphi(x)}{2} - 1\right)}$$

which is finite since $\log_\varphi(x) < 2$.

The Hard Case

Assume $x \in (\sqrt[n+1]{\varphi}, \sqrt[n]{\varphi}) \cap (\sqrt[m+1]{2}, \sqrt[m]{2}]$ where $n, m \in \mathbb{N}$ and $m \geq n \geq 1$. Since $x \in (\sqrt[m+1]{2}, \sqrt[m]{2}]$ we know that

$$\frac{x^m}{x^i} \leq \frac{2}{x^i} < \frac{x^{m+1}}{x^i}$$

so Bob's second bet on the "rewarding route" must be at most $\frac{1}{x^{i-m}}$. For Bob's following wagers, note that because $x < \sqrt[n]{\varphi}$ we know that

$$1 + x^n > x^{2n}$$

and because $x > \sqrt[n+1]{\varphi}$ we know

$$1 + x^{n+1} < (x^{n+1})^2 = x^{2n+2}.$$

Dividing by x^i in these inequalities shows that if Bob's last wager was x^n times his wager before that, his current wager can definitely be at least x^n times his previous wager and definitely not x^{n+2} times his previous wager.

The Hard Case

Since $m \geq n$, we can bound Bob's third wager on the "rewarding route" by either $\frac{1}{x^{i-(m+n)}}$ or $\frac{1}{x^{i-(m+n+1)}}$. Moreover, after this point we see that each of Bob's successive bets will be at most the previous bet multiplied by either x^n or x^{n+1}. The question now is when can we assume Bob's bets are going up by n factors of x and not $n+1$ factors of x? To answer this question we must break into two smaller cases.

The First Subcase

Assume $x \in [\sqrt[n+\frac{1}{2}]{\varphi}, \sqrt[n]{\varphi})$. Since $x \geq \sqrt[n+\frac{1}{2}]{\varphi}$, we know that

$$1 + x^n < 1 + x^{n+\frac{1}{2}} \leq (x^{n+\frac{1}{2}})^2 = x^{2n+1}.$$

Thus, we know that in this case Bob's wagers are always going to be x^n times his previous wager, and we may bound his third wager in the "rewarding route" by $\frac{1}{x^{i-(m+n)}}$.

The First Subcase

We see for this subcase that after k steps on the "rewarding route", Bob's winnings are bounded above by

$$\frac{2}{x^i} + \sum_{j=0}^{k-2} \frac{x^{m+jn}}{x^i} \le \sum_{j=0}^{k-1} \frac{x^{m+jn}}{x^i} = \frac{x^m}{x^i} \cdot \frac{x^{kn}-1}{x^n-1}$$

$$< \frac{x^m}{x^n-1} \cdot x^{kn-i} \le \frac{x^{m+4n}}{x^n-1} \cdot x^{i\left(n \log_\varphi(x) - 1\right)}.$$

As a result we see that Bob's total winnings over all stages $i \in \mathbb{N}$ are bounded by

$$\frac{x^{m+4n}}{x^n-1} \sum_{i=0}^{\infty} x^{i\left(n \log_\varphi(x) - 1\right)}$$

which we know is finite because $\log_\varphi(x) < \frac{1}{n}$.

The Second Subcase

Assume $x \in (\sqrt[n+1]{\varphi}, \sqrt[n+\frac{1}{2}]{\varphi})$. Whenever Bob's last wager is x^n times his wager before that, we allow his next wager can be x^{n+1} times his last wager; also we say that Bob's third wager in the "rewarding route" is bounded by $\frac{1}{x^{i-(m+n+1)}}$.
By the same calculations as above, we know that Bob's *next* wager cannot be x^{n+1} times his current wager, so it must be x^n times his current wager. Observe that this pattern must repeat: Bob's wagers alternate between being x^n times the previous wager and x^{n+1} times the previous wager.

The Second Subcase

We are now able to overestimate Bob's winnings by considering each of his wagers as $x^{n+\frac{1}{2}}$ times his previous wager and adding in an extra factor of $x^{\frac{1}{2}}$. This gives the upper bound

$$\frac{2}{x^i} + \sum_{j=0}^{k-2} \frac{x^{m+\frac{1}{2}+j(n+\frac{1}{2})}}{x^i} \leq \sum_{j=0}^{k-1} \frac{x^{m+\frac{1}{2}+j(n+\frac{1}{2})}}{x^i} = \frac{x^{m+\frac{1}{2}}}{x^i} \cdot \frac{x^{k(n+\frac{1}{2})} - 1}{x^{n+\frac{1}{2}} - 1}$$

$$< \frac{x^{m+\frac{1}{2}}}{x^{n+\frac{1}{2}} - 1} \cdot x^{k(n+\frac{1}{2})-i} \leq \frac{x^{m+4n+\frac{5}{2}}}{x^{n+\frac{1}{2}} - 1} \cdot x^{i\left((n+\frac{1}{2})\log_{\varphi}(x)-1\right)}.$$

Adding up these bounds for each stage $i \in \mathbb{N}$, we see that Bob's total winnings are bounded above by

$$\frac{x^{m+4n+\frac{5}{2}}}{x^{n+\frac{1}{2}} - 1} \sum_{i=0}^{\infty} x^{i\left((n+\frac{1}{2})\log_{\varphi}(x)-1\right)}$$

which we know is finite since $\log_{\varphi}(x) < n + \frac{1}{2}$.

Extending the Result

Try using the k-Fibonacci numbers!

Theorem (P.)

Let $A = \mathbb{N}$ *and* $B_x = \{0\} \cup \{x^{-n} : n \in \mathbb{N}\}$ *for* $x \in \mathbb{R}^+$*; then* A *singly evades* B_x*.*

Corollary (P.)

Let $A = \mathbb{N}$ *and* $B_x = \{0\} \cup \{x^{-n} : n \in \mathbb{Z}\}$ *for* $x \in \mathbb{R}^+$*; then* A *singly evades* B_x*.*

Conjecture

Let $A = \mathcal{F}^k$ *and* $B_x = \{x^{-n} : n \in \mathbb{N}\} \cup \{0\}$ *with* $k \in \mathbb{N} \cup \{\infty\}$ *and* $x \in \mathbb{R}^+ \setminus \{\sqrt[m]{\varphi_k} : m \in \mathbb{N}, m \geq 1\}$*. Then* A *singly evades* B_x*.*

Thanks for listening!

Relevant Papers

- Bavly and Peretz. How to gamble against all odds. *Games and Economic Behavior* 2014.
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