Randomness connecting to set theory and to reverse mathematics

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Goals

- Connect randomness and computability to cardinal characteristics in set theory:
  - define a dual $\Delta(A)$ of the Gamma operator,
  - show $\Delta(A) > 0 \to \Delta(A) = 1/2$ by dualising Monin's proof

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- Connect randomness and computability to cardinal characteristics in set theory:
  - define a dual $\Delta(A)$ of the Gamma operator,
  - show $\Delta(A) > 0 \to \Delta(A) = 1/2$ by dualising Monin's proof
- Connect randomness to reverse mathematics:
  - study the axiom power needed to verify equivalence of randomness notions;
  - study strength of randomness existence axioms in the setting of reverse mathematics

PART I:

The Γ and Δ operators, and
cardinal characteristics in set theory

The Γ operator

For $Z \subseteq \mathbb{N}$ the lower density is defined to be

$$\underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0,n)|}{n}.$$

Recall that

$$\gamma(A) = \sup_{X \text{ computable}} \underline{\rho}(A \leftrightarrow X)$$

The Γ operator was introduced by Andrews, Cai, Diamondstone, Jockusch and Lempp (2013):

$$\Gamma(A) = \inf\{\gamma(Y) \colon\ Y \leq_T A\}.$$

This only depends on the Turing degree of A.

Viewing $1 - \Gamma$ as a Hausdorff pseudodistance

For $Z \subseteq \mathbb{N}$ the upper density is defined by

$$\overline{\rho}(Z) = \limsup_n \frac{|Z \cap [0, n]|}{n}.$$

- For $X, Y \in 2^{\mathbb{N}}$ let $d(X, Y) = \overline{\rho}(X \triangle Y)$ be the upper density of the symmetric difference of X and Y
- this is a pseudodistance on Cantor space $2^{\mathbb{N}}$ (that is, two objects may have distance 0 without being equal).

Let $\mathcal{R} \subseteq \mathcal{A} \subseteq M$ for a pseudometric space(M, d). The Hausdorff distance is $d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S))$.

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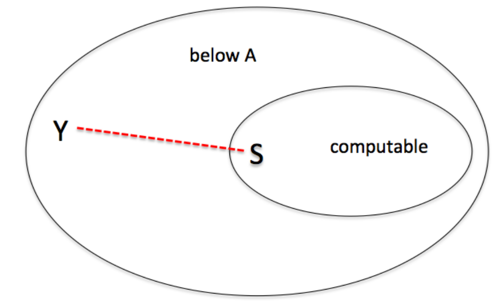
Given an oracle set A let $\mathcal{A} = \{Y : Y \leq_{\mathrm{T}} A\}$. Let $\mathcal{R} \subseteq \mathcal{A}$ denote the collection of computable sets. We have

$$1 - \Gamma(A) = d_H(\mathcal{A}, \mathcal{R}).$$



Δ operator, a dual to Γ

$$\begin{aligned} \delta(Y) &= \inf\{\underline{\rho}(Y \leftrightarrow S)\colon\ S \text{ computable}\} \\ \Delta(A) &= \sup\{\delta(Y)\colon\ Y \leq_T A\}. \end{aligned}$$

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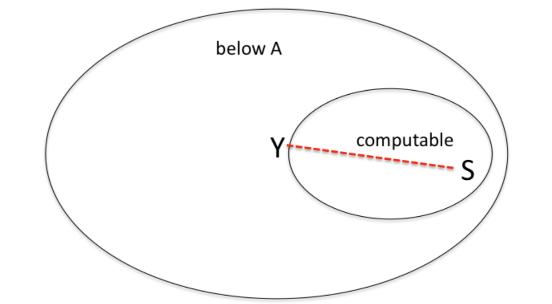
- $\Gamma(A)$ measures how well computable sets can approximate the sets that A computes.
  "$\Gamma(A) > p$" for fixed $p \in [0, 1)$ is a lowness property.
- $\Delta(A)$ measures how well the sets that A computes can approximate the computable sets.
  "$\Delta(A) > p$" is a highness property.

