A derivation on the field of d.c.e. reals

Joseph S. Miller
University of Wisconsin–Madison

(building on work of Barmpalias and Lewis-Pye)

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Random left-c.e. reals

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(1) α is a random left-c.e. real,

(2) α is the halting probability of a universal prefix-free machine,

(3) Any left-c.e. approximation to α converges at least as slowly as any left-c.e. approximation to any other left-c.e. real.

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The last of these conditions will be made precise in the next lemma. It is stronger than saying that α is "Solovay complete", but since we do not need Solovay reducibility below, we will not elaborate.

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All random left-c.e. reals are essentially equally hard to approximate.

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Let $\{\beta_s\}_{s\in\omega}$ and $\{\gamma_s\}_{s\in\omega}$ be left-c.e. approximations of β and γ, respectively. If we set $\alpha_s = \beta_s - \gamma_s$, then not only do we have $\lim_{s\to\infty} \alpha_s = \alpha$, but the variation of the approximation is finite, i.e.,

$$\begin{aligned}\sum_{s\in\omega} |\alpha_{s+1} - \alpha_s| &= \sum_{s\in\omega} |(\beta_{s+1} - \beta_s) - (\gamma_{s+1} - \gamma_s)| \\ &\leqslant \sum_{s\in\omega} |\beta_{s+1} - \beta_s| + \sum_{s\in\omega} |\gamma_{s+1} - \gamma_s| = \beta + \gamma < \infty.\end{aligned}$$

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We call $\{\alpha_s\}_{s\in\omega}$ a d.c.e. approximation of α. Such approximations characterize the d.c.e. reals.

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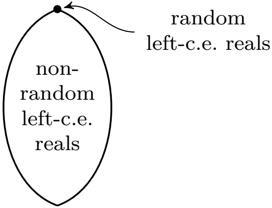
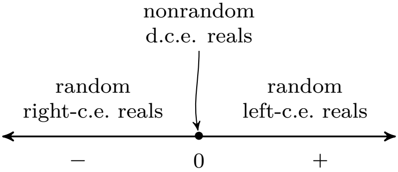
Comparison with Solovay degrees

The Solovay degrees are complementary to ∂.



The Solovay degrees
of left-c.e. reals



The range of ∂
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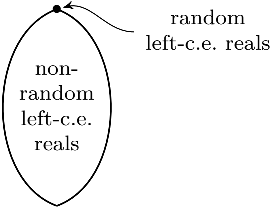

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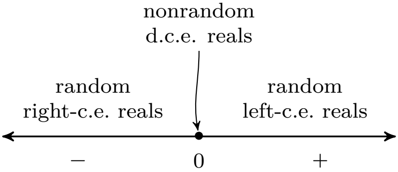

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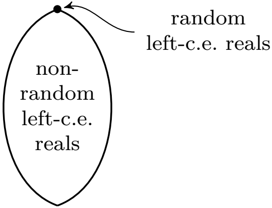

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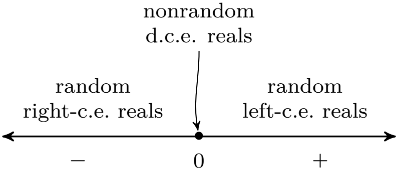

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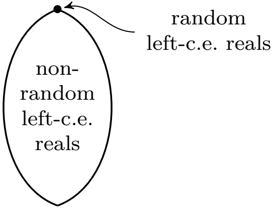

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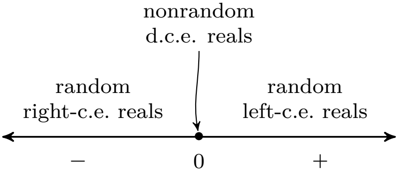

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- There is significant overlap, however, in what the two approaches tell us about the random left-c.e. reals.
- Rettinger and Zheng (2005) proved that all random d.c.e. reals are either left-c.e. or right-c.e.
- They also extended Solovay reducibility to the d.c.e. reals (with a slight modification). The top degree still contains all randoms.

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In fact, ∂ is a derivation on the field of d.c.e. reals:

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However, ∂ maps outside of the d.c.e. reals, so it does not make them a differential field.

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Since $\partial\Omega = 1$, we have $\partial e^\Omega = e^\Omega$.

Part II

Sketchy proofs

The main technical result

Lemma (Barmpalias and Lewis-Pye)

Let α and β be a left-c.e. reals with left-c.e. approximations $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$. If β is random, then

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Fix such stages $s < t$. So

$$\alpha_t - c\beta_t > \alpha - c\beta = \alpha - d\beta + (d - c)\beta > \alpha_s - d\beta_s + (d - c)\beta.$$

Rearranging, we have

$$\beta < \frac{\alpha_t - \alpha_s + d\beta_s - c\beta_t}{d - c}.$$

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The idea of the proof is to use such upper bounds to cover β with a Solovay test. The difficulty is that we cannot effectively determine which stages s and t satisfy our requirements, so we guess and update our guesses dynamically. □

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Similarly, in case (2), α is a right-c.e. real.

