Independence of normal numbers

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Randomness - aléatoire - Zufall - azar - rasgelelik - satunnaisuuden - slumpmässighet

In 1909 Émile Borel gave a definition of the most elementary form of randomness for a real number, thinking in the sequence of digits that determine its expansion.

Randomness - aléatoire - Zufall - azar - rasgelelik - satunnaisuuden - slumpmässighet

In 1909 Émile Borel gave a definition of the most elementary form of randomness for a real number, thinking in the sequence of digits that determine its expansion.

He called such reals normal numbers.

Normal numbers

A base is an integer greater than or equal to 2.

For a real number x in the unit interval, the expansion of x in base b is a sequence $a_1a_2a_3\ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = 0.a_1a_2a_3\ldots$$

where $x = \sum_{k\geq 1} a_k/b^k$, and x does not end with a tail of $b-1$.

Normal numbers

Definition (Borel, 1909)

A real number x is simply normal to base b if, in the expansion of x in base b, each digit occurs with limiting frequency equal to $1/b$.

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Equivalently: a real number x is normal to base b if, for every positive integer k, x is simply normal to base b^k.

A real number x is absolutely normal if x is normal to every base.

Existence

Theorem (Borel 1909)

The set of absolutely normal numbers in the unit interval has Lebesgue measure 1.

Not normal

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

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The numbers is the middle third Cantor set are not simply normal to base 3 (their expansions lack the digit 1).

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The rational numbers are not normal to any base.

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The numbers is the middle third Cantor set are not simply normal to base 3 (their expansions lack the digit 1).

The rational numbers are not normal to any base.

Liouville's constant $\sum_{n\geq 1} 10^{-n!}$ is not normal to any base.

Examples?

Problem (Borel 1909)

Give one example.

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Émile Borel 1871-1956 .

Normal to all bases

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

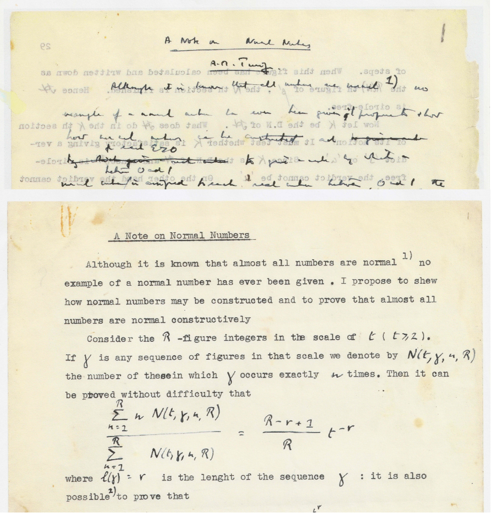
SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Normal to all bases

Turing, A. M. A Note on Normal Numbers. Collected Works of Alan M. Turing, Pure Mathematics, 117-119. Notes of editor J.L. Britton, 263-265. North Holland, 1992.



A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

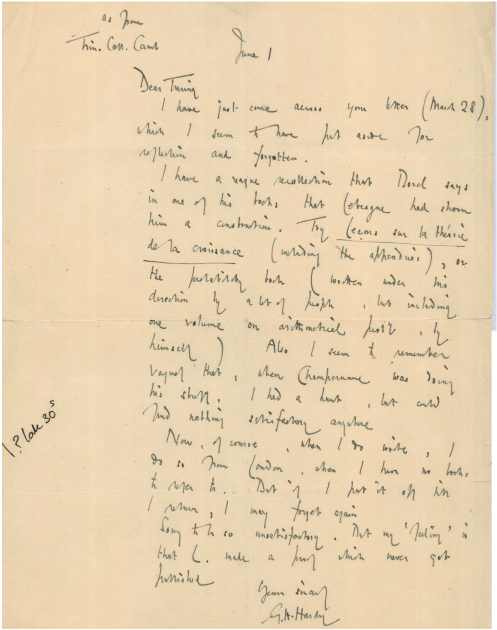
Consider the R -figure integers in the scale of t ($t \geqslant 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible [2] to prove that

Corrected and completed in Becher, Figueira and Picchi, 2007.