Interpreting $1 - \Delta(A)$ metrically

We can view $1 - \Delta(A)$ as a one-sided "dual" of the Hausdorff distance:

$$1 - \Delta(A) = d^*_H(\mathcal{A}, \mathcal{R}) = \inf_{Y \in \mathcal{A}} \sup_{S \in \mathcal{R}} d(Y, S).$$



Example: for the unit disc $D \subseteq \mathbb{R}^2$ we have $d^*_H(D, D) = 1$.

$$\begin{aligned}\delta(Y) &= \inf\{\underline{\rho}(Y \leftrightarrow S)\colon\ S \text{ computable}\}\\ \Delta(A) &= \sup\{\delta(Y)\colon\ Y \le_T A\}.\end{aligned}$$

Properties of δ and Δ (w. Merkle and Stephan)

- $\delta(Y) \le 1/2$ for each Y (by considering also the complement of S)
- Y Schnorr random $\Rightarrow \delta(Y) = 1/2$ (by law of large numbers)
- A computable $\Rightarrow \Delta(A) = 0$.
- $\Delta(A) = 0$ is possible for noncomputable A, e.g. if A is low and c.e., or 2-generic .

Cardinal characteristics and their analogs

We use analogs of cardinal characteristics in set theory. Consider a binary relation $R \subseteq \mathcal{X} \times \mathcal{Y}$ between sets, functions (or other objects encoded by reals).

- In set theory one lets

$$\mathfrak{b}(R) = \min\{|F| : F \subseteq \mathcal{X} \wedge \forall y \in \mathcal{Y} \exists x \in F\, [\neg xRy]\}$$

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- In computability we let

$$\mathcal{B}(R) = \{A\colon\ \exists y \leq_{\mathrm{T}} A \forall x \text{ computable } [xRy]\}$$

  (e.g., the same R yields highness.)

See Rupprecht, Thesis, 2010; Brooke, Brendle, Ng, N., 2014.

The highness classes $\mathcal{B}(\sim_p)$

Definition (Brendle and N.)

For $p \in [0, 1/2)$ let $S \sim_p Y$ if $\underline{\rho}(S \leftrightarrow Y) > p$, and

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The connection to Δ: for each $p \in [0, 1/2)$ we have

$$\Delta(A) > p \Rightarrow A \in \mathcal{B}(\sim_p) \Rightarrow \Delta(A) \geq p.$$

We will show that all the classes $\mathcal{B}(p)$ coincide, for $0 < p < 1/2$. Therefore:

$$\Delta(A) > 0 \Rightarrow \Delta(A) = 1/2.$$

A.e. avoiding a computable function

Definition ($\mathcal{B}(\neq^*, h)$)

For a computable function h, we let
$\mathcal{B}(\neq^*, h) = \{A\colon \exists f \leq_{\mathrm{T}} A, f < h \, \forall r \text{ computable } \forall^\infty n \, f(n) \neq r(n)\}$.

- This gets easier as h grows faster.
- In the extreme, $\mathcal{B}(\neq^*)$, i.e. the class obtained when we omit the computable bound, coincides with "high or diagonally noncomputable".

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Fact

A computes a Schnorr random $\Rightarrow A \in \mathcal{B}(\neq^*, 2^{\hat{h}})$ whenever $\hat{h}$ is computable and $\infty > \sum_n 1/\hat{h}(n)$. E.g. $\hat{h}(n) = n^2$.

Recall: for $p \in (0, 1/2)$ let

$$\mathcal{B}(\sim_p) = \{A\colon\ \exists Y \leq_{\mathrm{T}} A\,\forall S \text{ computable } \underline{\rho}(S \leftrightarrow Y) > p\}.$$

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Theorem (N., dual form of Monin's result)

$\mathcal{B}(\sim_p) = \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

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Corollary

$\Delta(A) > 0 \Leftrightarrow \Delta(A) = 1/2 \Leftrightarrow A \in \mathcal{B}(\neq^*, 2^{(2^n)})$.

Show the harder inclusion $\mathcal{B}(\sim_p) \supseteq \mathcal{B}(\neq^*, 2^{(2^n)})$:

Relation 1: Let $q > p$ such that $q < 1/2$. For $h(n) = 2^{\hat{h}(n)}$ and functions $x, y < h$, view $x(n)$ as string of length $\hat{h}(n)$.

$$x \neq^*_{\hat{h},q} y \Leftrightarrow \forall^\infty n \,|\{i < \hat{h}(n) \colon x(n)(i) \neq y(n)(i)\}| \geq nq.$$

Four steps:

1. there is k such that where $\hat{h}(n) = \lfloor 2^{n/k} \rfloor$
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Relation 2: Let $L \in \mathbb{N}$ and u be a function. For a trace s consisting of L-element sets, and a function $y < u$, let

$$s \not\ni^*_{u,L} y \Leftrightarrow \forall^\infty n [s(n) \not\ni y(n)].$$

Four steps:

2. There are $L \in \mathbb{N}$, $\epsilon > 0$ such that where $u(n) = 2^{\lfloor \epsilon \hat{h}(n) \rfloor}$, we have $\mathcal{B}(\neq^*_{\hat{h},q}) \supseteq \mathcal{B}(\not\ni^*_{u,L})$.

