DAC
Algebra and Analysis

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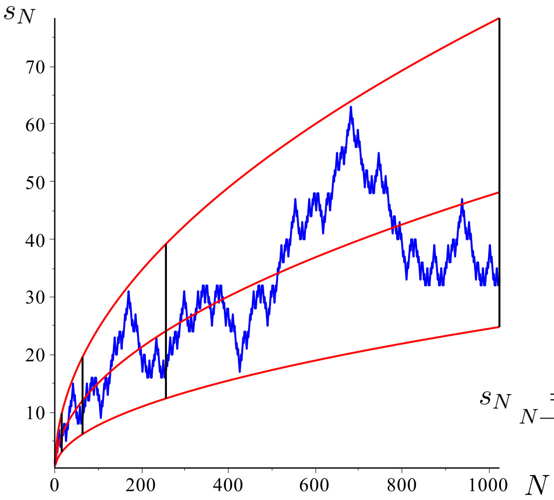
Centre International de Rencontres Mathématiques

All is available at

`http://specfun.inria.fr/dumas/Research/DAC/`



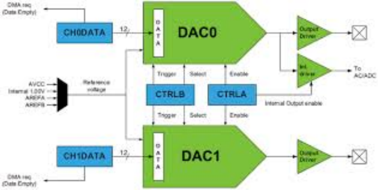
$$s_N \underset{N\to+\infty}{=} \sqrt{N}\varphi(\log_4 N) + O(1)$$

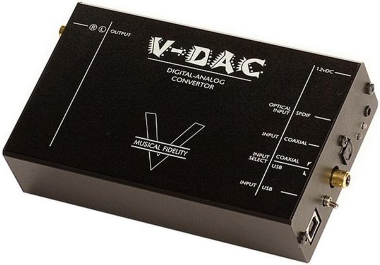
Part I

Algebra

Overview of Part I

What is a DAC recurrence?





Rank	Abbr.	Meaning
	DAC	Design Automation Conference
	DAC	Digital-to-Analog Converter
	DAC	Development Assistance Committee (OECD)
	DAC	Discretionary Access Control
	DAC	District Advisory Council
	DAC	Data Access Component
	DAC	Downhill Assist Control (automobiles)
	DAC	Department of Arts and Culture (South Africa)
	DAC	Divide and Conquer

What is a DAC recurrence?

Karatsuba's polynomial multiplication

$$a = a_0(x) + x^k a_1(x), \qquad b = b_0(x) + x^k b_1(x)$$

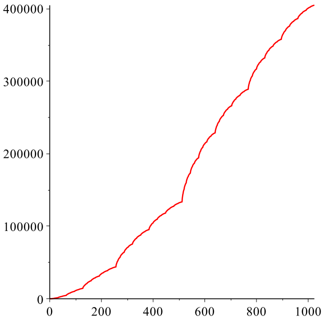
$$ab = a_0 \times b_0 + x^k((a_0 + a_1) \times (b_0 + b_1) - a_0 \times b_0 - a_1 \times b_1) + x^{2k} a_1 \times b_1$$

What is a DAC recurrence?

Karatsuba's polynomial multiplication

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1)$$

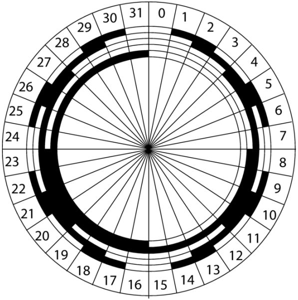
$n \geq 2$, with $u_0 = 0$, $u_1 = 1$



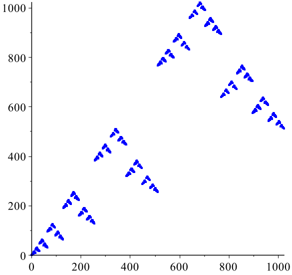
floor and ceil type

What is a DAC recurrence?

Gray code as usual binary code



n	$\mathrm{bin}(n)$	$\mathrm{gray}(n)$	u_n
0	00	00	0
1	01	01	1
2	10	11	3
3	11	10	2
4	100	110	6
5	101	111	7
6	110	101	5
7	111	100	4



What is a DAC recurrence?

Gray code as usual binary code

$$
\begin{aligned}
u_{4n} &= 2u_{2n}, \\
u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1}, \\
u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1}, \\
u_{4n+3} &= 2u_{2n+1},
\end{aligned}
\qquad \text{with } u_0 = 0.
$$

by case type

What is a DAC recurrence?

From floor and ceil type to by case type: obvious!

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1), \qquad n \geq 2, \qquad \text{with } u_0 = 0,\ u_1 = 1,$$

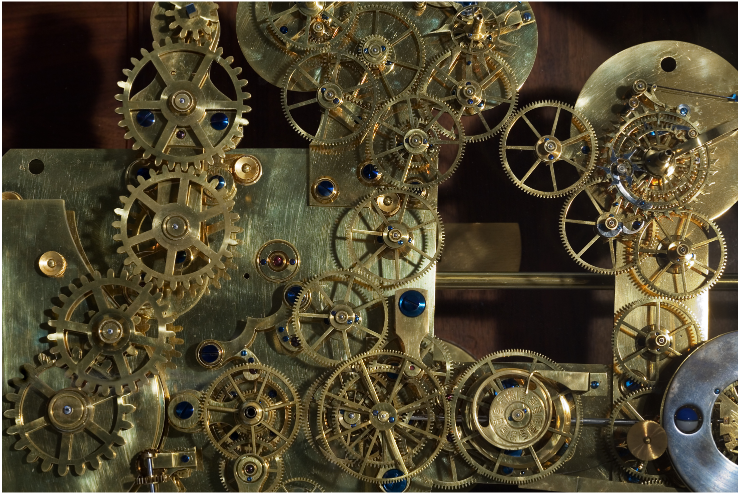
$$\begin{aligned} u_{2n} &= 3u_n + 8n - 4, & \text{with } u_0 &= 0 \\ u_{2n+1} &= 2u_{n+1} + u_n + 8n, & \text{with } u_1 &= 1 \end{aligned}$$

What is a DAC recurrence?