Letter exchange between Turing and Hardy (AMT/D/5)



June 1

Dear Turing,

I have just came across your letter (March 28) which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction.
Try Leçons sur la théorie de la croissance (including the appendices), or the productivity book (written under his direction by a lot of people, but including one volume on arithmetical prosy, by himself).

Also I seem to remember vaguely that when Champernowne was doing his stuff I had a hunt, but could not find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till my return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that Lebesgue made a proof which never got published.

Yours sincerely,

G.H. Hardy

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2002 Recursive formulation of Sierpinski's. Exponential complexity. Becher and Figueira

2007 Turing's algorithm has exponential complexity. Becher, Figuiera and Picchi

2013 Polynomial complexity. Mayordomo and Lutz (martingales);Figueira and Nies (martingales)
Becher, Heiber and Slaman (nearly quadratic complexity)

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2016 Levin's for low discrepancy has exponential complexity. Alvarez and Becher

2016 Discrepancy for numbers obtained by algorithms above. Scheerer; Madritsch, Scheerer and Tichy.

Normal to all bases

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

$$0.4031290542003809132371428380827059102765116777624189775110896366\ldots$$

Normal to a given base

Theorem (Champernowne, 1933)
$0.123456789101112131415161718192021\ldots$ *is normal to base* 10.

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$0.12345678910111213141516171819202 1 \ldots$ *is normal to base* 10.

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1935 squares Besicovitch

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2000 de Bruijn words Ugalde; Alvarez, Becher, Ferrari and Yuhjtman 2016.

Theorem (Bailey and Borwein 2012)
Stoneham number $\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \; 2^{3^k}}$ *is normal to base* 2 *but not simply normal to base* 6.

Normal words

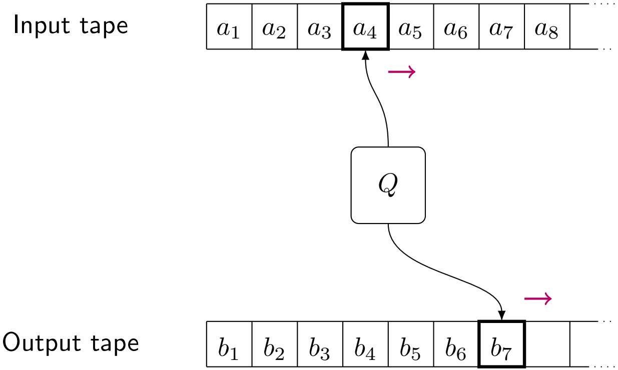
In this work we worry just about a single base, so, instead of real numbers we consider infinite words.

Finite transducer

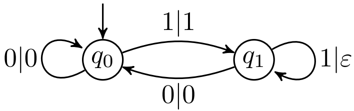
A finite transducer is a finite automaton $\mathcal{T} = \langle Q, A, B, \delta, q_0 \rangle$ that has an input and an output tape, where

Q is a finite set of states, q_0 is the initial
A and B are input and output alphabets (finite)

The transition δ determines finitely many transitions $p \xrightarrow{a|v} q$, for $p, q \in Q$, $a \in A$ and $v \in A^*$.

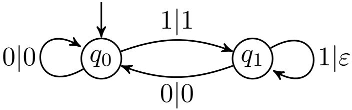


Example of a finite transducer



If the input is $010011000111\cdots$, the output is $01001000100\cdots$.

Example of a finite transducer



If the input is $010011000111\cdots$, the output is $01001000100\cdots$.
Blocks of 1s become a single 1.