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2. There are $L \in \mathbb{N}$, $\epsilon > 0$ such that where $u(n) = 2^{\lfloor \epsilon \hat{h}(n) \rfloor}$, we have $\mathcal{B}(\neq^*_{\hat{h},q}) \supseteq \mathcal{B}(\not\ni^*_{u,L})$.
3. $\mathcal{B}(\not\ni^*_{u,L}) \quad \supseteq \quad \mathcal{B}(\not\ni^*_{2^{(L2^n)},L})$.
4. Finally, $\mathcal{B}\langle \not\ni^*, 2^{(L2^n)}, L\rangle \supseteq \mathcal{B}(\neq^*, 2^{(2^n)})$

Separations?

Recall: For a computable function h, we let
$\mathcal{B}(\neq^*, h) = \{A\colon \exists f \leq_{\mathrm{T}} A, f < h \, \forall r$ computable
$\forall^\infty n \, f(n) \neq r(n)\}$.

It is easy to show $\mathcal{B}(\sim_0) \subseteq B(\neq^*, 2^{n!})$.

Question

Is $\mathcal{B}(1/4) \subset \mathcal{B}(\sim_0)$? Is $\mathcal{B}(\neq^*, 2^{(2^n)}) \subset \mathcal{B}(\neq^*, 2^{n!})$?

In fact we don't know much about any separations
$\mathcal{B}(\neq^*, g) \subset \mathcal{B}(\neq^*, h)$ for $g << h$.

Maybe set theory can help: the analog of $\mathcal{B}(\neq^*, h)$ is the cardinal characteristic

$\mathfrak{b}(\neq^*_h) =$ the least size of a set F of functions such that for each h-bounded function y,
there is a function $x \in F$ with $\exists^\infty n \, [x(n) = y(n)]$.

Separating the $\mathfrak{b}(\neq^*_h)$ in a suitable model of ZFC

$\mathfrak{b}(\neq^*_h)$ = the least size of a set F of functions such that for each h-bounded function y, there is a function x in F with $\exists^\infty n\ [x(n) = y(n)]$.

Theorem (Kamo and Osuga 2014, special case)

Let $\langle \lambda_n \rangle_{n<\omega}$ be a strictly increasing sequence of regular cardinals $> \aleph_0$, e.g. $\lambda_n = \aleph_{n+1}$.

There is a forcing notion $\mathbb{P}$ with the countable (anti)chain condition that forces:
there is a sequence of functions $\langle h_n \rangle_{n<\omega}$ in the ground model such that $\mathfrak{b}(\neq^*_{h_n}) = \lambda_n$ for each n.

The c.c.c. implies that cardinals remain cardinals.

PART II:

Randomness, analysis, reverse mathematics

Systems based on randomness notions

Let C denote a randomness notion. We study the strength of the subsystem of second-order arithmetic

$$\mathsf{C}_0 = \mathsf{RCA}_0 + \forall X \exists Y \, [Y \in \mathsf{C}^X].$$

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Notation:

- MLR is ML-randomness,
- CRand is computable randomness,
- SRand is Schnorr randomness.

$$\mathsf{MLR} \Rightarrow \mathsf{CRand} \Rightarrow \mathsf{SRand}$$

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$$\mathsf{MLR} \Rightarrow \mathsf{CRand} \Rightarrow \mathsf{SRand}$$

We will also use C to denote the axiom $\forall X \exists Y \, [Y \in \mathsf{C}^X]$.

Theorem (Simpson and X. Yu, 1990)

MLR is equivalent to WWKL over RCA_0.

Formalising randomness notions

Care has to be taken how to formalise the corresponding systems. For instance we can't assume measure theory to define MLR, as this needs WWKL.

MLR

- A ML-test relative to X is given by an X-computable sequence of trees $\langle T_i \rangle_{i \in \mathbb{N}}$ such that $\mu[T_i] \geq 1 - 2^{-i}$, where $\mu[T_i]$ denotes $\lim_n 2^{-n} |T_i^{=n}|$ (relative size of the n-th level). It simulates the ML-test $\left\langle 2^{\mathbb{N}} - [T_i] \right\rangle_{i \in \mathbb{N}}$
- Y is ML-random in X if for each such sequence, $Y \in [T_i]$ for some i.

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CRand

Y is computably random in X if each martingale computable in X fails on Y.

Equivalences in the framework of reverse maths

Theorem (Yokoyama and N.)

