Last Passage Percolation, KPZ, and Competition Interfaces

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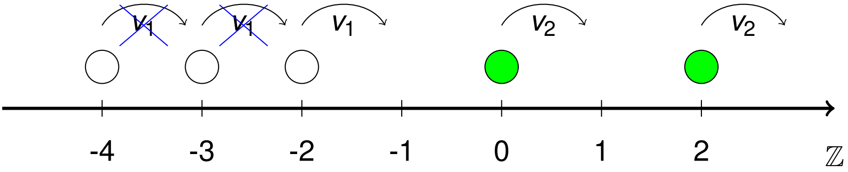
Totally asymmetric simple exclusion process (TASEP)



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- We will also consider particle-dependent speeds

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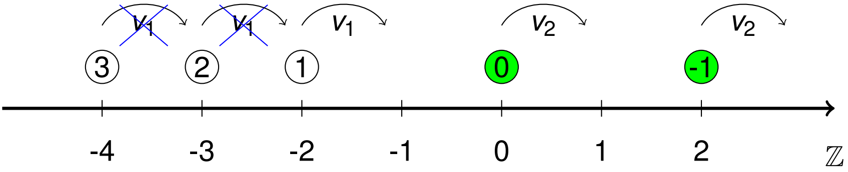

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- We will also consider particle-dependent speeds

We number particles from right to left

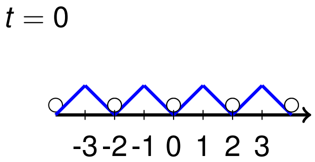
$$\ldots < x_3(0) < x_2(0) < x_1(0) < 0 \leq x_0(0) < x_{-1}(0) < \ldots$$

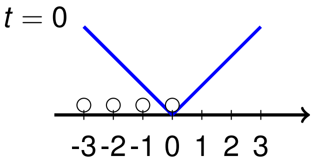
$$x_k(t) = \text{position of particle } k \text{ at time } t$$

TASEP - a KPZ growth model

Set $h(0,0) = 0$ and

$$h(x+1,0) - h(x,0) = \begin{cases} -1 & \text{if } x+1 \text{ is occupied at time } 0 \\ 1 & \text{otherwise} \end{cases}$$





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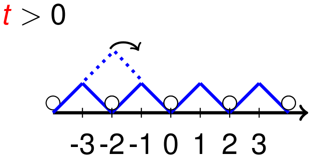

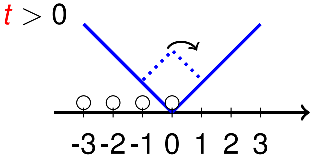

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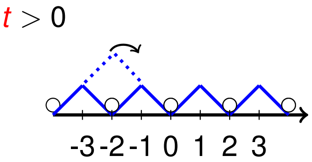
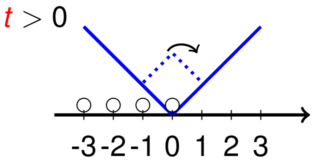


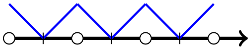


Hydrodynamic theory identifies TASEP as a KPZ model

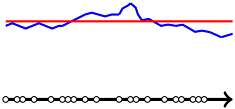
Flat TASEP and the Airy$_1$ process

TASEP with a flat geometry ($\partial_\xi^2 h_{\rm ma} = 0$) for periodic initial data:

$t = 0$



$t \gg 0$



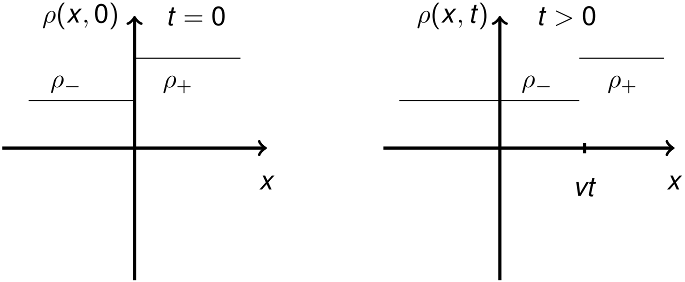
For flat TASEP we have [BFPS '07] in the sense of fin. dim. distr.

$$\lim_{t\to\infty} \frac{x_{t/4+\xi t^{2/3}}(t) + 2\xi t^{2/3}}{-t^{1/3}} = \mathcal{A}_1(\xi),$$

with $\mathcal{A}_1(\xi)$ the Airy$_1$ process with one-point distribution given by the F_1 (GOE) Tracy-Widom distribution from random matrix theory.