Compression ratio

Let $\mathcal{T} = \langle Q, A, B, \delta, q_0 \rangle$ be a finite transducer. For input $x = a_1 a_2 \ldots$ a run in $\mathcal{T}$ is a sequence of transitions starting at q_0,

$$q_0 \xrightarrow{a_1|v_1} q_1 \xrightarrow{a_2|v_2} q_2 \xrightarrow{a_3|v_3} \cdots$$

Definition

The compression ratio of $x = a_1 a_2 \ldots$ in $\mathcal{T}$ is

$$\rho_{\mathcal{T}}(x) = \liminf_{n \to \infty} \frac{|v_1 v_2 \cdots v_n|}{n} \frac{\log |B|}{\log |A|}.$$

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The compression ratio of $x = a_1 a_2 a_3 \cdots$ is

$$\rho(x) = \inf \{ \rho_{\mathcal{T}}(x) : \mathcal{T} \text{ is deterministic and one-to-one} \}$$

The random words for finite automata

Theorem

normality $\Leftrightarrow$ *no finite-state martingale success*

(Schnorr and Stimm 1971)

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normality	$\Leftrightarrow$	*no finite-state martingale success*
		(Schnorr and Stimm 1971)
incompressibility	$\Leftrightarrow$	*no finite-state martingale success (dimension* 1*)*
		(Dai, Lathrop, Lutz and Mayordomo 2004)
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normality	⇔	*incompressibility (direct)* (Becher and Heiber 2013)
normality	⇔	*incompressibility non-deterministic or one counter* (Becher, Carton and Heiber 2015)
normality	⇔	*incompressibility two-way transducers* (Carton and Heiber 2015)

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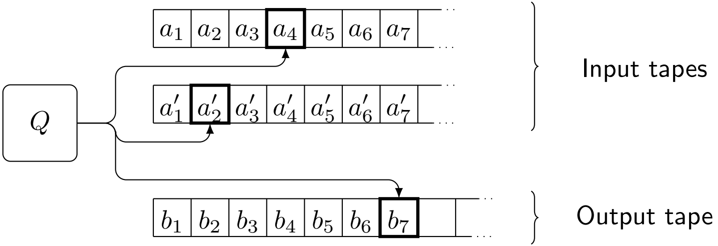
Problem

Can deterministic push-down one-to-one transducers compress some normal word?

The definition of independence

Two input transducers



The content of the first input tape is the input.

The content of the second input tape is used as an oracle.

Transducer $\mathcal{T}$ is one-to-one if for each oracle y fixed, the function $x \mapsto \mathcal{T}(x, y)$ is one-to-one.

Two input tranducer

A finite finite transducer with two inputs is a finite automata $\mathcal{T} = \langle Q, A, B, \delta, q_0 \rangle$, such that the transition function is $\delta : Q \times A \times A \to Q \times \{0,1\} \times \{0,1\} \times B^*$.
If $\delta(p, a, a') = (q, d, d', v)$ then

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Let $x = a_1 a_2 a_3 \cdots$ and $x' = a'_1 a'_2 a'_3 \cdots$ be two infinite words. We write

$$\langle p, m, m' \rangle \xrightarrow{a_m, a'_{m'} | v} \langle q, n, n' \rangle$$

if $\delta(p, a_m, a'_{m'}) = (q, d, d', v)$ and $n = m + d$ and $n' = m' + d'$.

Conditional compression ratio

A run of $\mathcal{T}$ with x and x' is a sequence of transitions, with $m_0 = m'_0 = 1$,

$$\langle p_0, m_0, m'_0 \rangle \xrightarrow{a_{m_0}, a'_{m'_0} | v_1} \langle p_1, m_1, m'_1 \rangle \xrightarrow{a_{m_1}, a'_{m'_1} | v_2} \langle p_2, m_2, m'_2 \rangle \cdots$$

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$$\rho_{\mathcal{T}}(x/x') = \liminf_{n \to \infty} \frac{|v_1 v_2 \cdots v_n|}{m_n}.$$

Note that $\rho_{\mathcal{T}}(x/x')$ does not depend on m'_n.

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$$\rho(x/y) = \inf \{\rho_{\mathcal{T}}(x/y) : \mathcal{T} \text{ is deterministic and one-to-one}\}$$

Recall $\mathcal{T}$ is one-to-one if for each y fixed, $x \mapsto \mathcal{T}(x, y)$ is one-to-one.

