Carleson's Theorem and Schnorr randomness

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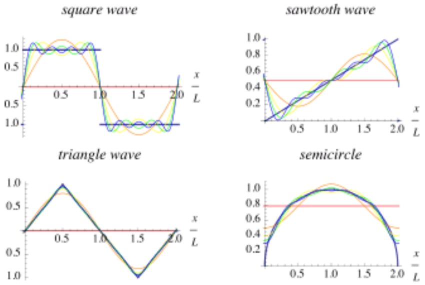
Fourier series: from undergraduate differential equations onward



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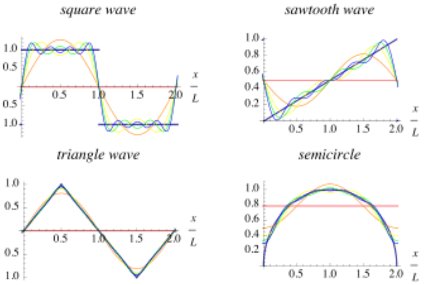

We'll work in the complex version of $L^p[-\pi, \pi]$: the space of all measurable $f : [-\pi, \pi] \to \mathbb{C}$ such that $\int_{-\pi}^{\pi} |f(t)|^p \, dt < \infty$.

Fourier series: convergence

Question (Fourier)

Does the Fourier series of a continuous function converge pointwise to the function?

Theorem (Dirichlet)

If f is continuously differentiable, then its Fourier series converges to f everywhere.

Theorem (du Bois-Reymond 1876)

There is a continuous function whose Fourier series diverges at a point.

Conjecture (Lusin 1913)

If f is a function in L^2, then its Fourier series converges to f almost everywhere.

Theorem (Carleson 1966, Hunt 1968)

Suppose $1 < p < \infty$. If f is a function in $L^p[-\pi, \pi]$, then its Fourier series converges to f almost everywhere.

Definitions

For all $n \in \mathbb{Z}$ and $f \in L^1[-\pi, \pi]$,

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{int}\, dt,$$

and for all $f \in L^1[-\pi, \pi]$ and $N \in \mathbb{N}$,

$$S_N(f) = \sum_{n=-N}^{N} c_n(f) e^{int}.$$

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$S_N(f)$ is the $(N+1)^{st}$ partial sum of f's Fourier series. We say $f \in L^1[-\pi, \pi]$ is *analytic* if $c_n(f) = 0$ whenever $n < 0$.

A *trigonometric polynomial* is a function in the linear span of $\{e^{int} \mid n \in \mathbb{Z}\}$, and the *degree* of such a polynomial p is the smallest $d \in \mathbb{N}$ such that $S_d(p) = p$.

Main theorems

Suppose $p > 1$ is a computable real.

Theorem 1

If $t_0 \in [-\pi, \pi]$ is Schnorr random and f is a computable vector in $L^p[-\pi, \pi]$, then the Fourier series for f converges at t_0.

Theorem 2

If $t_0 \in [-\pi, \pi]$ is not Schnorr random, then there is a computable function $f : [-\pi, \pi] \to \mathbb{C}$ whose Fourier series diverges at t_0.

Note: There are incomputable functions that are computable as vectors, so Theorem 2 is stronger than Theorem 1's converse.

A computable analysis primer (I)

- A point $z \in \mathbb{C}$ is *computable* if there is an algorithm that, given $k \in \mathbb{N}$, computes a rational point q such that $|q - z| < 2^{-k}$.
- A trigonometric polynomial τ is *rational* if each of its coefficients is a rational point.

A computable analysis primer (II)

Let $p \geq 1$ be a computable real and $f \in L^p[-\pi, \pi]$.

- f is a *computable vector of* $L^p[-\pi, \pi]$ if there is an algorithm that, given $k \in \mathbb{N}$, computes a rational polynomial τ such that $||f - \tau||_p < 2^{-k}$.
- $f : \mathbb{C} \to \mathbb{C}$ is *computable* if there is an algorithm P such that
  - whenever P is given an open rational rectangle as input, it either does not halt or returns an open rational rectangle,
  - when P halts on an open rational rectangle R, the rectangle it outputs contains $f(z)$ for every $z \in R \cap \text{dom}(f)$, and
  - when U is a neighborhood of $z \in \text{dom}(f)$ and V is a neighborhood of $f(z)$, there is an open rational rectangle R such that $z \in R \subseteq U$ and $P(R)$ is a rational rectangle in V.

Some facts

Proposition

Suppose $p \geq 1$ is a computable real and $f \in L^p[-\pi, \pi]$.

1. *If f is a computable vector, then $||f||_p$ and $\langle c_n(f) \rangle_{n \in \mathbb{Z}}$ are computable.*
2. *If $p = 2$, then f is computable if both $||f||_p$ and $\langle c_n(f) \rangle_{n \in \mathbb{Z}}$ are computable.*

Corollary

There is an incomputable vector $f \in L^2[-\pi, \pi]$ such that $\langle c_n(f) \rangle_{n \in \mathbb{Z}}$ is computable.

