On totally ω-c.e. degrees and complex left-c.e. reals

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Overview

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Definition

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ *be a v.s.a. A c.e. set A is called* $\mathcal{F}$-array noncomputable *(*$\mathcal{F}$*-a.n.c.) if for all c.e. sets B,*

$$\exists^\infty n \, (A \cap F_n = B \cap F_n).$$

A is called array noncomputable *(a.n.c.) if it is* $\mathcal{F}$*-a.n.c. for some v.s.a.* $\mathcal{F}$*.*

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Some simple observations on array noncomputable sets and degrees.

- For every v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$, there is an $\mathcal{F}$-a.n.c. c.e. set A (take $A = \bigcup_{n \in \omega} W_n \cap F_n$).
- For every c.e. set A, there is a v.s.a. $\mathcal{F}$ such that A is not $\mathcal{F}$-a.n.c. (for A noncomputable, take a computable infinite subset D of A and a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that $\forall n \ D \cap F_n \neq \emptyset$.

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An important example of a multiple permitting argument shown by Downey, Jokusch and Stob is that the choice of the v.s.a. $\mathcal{F}$ in an a.n.c. degree is independent up to equivalence.

Theorem (Downey,Jockusch, Stob)

Let $\mathcal{F}$ and $\mathcal{E}$ be very strong arrays and let A be c.e. and $\mathcal{F}$-a.n.c. Then there exists a $\mathcal{E}$-a.n.c. c.e. set $B \in deg(A)$.

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Theorem (Downey, Jockusch, Stob)

The following are equivalent for a c.e. degree $\mathbf{a}$.

- $\mathbf{a}$ *is a.n.c.*
- *For every computable order* h *there exists a function* $g \leq_T \mathbf{a}$ *such that for all computable approximations* $\{g_s\}_{s \geq 0}$ *of* g, *there exists* x *such that*

$$|\{s : g_{s+1}(x) \neq g_s(x)\}| > h(x)$$

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Definition

A degree **a** *is* not totally ω-c.e. *if there exists a function $g \leq_T \mathbf{a}$ which is* not ω-c.e., *i.e., for all computable orders h and all computable approximations $\{g_s\}_{s\geq 0}$ of g, there exists x such that*

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Statement of the Theorem

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Theorem (Ambos-Spies, Losert, Monath)

For a c.e. degree $\mathbf{a}$*, the following are equivalent.*

1. $\mathbf{a}$ *is not totally* ω*-c.e.*
2. *There is a left-c.e. real* $A \in \mathbf{a}$ *such that* A *is not cl-reducible to any left-c.e. complex real.*

(A left-c.e. real is complex if there exists a computable order h such that $C(A{\upharpoonright}n) \geq h(n)$.)

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(A left-c.e. real is complex if there exists a computable order h such that $C(A \upharpoonright n) \geq h(n)$.) This theorem proves a question by Greenberg which is related to the following characterization of a.n.c. degrees in [BDG10].

Theorem (Barmpalias, Downey, Greenberg)

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- $\mathbf{a}$ *is a.n.c.*
- *There is a left-c.e. real* $A \in a$ *which is not cl-reducible to any 1-random left-c.e. real.*

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In the remainder of the talk, we describe the main steps how to prove the implication ❶ $\Rightarrow$ ❷. The proof is based on the following observations.

Proposition

For a left-c.e. real A, the following are equivalent.

- *A is complex.*
- *A is wtt-hard, i.e., every c.e. set is wtt-reducible to A.*

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- *A is complex.*
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The proposition follows from the following facts.

- By a theorem of Kanovich [DH10, Theorem 8.16.7], a c.e. set is complex if and only if it is *wtt*-complete.
- Every left-c.e. real is *wtt*-equivalent to a c.e. set.
- Complex sets are closed upwards under *wtt*-reductions.

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In order to further reformulate the Theorem, let us define the notion of a maximal pair of left-c.e. reals.

Definition

A maximal pair *of left-c.e. reals (maximal pair for short) is a pair* (A, B) *of left-c.e. reals* (A, B) *such that there is no left-c.e. real* C *with* $A \leq_{cl} C$ *and* $B \leq_{cl} C$*.*

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Proposition

For a left-c.e. real A, the following are equivalent.

(i) *A is not cl-reducible to any wtt-hard left-c.e. real.*

(ii) *For any infinite, computable set* $D \subseteq \omega$*, there is a* c.e. *set* $B \subseteq D$ *such that* (A, B) *is a maximal pair of left-c.e. reals.*

The Theorem reformulated

Hence, what we actually show is the following.

Theorem

Let $\mathbf{a}$ *be a c.e. and not totally* ω*-c.e. degree. Then there exists a left-c.e. real* $A \in \mathbf{a}$ *such that for every infinite computable set* D*, there exists a c.e. subset* B *of* D *such that* (A, B) *is a maximal pair in the left-c.e. reals.*

In the following, we always refer to this theorem.

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We use a refined version of Fan's construction for the Theorem. As a first simple observation, it is easy to see that in Fan's construction, one can construct the c.e. half such that it is a subset of a given infinite, computable set D. Hence, we get the statement

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$$\forall D \text{ inf., comp. } \exists A \text{ left-c.e. } \exists B \subseteq D \text{ c.e. } ((A, B) \text{ is a maximal pair}).$$

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The proof of the above statement is split in two parts.

- First, the existence of a left-c.e. real with a strong similarity property in any c.e. not totally ω-c.e. degree (which is related to the definition of $\mathcal{F}$-a.n.c. sets).

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- First, the existence of a left-c.e. real with a strong similarity property in any c.e. not totally ω-c.e. degree (which is related to the definition of $\mathcal{F}$-a.n.c. sets).
- Second, a maximal pair property which is satisfied for every left-c.e. real that has the above strong similarity property.