Shocks

- Discontinuities of the particle density are called **shocks**

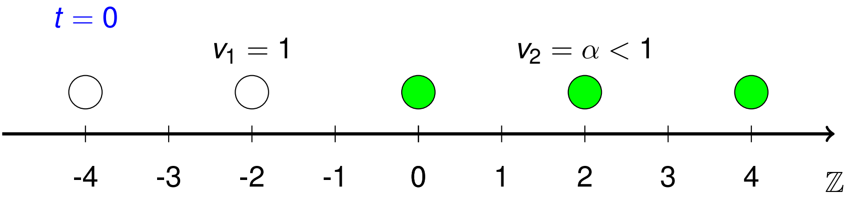


- Initial condition: $\mathrm{Ber}(\rho_+)$ on $\mathbb{N}$ and $\mathrm{Ber}(\rho_-)$ on $\mathbb{Z}_-$.
  - for $\rho_- < \rho_+$ there is a shock with speed $v = 1 - (\rho_+ + \rho_-)$
  - one can identify the microscopic shock with the position Z_t of a particle fluctuating around vt:

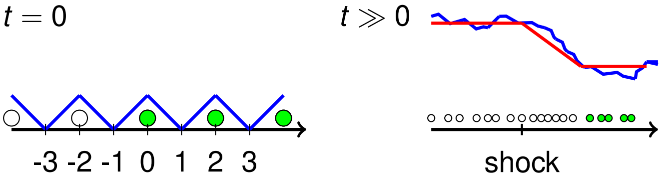
$$\lim_{t\to\infty} \frac{Z_t - vt}{t^{1/2}} \sim \mathcal{N}(0, \mu^2) \qquad \text{(see Lig '99)}$$

Question: What are the shock fluctuations for **non-random initial configuration (IC)**?

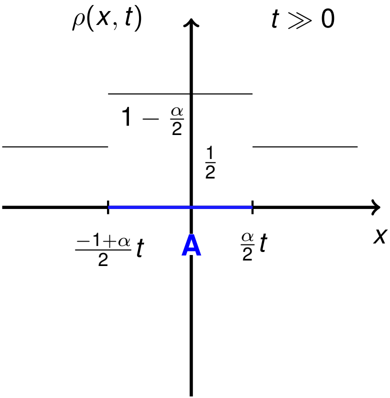
Two Speed TASEP with periodic IC



This leads to a wedge limit shape:



Shock as particle position



- The last slow particle is macroscopically at position $(1-\rho)\alpha t = \frac{\alpha}{2}t$.
- Behind it is a jam region A of increased density $\rho = 1-\alpha/2$.
- The particle ηt, with $\eta = \frac{2-\alpha}{4}$ is at the macro shock position.

Inside the constant density regions, $\eta' \neq \eta$, the fluctuations of $x_{\eta' t}$ are governed by the F_1 GOE Tracy-Widom distribution and live in the $t^{1/3}$ scale.

Goal: Determine the large time fluctuations of the (rescaled) particle position $x_{n(t)}$ around the shock:

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_{n(t)} - vt}{t^{1/3}} \leq s\right) =?$$

where vt is the macroscopic position of $x_{n(t)}$.

For arbitrary fixed IC, the law of $x_{n(t)}$ is given as a Fredholm determinant of a kernel K_t [BFPS '07],

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_{n(t)} - vt}{t^{1/3}} \leq s\right) = \lim_{t\to\infty} \det(1 - \chi_s K_t \chi_s)_{\ell^2(\mathbb{Z})}, \quad (1)$$

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Problem: K_t is diverging for our example (but its Fredholm determinant will still converge), so one cannot analyze (1) directly

Product structure for Two-Speed TASEP

Theorem (At the F_1–F_1 shock, Ferrari, N. '14)

Let $x_n(0) = -2n$ for $n \in \mathbb{Z}$. For $\alpha < 1$ let $\eta = \frac{2-\alpha}{4}$ and $v = -\frac{1-\alpha}{2}$. Then it holds

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_{\eta t+\xi t^{1/3}}(t) - vt}{t^{1/3}} \leq s\right) = F_1\left(\frac{s-2\xi}{\sigma_1}\right) F_1\left(\frac{s-\frac{2\xi}{2-\alpha}}{\sigma_2}\right),$$

where $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = \frac{\alpha^{1/3}(2-2\alpha+\alpha^2)^{1/3}}{2(2-\alpha)^{2/3}}$.

