Borel Isomorphism and Computability

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Computability, Randomness and Applications

CIRM Seminar, Marseille, France, June 21, 2016

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Definition (Kuratowski 1934, Jayne 1974)

We say that X is α-th level Borel isomorphic to Y if there exists a bijection f between X and Y preserving the Borel hierarchy above $\utilde{\Sigma}^0_{1+\alpha}$, that is,

$$A \text{ is } \utilde{\Sigma}^0_{1+\alpha} \text{ in } X \iff f[A] \text{ is } \utilde{\Sigma}^0_{1+\alpha} \text{ in } Y.$$

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- homeomorphism $=$ $\mathbf{0}$-th level Borel isomorphism.
- If $\alpha \leq \beta$, then every α-th level Borel isomorphism is β-th level Borel isomorphism.

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Let X and Y be uncountable Polish spaces.

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3. (Jayne-Rogers 1970s) If X can be written as a countable union of finite dimensional subspaces (e.g., $X = \omega^\omega, \mathbb{R}^n$ for $n \in \omega$, $\coprod_n \mathbb{R}^n$),
   - X is second-level Borel isomorphic to $\mathbb{R}$.
   - X is *not* finite-level Borel isomorphic to $[0, 1]^{\mathbb{N}}$.

How many Polish spaces are there up to α-th level Borel isomorphism?

- There are continuum many Polish spaces up to first level Borel isomorphism
- There are at least two Polish spaces up to n-th level Borel isomorphism for any $n < \omega$
- There is only one Polish space up to α-th level Borel isomorphism for any $\alpha \geq \omega$

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Second Level Borel Isomorphism Problem

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Is there a third Polish space up to second-level Borel isomorphism?

Such a third Polish space must be infinite dimensional.
Therefore, the *second-level Borel isomorphism problem* is inescapably tied to *infinite dimensional* topology.

Main Theorem (K. and Pauly)

There are continuum many compact metric spaces up to n-th level Borel isomorphism for any $n < \omega$.

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There are continuum many compact metric spaces up to ***n***-th level Borel isomorphism for any $n < \omega$.

Some Corollary to Banach Space Theory

- (Bade, Dashiell, Jayne, and others in 1970s) $B^*_\xi(X)$: the space of bounded real valued Baire ξ functions on X endowed with supremum norm and pointwise ring operation.
- (Jayne 1974) There is an analog of the Gel'fand-Kolmogorov Theorem in the Baire hierarchy, that is, TFAE for realcompact spaces X and Y:
  - Baire isomorphic at level (η, ξ).
  - $B^*_\xi(X)$ and $B^*_\eta(Y)$ are linearly isometric (ring isomorphic, etc.)

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    - Baire isomorphic at level (η, ξ).
    - $B^*_\xi(X)$ and $B^*_\eta(Y)$ are linearly isometric (ring isomorphic, etc.)
- Thus, our main theorem also implies the existence of $2^{\aleph_0}$ many mutually non-linearly-isometric (non-ring-isomorphic, etc.) Banach algebras of the form $B^*_n(X)$ for a compact mertic space X.
- Our result also gives a negative solution to Motto Ros' problem asking whether for any Polish space X, the Banach space $B^*_2(X)$ of Baire-two functions is linearly isometric to $\mathbb{R}^n$ for some $n \in \omega \cup \{\omega\}$.

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Idea

1. There are a number of descriptive set-theoretic attempts to generalize the Jayne-Rogers Theorem (e.g., by Pawlikowski-Sabok (2012), Motto Ros (2013), et al.)
2. K. (2015) used the Shore-Slaman Join Theorem for Turing degrees to show some variant of the Jayne-Rogers Theorem for finite dimensional Polish spaces.
3. To prove its infinite-dimensional version, Gregoriades-K.-Ng showed the Shore-Slaman Join Theorem for continuous degrees by introducing the weighted version of Kumabe-Slaman forcing.

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4. The above results show that finite-level Borel isomorphisms are exactly σ-continuous isomorphisms of finite Borel rank.

   (Here, a function is *σ-continuous* if it is written as the union of countably many partial continuous functions. This notion was introduced by Lusin in 1920s.)