The definition of independence

Definition

The two words x and y are independent if they satisfy

$$\rho(x) = \rho(x/y) > 0 \text{ and } \rho(y) = \rho(y/x) > 0.$$

It means that y does not help to compress x and x does not help to compress y.

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Theorem (Becher and Carton 2016)

The set $\{(x, y) : x \text{ and } y \text{ are independent}\}$ has Lebesgue measure 1.

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It means that y does not help to compress x and x does not help to compress y.

Theorem (Becher and Carton 2016)

The set $\{(x, y) : x$ and y are independent$\}$ has Lebesgue measure 1.

Lemma

For each normal y, the set $\{x : \rho(x/y) < 1\}$ has Lebesgue measure 0.

Splitter

We write ϵ for the empty word.

Definition

A splitter is a deterministic transducer $T = \langle Q, A, \delta, q_0 \rangle$ with one input tape and two output tapes, The transition function is $\delta : Q \times A \to Q \times A \cup \{\epsilon\} \times A \cup \{\epsilon\}$. Hence, transitions have the form

$$p \xrightarrow{a|a,\epsilon} q \quad \text{or} \quad p \xrightarrow{a|\epsilon,a} q$$

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The splitter can be turned into a shuffler by exchanging input and output. We consider deterministic shufflers as follows.

Shuffler

A shuffler is a deterministic two input finite transducer which shuffles two input words into a new word. Whether the next digit is taken from the first or the second input word only depends the current state.

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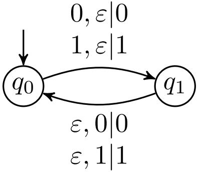
Definition
A shuffler is a finite transducer $\mathcal{T} = \langle Q, A, \delta, q_0 \rangle$ with two input tapes and one output tape. The transition function is $\delta : Q \times A \cup \{\epsilon\} \times A \cup \{\epsilon\} \to Q \times A$. A shuffler reads a symbol from either the first or the second input tape depending on the current state and copies it to the output tape, so transitions have the form

$$p \xrightarrow{a,\epsilon|a} q \quad \text{or} \quad p \xrightarrow{\epsilon,a|a} q.$$

For each state q, all incoming transitions have the same type.

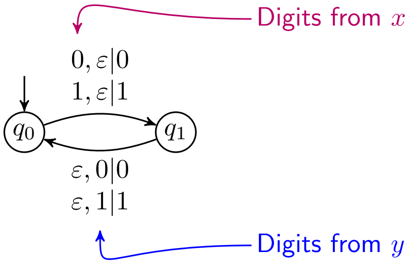
Example of a Shuffler



$$x = 0011010001 \cdots$$

$$y = 0100011000 \cdots$$

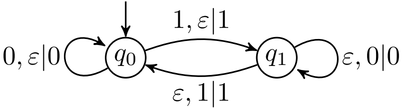
Example of a Shuffler



Input words $\begin{cases} x = 0011010001 \cdots \\ y = 0100011000 \cdots \end{cases}$

Output word $\quad x \vee y = 00011010001101000010 \cdots$

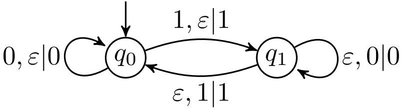
Example of another shuffler



Input words $\begin{cases} x = 001\,1\,01\,0001\ \cdots \\ y = 01\,0001\,1\,0001\ \cdots \end{cases}$

Output word $\quad z = 00101100010110001 0001 \cdots$

Example of another shuffler



Input words $\begin{cases} x = 001\,1\,01\,0001\ \cdots \\ y = 01\,0001\,1\,0001\ \cdots \end{cases}$

Output word $z = 0010110001011000100001\cdots$

It alternates (possibly empty) blocks of 0s followed by a 1, from each sequence.

Shuffling independent words

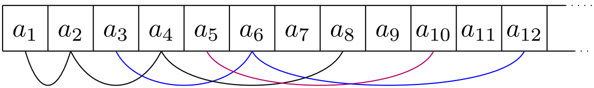
Theorem (Becher and Carton 2016)

Shuffling two normal independent words yields a normal word.