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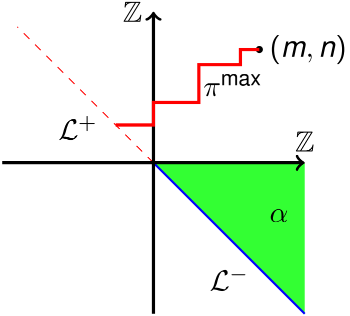
where $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = \frac{\alpha^{1/3}(2-2\alpha+\alpha^2)^{1/3}}{2(2-\alpha)^{2/3}}$.

One recovers GOE by changing $s \to s + 2\xi$ and $\xi \to +\infty$, resp. by $s \to s + 2\xi/(2-\alpha)$ and $\xi \to -\infty$

TASEP as Last Passage Percolation (LPP)

- Let $\omega_{i,j}$, $(i,j) \in \mathbb{Z}^2$, be independent weights, $\mathcal{L} \subseteq \mathbb{Z}^2$
  $\pi : \mathcal{L} \to (m,n)$ an up-right path
- $L_{\mathcal{L}\to(m,n)} = \max_\pi \sum_{\omega_{i,j}\in\pi} \omega_{i,j} = \sum_{\omega_{i,j}\in\pi^{\max}} \omega_{i,j}$



$\mathcal{L} = \{(u,-u) : u \in \mathbb{Z}\} = \mathcal{L}^+ \cup \mathcal{L}^-$
$\omega_{i,j} \sim \exp(1)$ (white), $\exp(\alpha)$ (green).

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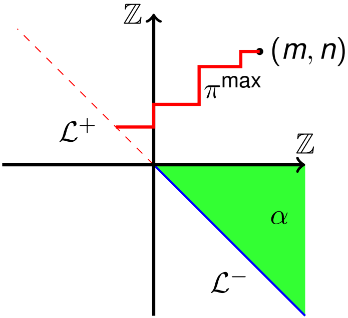
Link: $\mathbb{P}\left(L_{\mathcal{L}\to(m,n)} \le t\right) = \mathbb{P}\left(x_n(t) \ge m-n\right),$

$\omega_{i,j} \sim \exp(v_j)\mathbf{1}_{(i,j)\in\mathcal{L}^c}$, $\mathcal{L} = \{(k + x_k(0), k) : k \in \mathbb{Z}\}$



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Last Passage Percolation in combinatorics

There is a bijection between integer matrices

$$\mathcal{M}^k_{M,N} = \{A |\, A = (a_{i,j})_{\substack{1\le i\le M\\ 1\le j\le N}},\ a_{i,j} \in \mathbb{N}_0,\ \sum_{i,j} = k\}$$

and generalized permutations σ

$$\{\sigma : \sigma = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_{k-1} & i_k \\ j_1 & j_2 & j_3 & \cdots & j_{k-1} & j_k \end{pmatrix}, i_l \in [N], j_l \in [M],\ \text{either}\ \ i_l < i_{l+1}$$
$$\text{or}\ \ i_l = i_{l+1}, j_l \le j_{l+1}\}$$

where $[M] = \{1, 2, \ldots M\}$. Call $\binom{i_{r_1}}{j_{r_1}} \cdots \binom{i_{r_m}}{j_{r_m}}$ an increasing subsequence of length m if $r_1 < r_2 < \cdots < r_m$ and $j_1 \le j_2 \cdots \le j_m$, and denote $\ell(\sigma)$ a longest increasing subsequence.

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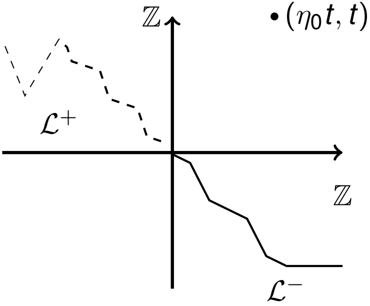
If we set $\omega_{i,j} = a_{i,j}$ then under the above bijection

$$L_{\{(1,1)\}\to(M,N)} = \ell(\sigma).$$

Generic Theorem



Assume that there exists some μ such that

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}^+\to(\eta_0 t,t)} - \mu t}{t^{1/3}} \le s\right) = G_1(s),$$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}^-\to(\eta_0 t,t)} - \mu t}{t^{1/3}} \le s\right) = G_2(s).$$

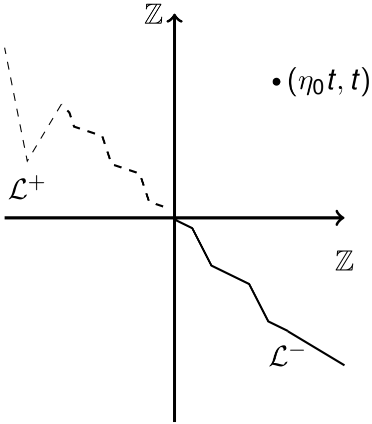
Theorem (Ferrari, N. '14)

Under some assumptions we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}\to(\eta_0 t,t)} - \mu t}{t^{1/3}} \le s\right) = G_1(s)G_2(s),$$

where $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$.