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   (Here, a function is *σ-continuous* if it is written as the union of countably many partial continuous functions. This notion was introduced by Lusin in 1920s.)
5. K.-Pauly clarified that the degree structure (relative to an oracle) on a Polish space is invariant under σ-continuous isomorphisms.

Decomposition Theorem (Gregoriades-K.-Ng)

$\mathcal{A}$: analytic subset of a Polish space; $\mathcal{Y}$: separable metrizable.

Suppose that $f : \mathcal{A} \to \mathcal{Y}$ satisfies that $S \in \underset{\sim}{\Sigma}^0_{1+\eta} \implies f^{-1}[S] \in \underset{\sim}{\Sigma}^0_{1+\xi}$.

Then there is a $\underset{\sim}{\Pi}^0_{1+\xi}$-cover $(\mathcal{A}_n)_{n\in\omega}$ of $\mathcal{A}$ such that $(\forall n)(\exists\theta$ with $\theta + \eta \leq \xi)$ the restriction $f \upharpoonright \mathcal{A}_n$ is of Baire class θ.

Corollary (Pawlikowski-Sabok 2012; Motto Ros 2013)

X: an analytic subset of a Polish space; Y: separable metrizable. The following are equivalent for a function $f : X \to Y$:

1. f is an n-th level Borel function for some $n < \omega$.
2. f is $\underset{\sim}{\Pi}^0_n$-piecewise continuous for some $n < \omega$.

Sketch of Proof when X and Y are countable-dimensional

$$f^{-1}\underset{\sim}{\Sigma}^0_{m+1} \subseteq \underset{\sim}{\Sigma}^0_{n+1} \implies \text{piecewise Baire } n-m.$$

1. Suppose: $A \in \underset{\sim}{\Sigma}^0_{m+1}(Y) \Rightarrow f^{-1}[A] \in \underset{\sim}{\Sigma}^0_{n+1}(X)$.
2. By the Louveau Separation Theorem,
$$f^{-1}[\cdot] : \underset{\sim}{\Sigma}^0_m(Y) \to \underset{\sim}{\Delta}^0_{n+1}(X) \text{ is Borel.}$$
3. Then we have the following inequality for Turing degrees:
$$(\forall w \geq_T z)\ (f(x) \oplus w)^{(m)} \leq_T (x \oplus w^{(\xi)})^{(n)}.$$
4. By using the Friedberg Jump Inversion Theorem:
$$(\forall a, b)(\exists c \geq_T a)\ [(b \oplus a^{(\xi)}) \equiv_T c^{(\xi)}]$$
and the Shore-Slaman Join Theorem:
$$(\forall x)(\forall y)\ (y \not\leq_T x^{(n)} \rightarrow (\exists g)\ [g \geq_T x \ \&\ g^{(n+1)} \leq_T g \oplus y],$$
we obtain the following inequality for Turing degrees:
$$f(x) \leq_T (x \oplus z^{(\xi)})^{(n-m)}.$$
5. Hence, f is decomposable into countably many Baire $n-m$ functions $(x \mapsto \Phi_e^{(x\oplus z^{(\xi)})^{(n-m)}})_{e\in\mathbb{N}}$, where Φ_e^p is the e-th Turing machine computation with oracle p □

$$f^{-1}\underset{\sim}{\Sigma}^0_{m+1} \subseteq \underset{\sim}{\Sigma}^0_{n+1} \implies \text{piecewise Baire } n - m.$$

We also need the Shore-Slaman Join Theorem for continuous degrees. Recall that the Shore-Slaman Join Theorem is:

$$\mathcal{D}_T \models (\forall x)(\forall y \not\leq x^{(n)})(\exists g \geq x)\ g^{(n+1)} = y \oplus g = y \oplus x^{(n+1)}.$$

Unfortunately, this is false in the continuous degrees.
However, we can still have the following weaker version:

Theorem (Gregoriades-K.-Ng)

The continuous degrees satisfy the following sentence:

$$(\forall x)(\forall y \not\leq x^{(n)})(\exists g \geq x)\ y \oplus g = y \oplus x^{(n+1)}.$$

Proof: By a "*weighted*" version of Kumabe-Slaman forcing. □

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Second Level Borel Isomorphism Problem

Is there a Polish space which is second-level Borel isomorphic neither to $\mathbb{R}$ nor to $\mathbf{[0,1]}^{\mathbb{N}}$?