A peculiar normal word

Theorem (Becher, Carton and Heiber 2016)

There is a binary normal word $x = a_1 a_2 a_3 \cdots$ *such that* $a_{2n} = a_n$*, for every* $n \geqslant 1$.



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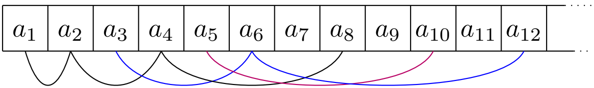

Since x is normal and $x = \mathrm{even}(x)$,

Corollary

There is a normal word x *such that* $\mathrm{odd}(x)$ *and* $\mathrm{even}(x)$ *are not independent.*

Toeplitz transformation: $x \mapsto T(x)$

K. Jacobs and M. Keane. 0-1 sequences of Toeplitz type, 1969.

Let $x = a_1 a_2 a_3 \cdots$, then $T(x) = b_1 b_2 b_3 \cdots$ where

$$b_n = a_m \quad \text{if } n = 2^k(2m-1) \text{ for some } k \geqslant 0.$$

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x :	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	$\cdots$

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$k = 0$	a_1		a_2		a_3		a_4		a_5		a_6		a_7		a_8		a_9	

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$k = 0$	a_1		a_2		a_3		a_4		a_5		a_6		a_7		a_8		a_9	
$k = 1$	a_1	a_1	a_2		a_3	a_2	a_4		a_5	a_3	a_6		a_7	a_4	a_8		a_9	a

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$k=1$	a_1	a_1	a_2		a_3	a_2	a_4		a_5	a_3	a_6		a_7	a_4	a_8		a_9	a
$k=2$	a_1	a_1	a_2	a_1	a_3	a_2	a_4		a_5	a_3	a_6	a_2	a_7	a_4	a_8		a_9	a

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$k=2$	a_1	a_1	a_2	a_1	a_3	a_2	a_4		a_5	a_3	a_6	a_2	a_7	a_4	a_8		a_9	a
$k=3$	a_1	a_1	a_2	a_1	a_3	a_2	a_4	a_1	a_5	a_3	a_6	a_2	a_7	a_4	a_8		a_9	a

Many peculiar normal words

Theorem (Becher, Carton and Heiber 2016)

Let p be any positive integer. There is a binary normal word $x = a_1 a_2 \ldots$ such that, for every n, $a_n = a_{pn}$.

Conjecture

Let $T(x)$ be such that for all k, for all m, $(T(x))_{2^k(2m-1)} = x_m$.

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For x the Champernowne's word $01\,00011011\,000001\cdots$. experiments suggest that $T(x)$ is normal.

Conjecture

The set $\{x : T(x)$ is normal $\}$ has Lebesgue measure 1.

Selecting

Let L be a set of finite words. If $x = a_1 a_2 a_3 \cdots$, then $x \restriction L$ is the word $a_{i_1} a_{i_2} a_{i_3} \cdots$ where $i_1 < i_2 < i_3 \cdots$ and $\{i_1, i_2, i_3, \ldots\}$ is the set $\{i : a_1 \cdots a_{i-1} \in L\}$.

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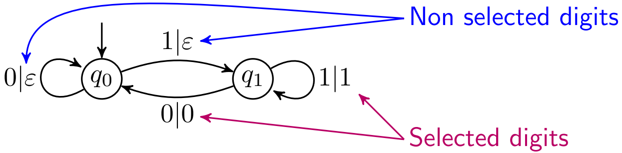

Input $\quad x = 00110100011100\cdots$

Output $\quad z = 100110\cdots$

Selection preserves normality

Theorem (Agafonoff 1968)

If x is normal and L is rational (accepted by a finite automaton) then $x \upharpoonright L$ is normal.

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Selections based on linear languages (recognized by one-turn pushdown automata) or deterministic one-counter languages do not preserve normality, Merkle and Reimann 2003.

Selecting with a two input finite transducer

Agafonoff's theorem says that selection by finite automata preserves normality.