Proof.

Let $\langle r_n \rangle_{n \in \mathbb{N}}$ be a computable sequence of positive rational numbers such that $\sum_{n=0}^{\infty} r_n^2$ is incomputable (Specker). If $f = \sum_{n=0}^{\infty} r_n e^{int}$, then $||f||_2^2 = \sum_{n=0}^{\infty} r_n^2$ and f is incomputable. □

Theorem 1

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Definition

Suppose $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence of functions on $[-\pi, \pi]$. A function $\eta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a *modulus of almost-everywhere convergence* for $\langle f_n \rangle_{n \in \mathbb{N}}$ if, for all k and m,

$$\mu(\{t \in [-\pi, \pi] \mid \exists M, N \geq \eta(k, m) |f_N(t) - f_M(t)| \geq 2^{-k}\}) < 2^{-m}.$$

Two lemmas

Lemma

Suppose p is a computable real such that $p > 1$, and suppose f is a computable vector in $L^p[-\pi, \pi]$. Then $\langle S_N(f) \rangle_{n \in \mathbb{N}}$ has a computable modulus of almost-everywhere convergence.

Lemma

Assume $\langle f_n \rangle_{n \in \mathbb{N}}$ is a uniformly computable sequence of functions on $[-\pi, \pi]$ for which there is a computable modulus of almost-everywhere convergence. Then the sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ converges at every Schnorr random real.

First lemma

We must construct our $\eta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let k and m be given.

- Compute a rational trigonometric polynomial $\tau_{k.m}$ so $||f - \tau_{k,m}||_p \leq 2^{-(m+k+3)} C^{-1}$ where

$$||\sup_N |S_N(f)|\,||_1 \leq C||f||_p$$

  (Fefferman's inequality).
- Set $\eta(k, m)$ to be the degree of $\tau_{k,m}$.

For $g \in L^p[-\pi, \pi]$, let

$$\widehat{E}_k(g) = \{t \in [-\pi, \pi] \mid \sup_N |S_N(g)(t)| > 2^{-k}\}.$$

- Lots of manipulations.
- Fefferman's inequality:

$$||\sup_N |S_N(f - \tau_{k,m})|\,||_1 \leq 2^{-(m+k+3)}$$

- Chebyshev's inequality:

$$\mu(\widehat{E}_{k+2}(f - \tau_{k,m})) \leq 2^{-(m+k+3)} 2^{k+2} < 2^{-m}$$

Second lemma

Lemma

Assume $\langle f_n \rangle_{n\in\mathbb{N}}$ *is a uniformly computable sequence of functions on* $[-\pi, \pi]$ *for which there is a computable modulus of almost-everywhere convergence. Then the sequence* $\langle f_n \rangle_{n\in\mathbb{N}}$ *converges at every Schnorr random real.*

Definition (Miyabe)

A *Schnorr integral test* is a lower semicomputable function $T : [-\pi, \pi] \to [0, \infty]$ so that $\int_{-\pi}^{\pi} T \, d\mu$ is a computable real. A point $x \in [-\pi, \pi]$ is Schnorr random if and only if $T(x) < \infty$ for every Schnorr integral test T.

Define a Schnorr integral test:

- Let η be a computable modulus of almost-everywhere convergence for $\langle f_n \rangle_{n \in \mathbb{N}}$. Let $N_k = \eta(k, k)$.
- For each $k \in \mathbb{N}$ and each $t \in [-\pi, \pi]$, define

$$g_k(t) = \min\{1, \max\{|f_M(t) - f_N(t)| \mid N_k < M, N \leq N_{k+1}\}\}.$$

- $\langle g_k \rangle_{k \in \mathbb{N}}$ is computable. Set $T = \sum_{k=0}^{\infty} g_k$.

Show that T is a Schnorr integral test:

- T is clearly lower semicomputable.
- T is computable: Lots of manipulation.

Claim

$T(t_0) = \infty$ whenever $\langle f_n(t_0)\rangle_{n\in\mathbb{N}}$ diverges.

Suppose $\langle f_n(t_0)\rangle_{n\in\mathbb{N}}$ diverges.

- There is a k_0 such that $\limsup_{M,N} |f_M(t_0) - f_N(t_0)| \geq 2^{-k_0}$. So: show that for all k_1,

$$\sum_{k=k_1}^{\infty} g_k(t_0) \geq 2^{-k_0}.$$

- By the choice of k_0, there are M and N such that $N_{k_1} \leq M < N$ and

$$2^{-k_0} \leq |f_M(t_0) - f_N(t_0)|.$$

- Form a telescoping sum and apply the Triangle Inequality:

$$|f_M(t_0) - f_N(t_0)| \leq \sum_{k=k_1}^{\infty} g_k(t_0).$$

Theorem 2

Theorem 2

If $t_0 \in [-\pi, \pi]$ is not Schnorr random, then there is a computable function $f : [-\pi, \pi] \to \mathbb{C}$ whose Fourier series diverges at t_0.