On the assumptions



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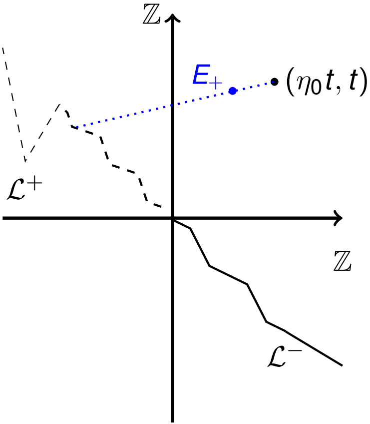

I. Assume that we have a point $E^+ = (\eta_0 t - \kappa t^\nu, t - t^\nu)$ such that for some μ_0, and $\nu \in (1/3, 1)$ it holds

$$\frac{L_{\mathcal{L}^+ \to E_+} - \mu t + \mu_0 t^\nu}{t^{1/3}} \to G_1$$

$$\frac{L_{E^+ \to (\eta_0 t, t)} - \mu_0 t^\nu}{t^{\nu/3}} \to G_0,$$

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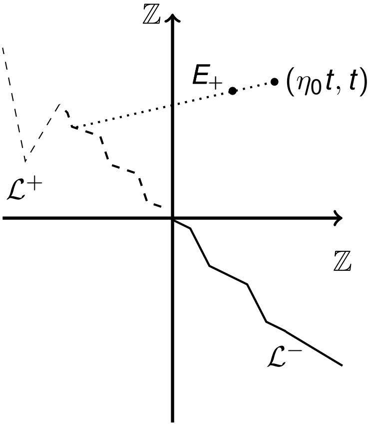

I. *Slow Decorrelation*

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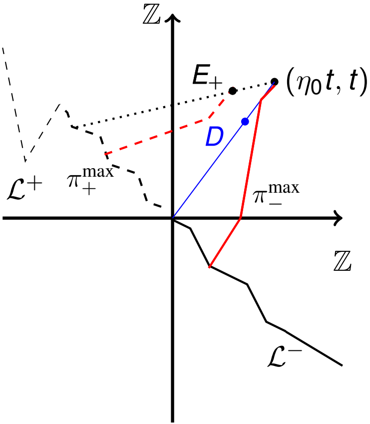

I. *Slow Decorrelation*

II. Assume there is a point $D = (\eta_0(t - t^\beta), t - t^\beta)$ with $\eta_0 t^\beta \leq \kappa t^\nu$ such that $\pi_+^{\max}$ and $\pi_-^{\max}$ cross $\overline{(0,0)D}$ with vanishing probability.

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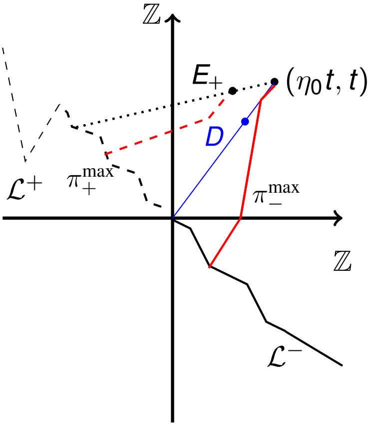

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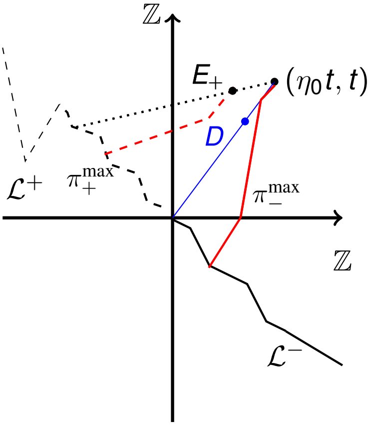