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Definition

A sequence $\mathcal{F} = \{F_n\}_{n \geq 0}$ *is called a* very strong array of intervals *(v.s.a.i.) if* $\mathcal{F}$ *is a v.s.a. and for every* $n \geq 0$, F_n *is an interval and* $\max F_n < \min F_{n+1}$.

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ *be a v.s.a.i. Then two sets* A *and* B *are called* $\mathcal{F}$-similar, *denoted as* $A \sim_{\mathcal{F}} B$, *if*

$$\exists^{\infty} n \, (A \cap F_n = B \cap F_n).$$

In this notation, a c.e. set A is $\mathcal{F}$-a.n.c. for a v.s.a. if it is $\mathcal{F}$-similar to every c.e. set.

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The important observation now is that we do not need A to be similar to *all* left-c.e. reals (on some v.a.s.i. $\mathcal{F}$), but to only those left-c.e. reals whose approximation is compatible with the F_n's in the following sense.

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As a first observation, we showed the following.

Theorem (Ambos-Spies, Losert, Monath)

*Let **a** be a c.e. a.n.c. degree. Then for every v.s.a.i. $\mathcal{F}$, there exists an $\mathcal{F}$-left-c.e. real $A \in$ **a** which is $\mathcal{F}$-a.n.c. w.r.t. to $\mathcal{F}$-left-c.e. reals.*

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A left-c.e. real A has the universal similarity property, *if it is $\mathcal{F}$-a.n.c w.r.t. to $\mathcal{F}$-left-c.e. reals for all v.s.a.i. $\mathcal{F}$.*

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Lemma (First Main Lemma)

Let $\mathbf{a}$ be a c.e. Turing degree which is not totally ω-c.e. Then there exists a left-c.e. real $A \in \mathbf{a}$ which has the universal similarity property.

Note that the converse also holds.

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So far, we showed the existence of left-c.e. reals in c.e. not totally ω-c.e. degrees that have the universal similarity property. For the maximal pair property, we need the following observations.

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So far, we showed the existence of left-c.e. reals in c.e. not totally ω-c.e. degrees that have the universal similarity property. For the maximal pair property, we need the following observations.

- In Fan's construction (as well as in Yu/Ding's construction) of a maximal pair, every requirement can be satisfied locally on a sufficiently large interval.

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Lemma (Second Main Lemma)

Let D be an infinite computable set. There is a v.s.a.i. $\mathcal{F}$ ($\mathcal{F} = \mathcal{F}_D$) such that for any left-c.e. real A which is $\mathcal{F}$-a.n.c. w.r.t. $\mathcal{F}$-left-c.e. reals, there exists a c.e. set $B \subseteq D$ such that (A, B) is a maximal pair in the left-c.e. reals.

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- Let D be computable and infinite. By the Second Main Lemma, fix a v.s.a.i. $\mathcal{F}$ and a c.e. subset B of D such that every real A' which is $\mathcal{F}$-a.n.c. w.r.t. $\mathcal{F}$-left-c.e. reals forms a maximal pair together with B.

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- Since A is $\mathcal{F}$-a.n.c. w.r.t. $\mathcal{F}$-left-c.e. reals, this completes the proof.

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In this way, we get a simplified, uniform and modular approach.

Examples

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Theorem (Barmpalias, Downey, Greenberg [BDG10])

Let $\mathbf{a}$ *be a c.e. and not totally* ω*-c.e. degree. Then there is a c.e. set* $A \in \mathbf{a}$ *that is not wtt-reducible to any hypersimple set.*

Theorem (extending Brodhead, Downey, Ng [BDN12])

Let $\mathbf{a}$ *be a c.e. and not totally* ω*-c.e. degree. There is a left-c.e. real* $A \in \mathbf{a}$ *which is CB-random.*

(A real A is *CB-random* if it passes all Martin-Löf Tests where the sizes of its components are computably bounded).

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Alternative proofs: show that, for any left-c.e. real A with the universal similarity property, A is *wtt*-reducible to any h-simple set and is CB-random, respectively.

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Theorem (Barmpalias, Downey, Greenberg [BDG10])

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By using the observation

Theorem (Ambos-Spies, Losert, Monath)

Let $\mathbf{a}$ *be a c.e. a.n.c. degree. Then for every v.s.a.i.* $\mathcal{F}$*, there exists a* $\mathcal{F}$*-left-c.e. real* A *which is* $\mathcal{F}$*-a.n.c. w.r.t. to* $\mathcal{F}$*-left-c.e. reals.*

and the Second Main Lemma (for $D = \omega$) we can give an alternative proof of the above BDG-theorem.

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Let **a** be c.e. and a.n.c.

- By the Second Main Lemma (applied to $D = \omega$), fix a v.s.a.i. $\mathcal{F}$ and a c.e. set B such that every left-c.e. real A' which is $\mathcal{F}$-a.n.c. w.r.t. to $\mathcal{F}$-left-c.e. reals forms a maximal pair together with B.

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- Since **a** is a.n.c., it contains an $\mathcal{F}$-left-c.e. real A which is $\mathcal{F}$-a.n.c. w.r.t. to $\mathcal{F}$-left-c.e. reals.

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- Since **a** is a.n.c., it contains an $\mathcal{F}$-left-c.e. real A which is $\mathcal{F}$-a.n.c. w.r.t. to $\mathcal{F}$-left-c.e. reals.
- Since any c.e. set is cl-reducible to any 1-random left-c.e. real, A cannot be cl-reducible to any 1-random left-c.e. real, since (A, B) is a maximal pair.

□

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Thank you for your attention!