The proof follows a construction of Kahane and Katznelson.

Three lemmas

Lemma

Suppose G is a computably compact subset of the unit circle so that $\lambda(G)$ is computable and smaller than 2π. Then there is a computable function ψ from $\mathbb{D} \cup G$ into the horizontal strip $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ that is analytic on $\mathbb{D}$ and has the property that $Re(\psi(\zeta)) \geq -\frac{3}{4} \ln(\lambda(G)(2\pi)^{-1})$ for all $\zeta \in G$. Furthermore, we may choose ψ so that $\psi(0) = 0$.

Lemma

Suppose G is a computably compact subset of $[-\pi, \pi]$ so that $\lambda(G)$ is computable and smaller than 2π. Then there is a computable and analytic trigonometric polynomial R so that $Re(R(t)) \geq -\frac{1}{2} \ln(\lambda(G)/(2\pi))$ for all $t \in G$ and so that $|Im(R(t))| < \pi$ for all $t \in [-\pi, \pi]$. Furthermore, we may choose R so that $R(0) = 0$.

Lemma

Suppose G is a computably compact subset of $[-\pi, \pi]$ so that $\lambda(G)$ is computable and smaller than 2π. Then there is a computable trigonometric polynomial p so that

$$\sup_N |S_N(p)(t)| \geq -\frac{1}{4\pi} \ln\left(\frac{\lambda(G)}{2\pi}\right)$$

for all $t \in G$ and so that $\|p\|_\infty < 1$.

The proofs of these lemmas are all (1) analytic and (2) uniform.

Now, given those lemmas...

Suppose t_0 isn't Schnorr random. Then there is a Schnorr test $\langle V_n \rangle_{n\in\mathbb{N}}$ such that $t_0 \in \cap V_n$.

Compute an array of closed rational intervals $\langle I_{n,j} \rangle_{n,j\in\mathbb{N}}$ such that

- $V_{2^n} = \bigcup_j I_{n,j}$ and
- $\mu(I_{n,j} \cap I_{n,j'}) = 0$ when $j \neq j'$.

Compute for each n a sequence $m_{n,0} < m_{n,1} < \ldots$ such that

$$\mu\left(V_{2^n} - \bigcup_{j \le m_{n,k}} I_{n,j}\right) < 2^{-(2^{n+k+1})}$$

for all n and k.

Define

$$G_{n,0} = \bigcup_{j \le m_{n,1}} I_{n,j} \cap [-\pi, \pi]$$
$$G_{n,k} = \bigcup_{m_{n,k} < j \le m_{n,k+1}} I_{n,j} \cap [-\pi, \pi]$$

Then $\mu(G_{n,k}) < 2^{-(2^{n+k})}$.

Given n and k, use the third lemma to get a trigonometric polynomial p such that $||p||_\infty < 1$ and

$$\sup_N |S_N(p)(t)| > -\frac{1}{4\pi} \ln\left(\frac{\mu(G_{n,k})}{2\pi}\right)$$

for all $t \in G_{n,k}$. Set $p_{n,k} = 2^{-(n+k+1)}p$. Then

$$\sup_N |S_N(p_{n,k})(t)| > \frac{1}{8\pi}.$$

Compute an array $\langle r_{n,k} \rangle$ that produces "nice" Fourier coefficients. Set

$$f = \sum_{n,k} e_{r_{n,k}} p_{n,k}.$$

Since $||p_{n,k}||_\infty < 2^{-(n+k+1)}$, f is computable.

Finally: Show that f's Fourier series diverges at t_0 by showing that

$$\sup_{M,N} |S_M(f)(t_0) - S_N(f)(t_0)| > \frac{1}{8\pi}.$$

Fix N_0 and choose n such that $\langle n, 0\rangle \geq N_0$ and k such that $t_0 \in G_{n,k}$.

The array was constructed so that there are M and N' so that $e_{r_{n,k}} p_{n,k} = S_{N'}(f) - S_M(f)$ and $M \geq \langle n, k\rangle \geq \langle n, 0\rangle$, and by our construction of $p_{n,k}$, there is an N such that $M \leq N \leq N'$ and $\sup_{M,N} |S_M(f)(t_0) - S_N(f)(t_0)| > \frac{1}{8\pi}$.

References

- Franklin, Johanna N.Y., McNicholl, Timothy H., and Rute, Jason. Algorithmic randomness and Fourier analysis. Submitted.

Thank you!