I. *Slow Decorrelation*

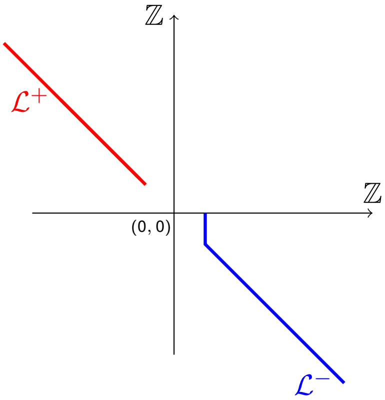
II. *No crossing*

Some remarks:

- I. is related to the universal phenomenon known as slow decorrelation [CFP '12]
- II. follows if we have that the 'characteristic lines' of the two LPP problems meet at $(\eta_0 t, t)$, together with the transversal fluctuations which are only $\mathcal{O}(t^{2/3})$ [Jo '00]
- III. An extension to joint laws

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{L_{\mathcal{L}\to(\eta t+u_k t^{1/3},t)} \leq \mu t + s_k t^{1/3}\}\right)$$

is available (Ferrari, N. '16) and based on controling local fluctuations in LPP



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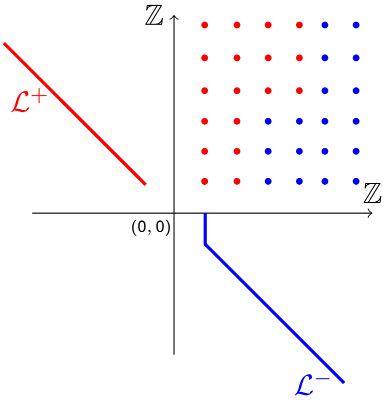

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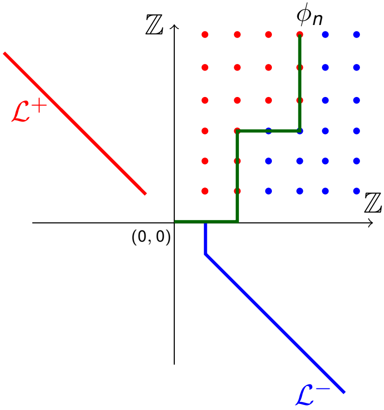
Paint $(k, l) \in \mathbb{Z}^2_{\geq 1}$ red if

$$L_{\mathcal{L}^+ \to (k,l)} > L_{\mathcal{L}^- \to (k,l)}$$

and blue if

$$L_{\mathcal{L}^- \to (k,l)} > L_{\mathcal{L}^+ \to (k,l)}$$



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The competition interface $\{\phi_n\}_{n\geq 0}$ is defined via $\phi_0 = (0,0)$ and

$$\phi_{n+1} = \begin{cases} \phi_n + (1,0) & \text{if} \quad \phi_n + (1,1) \quad \text{is} \quad \textit{red} \\ \phi_n + (0,1) & \text{if} \quad \phi_n + (1,1) \quad \text{is} \quad \textit{blue} \end{cases}$$

Some Properties of competition interfaces

- if (k, l) is red, then so are $(k, l + 1)$ and $(k - 1, l)$ (or they have no color)
- if (k, l) is blue, then so are $(k + 1, l)$ and $(k, l - 1)$ (or they have no color)
- for $\phi_n = (I_n, J_n)$ we have that $I_n + J_n = n$ and $(k, n - k)$ is red for $0 \leq k < I_n$ and blue for $I_n < k \leq n$.
- $I_n - J_n$ is again located at the shock, and is related to the position of a "second-class particle" Z_t

Theorem (Ferrari, N. '16)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{I_{\lfloor t \rfloor} - J_{\lfloor t \rfloor} - (\alpha - 1)t}{t^{1/3}} \leq s \right) = \mathbb{P}(\chi^{1,s}_{\rm GOE} - \chi^{2,s}_{\rm GOE} > 0)$$

where $\chi^{1,s}_{\rm GOE}, \chi^{2,s}_{\rm GOE}$ are independent random variables with shifted GOE distribution,

$$\mathbb{P}(\chi^{1,s}_{\rm GOE} \leq \tau) = F_{\rm GOE}\left((\tau + (2/(2-\alpha))^{4/3} s)/\sigma_1 \right),$$

$$\mathbb{P}(\chi^{2,s}_{\rm GOE} \leq \tau) = F_{\rm GOE}\left((\tau + (2/(2-\alpha))^{4/3} s/\alpha)/\sigma_2 \right),$$

where $\sigma_1 = \frac{2^{2/3}}{(2-\alpha)^{1/3}}$ and $\sigma_2 = \frac{2^{2/3}(2-2\alpha+\alpha^2)^{1/3}}{\alpha^{2/3}(2-\alpha)}$.

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