Over RCA_0, MLR is equivalent to the statements (suitably formulated)

- every continuous function of bounded variation is differentiable somewhere
- every continuous function of bounded variation is differentiable almost everywhere.

Original proof uses infinite pigeonhole principle $\mathsf{RT}^1_{<\infty}$ in one important place; we needed to get rid of that.

Equivalences in the framework of reverse maths

$C^X(\sigma)$ denotes plain Kolmogorov complexity of σ relative to oracle X.

Theorem (Shafer and N.)

Over $B\Sigma_2$, $\mathsf{2Rand}$ is equivalent to the statement
for each X there is Z such that $\exists^\infty n\, C^X(Z \mid n) \geq n - O(1)$.

- Right-to-left actually works over RCA
- left-to-right may as well (wip).

Other equivalences left to be done: e.g.
$\mathsf{2Rand} \leftrightarrow \mathsf{MLR} \cap \mathsf{Low}(\Omega)$.

An ω-model of CRand_0 without a d.n.c. function

- Every high set is Turing above a computably random set (N., Stephan and Terwijn 2005).
- By the proof of Lemma 4.1 in Cholak, Greenberg, et al. 06, for each set B of non-d.n.c. degree there is a set X, high relative to B, such that $B \oplus X$ is also not of d.n.c. degree.
- Iterating this in the standard way, we build an ω-model of CRand_0 without a set of d.n.c. degree.
- In particular, there is no ML-random set.

$\mathsf{RCA}_0 \vdash \mathsf{SRand} \to \mathsf{CRand}$

Let $\mathcal{M}$ be a model of $\mathsf{SR_0}$. Given a set X of $\mathcal{M}$, we want to find a set Y in $\mathcal{M}$ that is computably random in X. Let Z be a set of $\mathcal{M}$ that is Schnorr random in X.

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- Let $f(m) = \mu s. Z \in G_{m,s}$. Then $f \leq_{\mathrm{T}} Z \oplus X$, hence f exists in $\mathcal{M}$.

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- Use f to build a martingale L that dominates all X-computable martingales. Let Y be a set on which L does not succeed.

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$\mathsf{W2Rand}_0 \not\vdash \mathsf{2Rand}$ shown by an ω-model

- Take a weakly 2-random Z that does not compute a 2-random.
  For instance, a 2-random is not computably dominated.
  Any computably dominated ML-random Z is weakly 2-random and hence does not compute a 2-random.
- For each n let Z_n be the n-th column of Z, that is, $Z_n = \{k \colon \langle k, n \rangle \in Z\}$.
- Let $\mathcal{M} = (\omega, \mathcal{S})$ where $\mathcal{S}$ consists of all the sets Turing below the join of finitely many columns of Z.
- Z_n is weakly 2-random in any finite sum of columns not containing Z_n. So $\mathcal{M}$ is a model of $\mathsf{W2Rand}_0$.

Notions slightly stronger than MLR

An *h-Demuth test* for computable function h is an effective sequence $\langle \mathcal{U}_n \rangle$ of effectively open (Σ^0_1) subsets of Cantor space such that:

- For all n, the measure $\lambda(\mathcal{U}_n)$ of $\mathcal{U}_n$ is bounded by 2^{-n}
- there is an h-c.e. function mapping n to a Σ^0_1 index for $\mathcal{U}_n$.

A set Z is h-weakly Demuth random if $Z \notin \bigcap_n \mathcal{U}_n$ for every h-Demuth test. Z is balanced random if Z is $O(2^n)$ weakly Demuth random.

Proposition (Figueira et al. 2015)

Let $Z = Z_0 \oplus Z_1$ be ML-random. Then Z_0 or Z_1 is balanced random.

So MLR_0 + sufficient induction $\vdash$ $\mathsf{BalancedRd}$.

$\mathsf{WKL}_0 \not\vdash 2^{n \log\log n} - \mathsf{weakDemRd}$ via ω-model

Definition

For a computable function h, we say that a set Z is h-c.e. if there is a computable approximation such that $Z \upharpoonright n$ changes at most $h(n)$ times.

Proposition (Shafer and N.)

There is an ω-model $\mathcal{M}$ of $\mathtt{WKL}$ such that each set of $\mathcal{M}$ is superlow and k^n-c.e. for some $k \in \mathbb{N}$.

- An h-c.e. set is not h-weak Demuth random.
- So $\mathcal{M}$ satisfies $\mathtt{WKL}_0$, but not the axiom for weak h-Demuth randomness, for any function h dominating all the functions k^n (e.g. $h(n) = 2^{n \log\log n}$).

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