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Theorem (Becher and Carton 2016)

Selection by a finite automata on a normal word using an independent word as an oracle preserves normality.

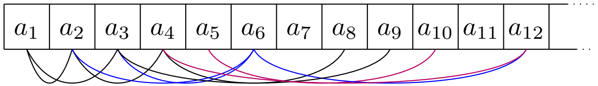
Open problems

Construct two independent normal words.

Open problems

Is there a normal word on a binary alphabet $x = a_1 a_2 \ldots$ satisfying, for every $n \geq 1$, $a_{2n} = x_n$ and $a_{3n} = a_n$?

We guess yes.



Open problems

Develop the theory of independence as uniform distribution modulo 1.

Theorem (D. Wall)

A real x is normal to base b if and only if the sequence $(b^n x)_{n\geq 0}$ is u.d. modulo 1.

Open problems

Develop the notion of independence of normality for shift spaces.

Definition

A sequence is normal in a shift of finite type if every block of A has a limiting asymptotic frequency equal to its Parry measure.

Theorem (Alvarez and Carton 2016)

Let X be a subshift of finite type. A sequence $x \in X$ is normal if and only if it is incompressible by a finite transducer.

Concluding remark

Little is known about the interplay between combinatorial, computational and Diophantine properties of the expansions of real numbers.
These investigations on normal numbers aim to make progress in this direction.

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The End

V. N. Agafonov. Normal sequences and finite automata. *Soviet Mathematics Doklady*, 9:324–325, 1968.

V. Becher, O. Carton, and P. A. Heiber. Normality and automata. *Journal of Computer and System Sciences*, 81(8):1592–1613, 2015.

V. Becher and P. A. Heiber. Normal numbers and finite automata. *Theoretical Computer Science*, 477:109–116, 2013.

É. Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Rendiconti del Circolo Matematico di Palermo*, 27:247–271, 1909.

C. Bourke, J. Hitchcock, and N. Vinodch. Entropy rates and finite-state dimension. *Theoretical Computer Science*, 349:392–406, 2005.

A. Broglio and P. Liardet. Predictions with automata. Symbolic dynamics and its applications. *Contemporary Mathematics*, 135:111–124, 1992. Also in Proceedings AMS Conference in honor of R. L. Adler. New Haven CT - USA 1991.

Y. Bugeaud. *Distribution Modulo One and Diophantine Approximation*. Series: Cambridge Tracts in Mathematics 193. Cambridge University Press, 2012.

Cristian S. Calude and Marius Zimand. Algorithmically independent sequences. *Information and Computation*, 208(3):292 – 308, 2010.

Olivier Carton and Pablo Ariel Heiber. Normality and two-way automata. *Information and Computation*, 241:264–276, 2015.

Adam Case and Jack H. Lutz. Mutual dimension. *ACM Trans. Comput. Theory*, 7(3):12:1–12:26, July 2015.

D. G. Champernowne. The construction of decimals normal in the scale of ten. *Journal of the London Mathematical Society*, 8:254–260, 1933.

J. Dai, J. Lathrop, J. Lutz, and E. Mayordomo. Finite-state dimension. *Theoretical Computer Science*, 310:1–33, 2004.

G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Sixth Edition 2008.

D. Huffman. A method for the construction of minimum-redundancy codes. In *Institute of Radio Engineers*, pages 1098–1102, 1952.

K. Jacobs and M. Keane. 0-1 sequences of Toeplitz type. *Z. Wahrsheinlichkeitstheorie verw. Geb.*, 13:123–131, 1969.

W. Merkle and J. Reimann. Selection functions that do not preserve normality. *Theory of Computing Systems*, 39(5):685–697, 2006.

M. G. O'Connor. An unpredictability approach to finite-state randomness. *Journal of Computer and System Sciences*, 37(3):324–336, 1988.

Dominique Perrin and Jean-Éric Pin. *Infinite Words*. Elsevier, 2004.

C. P. Schnorr and H. Stimm. Endliche automaten und zufallsfolgen. *Acta Informatica*, 1:345–359, 1972.