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We define a new approximation of α as follows. At stage s, check if α_s^* is contained in an *unused* interval $[c_n, d_n]$ for $n \leqslant s$. If so, mark that interval *used* and let $\alpha_{4s} = \alpha_{4s+3} = \alpha_s^*$, $\alpha_{4s+1} = c_n$, and $\alpha_{4s+2} = d_n$.

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Until we have proved independence from the approximation:

Notation. If α is a d.c.e. real with approximation $\{\alpha_s\}_{s\in\omega}$, let

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(2) If $\partial\{\alpha_s\} > 0$, then α is a left-c.e. real.

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Proof. For (2), if $\partial\{\alpha_s\} > 0$, then there is an $s* \in \omega$ such that $(\forall s \geqslant s*)\ \alpha_s < \alpha$. Hence, α is a left-c.e. real.

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Proof. For (2), if $\partial\{\alpha_s\} > 0$, then there is an $s* \in \omega$ such that $(\forall s \geqslant s*)\ \alpha_s < \alpha$. Hence, α is a left-c.e. real. Part (3) is similar. □

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Proof. Note that $\partial\{\alpha_s\} - \partial\{\alpha^*_s\} = \partial\{\alpha_s - \alpha^*_s\} = 0$, because $\{\alpha_s - \alpha^*_s\}_{s\in\omega}$ is a d.c.e. approximation of 0. □

Basic properties of ∂

Lemma

Let α be a d.c.e. real with d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$.

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This implies that $\partial\alpha > 1/c > 0$.

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Theorem

Let α be a d.c.e. real.

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(5) $$\begin{aligned}\partial\alpha &= \sup\{c \in \mathbb{Q} \colon \alpha - c\,\Omega \text{ is left-c.e.}\}\\ &= \inf\{c \in \mathbb{Q} \colon \alpha - c\,\Omega \text{ is right-c.e.}\}.\end{aligned}$$

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Note. We lose nothing by working with Ω as a fixed benchmark; it is easy to see that if β is a random d.c.e. real, then

$$\frac{\partial\alpha}{\partial\beta} = \frac{\partial\alpha/\partial\Omega}{\partial\beta/\partial\Omega}.$$

Part III

The field of nonrandom d.c.e. reals

The nonrandom d.c.e. reals

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Proof.

Let α and β be nonrandom d.c.e. reals. Then $\partial(\alpha + \beta) = \partial\alpha + \partial\beta = 0$, so $\alpha + \beta$ is not random. It is similarly easy to see that $\alpha - \beta$, $\alpha\beta$ and α/β are not random. So the nonrandom d.c.e. reals form a field.

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Now let $p(x)$ be a polynomial whose coefficients are nonrandom d.c.e. reals. Assume that α is a real root of $p(x)$. As mentioned, the d.c.e. reals form a real closed field (Ng 2006; Raichev 2005), so α must be a d.c.e. real. We need to show that α is nonrandom.

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We may assume that α has multiplicity one as a root of $p(x)$; otherwise, we could replace $p(x)$ with the greatest common divisor of $p(x)$ and $p'(x)$, which also has coefficients in the field of nonrandom d.c.e. reals.

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Therefore, we have

$$\partial\alpha = \frac{\partial p(\alpha)}{p'(\alpha)} = \frac{\partial 0}{p'(\alpha)} = 0,$$

so α is nonrandom. □

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Proof Sketch. If $c, d \in \mathbb{Q}$ are such that

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then $\alpha - c\beta$ is not random because $\alpha_s - c\beta_s$ is infinitely often above and infinitely often below it.

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then $\alpha - c\beta$ is not random because $\alpha_s - c\beta_s$ is infinitely often above and infinitely often below it. Similarly, $\alpha - d\beta$ is nonrandom. Therefore, their difference $(d - c)\beta$ is nonrandom. But this implies that β is nonrandom, which is a contradiction. □

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Definition

Call a d.c.e. real α variation nonrandom if it has a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$ such that the variation $\sum_{n\in\omega} |\alpha_{s+1} - \alpha_s|$ is not random. Otherwise, call α variation random.

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Proposition

TFAE for a d.c.e. real α:

- α is variation nonrandom,
- There are nonrandom left-c.e. reals β and γ such that $\alpha = \beta - \gamma$.

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Thank You!