Observation

1. (K.-Pauly) The degree structure (relative to an oracle) on a Polish space is invariant under finite-level Borel isomorphism.

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   - The degree structure on $\mathbb{R}$ is the *Turing degrees*.
   - The degree structure on $\mathbf{[0,1]}^{\mathbb{N}}$ is the *continuous degrees* (J. Miller 2004).
2. Thus, to solve the second (finite) level Borel isomorphism problem, it suffices to find a Polish space whose degree structure is strictly intermediate between the Turing degrees and the continuous degrees (relative to any oracle).

Definition

1. $\Gamma : 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *ω-left-CEA operator* if the output $\Gamma(x)$ is a sequence $(y_0, y_1, y_2, \dots)$ such that y_{n+1} is left-c.e. in $(x, y_0, y_1, \dots, y_n)$ uniformly in x and n.

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2. (Formal Definition) Γ is ω-left-CEA if there is a left-c.e. operator γ such that for all n,

$$x_n := \Gamma(y)(n) = \gamma(y, n, x_0, x_1, \dots, x_{n-1}).$$

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$$x_n := \Gamma(y)(n) = \gamma(y, n, x_0, x_1, \dots, x_{n-1}).$$
3. An ω-left-CEA operator $\Gamma : \mathbb{N} \times 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *universal* if $(\forall \Psi\ \omega\text{-left-CEA})(\exists e)\ \Psi = \lambda y.\Gamma(e, y)$.

Let ω**CEA** denote the graph of a universal ω-left-CEA operator.

Theorem (K.-Pauly)

The space ω**CEA** (as a subspace of Hilbert cube) is a Polish space which is finite-level Borel isomorphic neither to $\mathbb{R}$ nor to $\mathbf{[0,1]}^{\mathbb{N}}$.

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Remark

Furthermore, ω**CEA** is second-level Borel isomorphic to the following spaces:

- Rubin-Schori-Walsh (1979)'s strongly infinite dimensional totally disconnected Polish space.
- Roman Pol (1981)'s compactum: a compact metric space which is weakly infinite dimensional, but not countable dimensional (a solution to Alexandrov's problem in infinite dimensional topology).

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Idea of Proof

1. Why is the degree structure of the space **ωCEA** strictly intermediate between the Turing degrees and the continuous degrees (relative to any oracle)?

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   - The space ω**CEA** is large enough to have a point whose Turing lower cone is a non-principal Turing ideal.
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4. (J. Miller 2004) For every countable Scott ideal $\mathcal{I}$ there is a point in Hilbert cube whose Turing lower cone is exactly $\mathcal{I}$.
   - The Turing lower cone of a point in ω**CEA** cannot be closed under the ω-th Turing jump.

Main Theorem (K. and Pauly)

There are continuum many compact metric spaces up to *n*-th level Borel isomorphism for any $n < \omega$.

Idea

- An oracle $\underset{\sim}{\mathbf{\Pi}}^0_2$ singleton is a function whose graph is G_δ in Hilbert cube.
- K.-Pauly introduced the notion of almost arithmetical (aa) reducibility between oracle $\underset{\sim}{\mathbf{\Pi}}^0_2$ singletons.
- Introduce a method of constructing a Polish space $\mathcal{S}(\mathcal{G})$ from a countable set $\mathcal{G}$ of oracle $\mathbf{\Pi}^0_2$ singletons such that

  if the degree structure on $\mathcal{S}(\mathcal{G})$ is included in that of $\mathcal{S}(\mathcal{H})$ (relative to an oracle), then $\mathcal{G}$ is aa-included in $\mathcal{H}$.
- The finite level Borel isomorphism problem is reduced to the problem on aa-degrees for oracle $\underset{\sim}{\mathbf{\Pi}}^0_2$ singletons. The latter problem is easy!
- Although the spaces $\mathcal{S}(\mathcal{G})$ are not compact, one can easily see that the Lelek-compactification in infinite dimensional topology preserves the finite-level Borel structure of the space.

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