But from by case type to floor and ceil type?

$$\begin{aligned} u_{4n} &= 2u_{2n}, \\ u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1}, \\ u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1}, \\ u_{4n+3} &= 2u_{2n+1}, \end{aligned} \qquad \text{with } u_0 = 0.$$

Algebraic machinery



Algebraic machinery: Linear operators

$$u(x) = \sum_{n \geq 0} u_n x^n$$

$u(x)$ formal series in $\mathbb{K}[[x]]$ (u_n) sequence in $\mathbb{K}^{\mathbb{N}}$

Both are exactly the same object.

Algebraic machinery: Linear operators

$$u(x) = \sum_{n \geq 0} u_n x^n$$

radix $b \geq 2$ Mahler operator $Mu(x) = u(x^b)$

Algebraic machinery: Linear operators

$$u(x) = \sum_{n \geq 0} u_n x^n$$

radix $b \geq 2$ Mahler operator $Mu(x) = u(x^b)$

$0 \leq r < b$ section operator $T_{b,r}u(x) = \sum_{k \geq 0} u_{bk+r} x^k$

Algebraic machinery: Linear operators

$$u(x) = \sum_{n \geq 0} u_n x^n$$

radix $b \geq 2$	Mahler operator	$Mu(x) = u(x^b)$
$0 \leq r < b$	section operator	$T_{b,r}u(x) = \sum_{k \geq 0} u_{bk+r} x^k$
	forward shift	$Su(x) = \sum_{n \geq 0} u_{n+1} x^n$

Algebraic machinery: Linear operators

$$u(x) = \sum_{n\geq 0} u_n x^n$$

radix $b \geq 2$	Mahler operator	$Mu(x) = u(x^b)$
$0 \leq r < b$	section operator	$T_{b,r}u(x) = \sum_{k\geq 0} u_{bk+r} x^k$
	forward shift	$Su(x) = \sum_{n\geq 0} u_{n+1} x^n$
	backward shift	$xu(x) = \sum_{n\geq 0} u_{n-1} x^n$

Algebraic machinery: Linear operators

radix $b = 2$, by far the most usual case

Mahler operator

$$Mu(x) = u_0 \qquad + u_1 x^2 \qquad + u_2 x^4 \qquad + \cdots$$

Algebraic machinery: Linear operators

radix $b = 2$, by far the most usual case

Mahler operator

$$Mu(x) = u_0 \qquad + u_1 x^2 \qquad + u_2 x^4 \qquad + \cdots$$

section operator

$$T_{2,0}u(x) = u_0 \quad + u_2 x + u_4 x^2 \quad + u_6 x^3 + \cdots \qquad \text{even part}$$

$$T_{2,1}u(x) = u_1 \quad + u_3 x + u_5 x^2 \quad + u_7 x^3 + \cdots \qquad \text{odd part}$$

Algebraic machinery: Linear operators

radix $b = 2$, by far the most usual case

Mahler operator

$$Mu(x) = u_0 + u_1 x^2 + u_2 x^4 + \cdots$$

section operator

$$T_{2,0}u(x) = u_0 + u_2 x + u_4 x^2 + u_6 x^3 + \cdots \qquad \text{even part}$$

$$T_{2,1}u(x) = u_1 + u_3 x + u_5 x^2 + u_7 x^3 + \cdots \qquad \text{odd part}$$

forward shift

$$Su(x) = u_1 + u_2 x + u_3 x^2 + u_4 x^3 + \cdots$$

backward shift

$$xu(x) = u_0 x + u_1 x^2 + u_2 x^3 + u_3 x^4 + \cdots$$

Algebraic machinery: Basic functional properties

$T_{b,0}M = 1, \quad T_{b,r}M = 0$	$1 \leq r < b$	obvious
$Mx = x^b M$		obvious
$ST_{b,r} = T_{b,r}S^b$		the same, but...

Algebraic machinery: Basic functional properties

$$ST_{b,r} = T_{b,r}S^b, \qquad \text{the same, but\ldots}$$

$$T_{b,r}u(x) = u_r \qquad + u_{b+r}x + u_{2b+r}x^2 \quad + u_{3b+r}x^3 + \cdots$$

$$S^b u(x) = u_b \qquad + u_{b+1}x + u_{b+2}x^2 \qquad + u_{b+3}x^3 + \cdots$$

Algebraic machinery: Basic functional properties

$$ST_{b,r} = T_{b,r}S^b, \qquad \text{the same, but}\ldots$$

$$T_{b,r}u(x) = u_r \qquad +u_{b+r}x + u_{2b+r}x^2 \quad +u_{3b+r}x^3 + \cdots$$

$$ST_{b,r}u(x) = u_{b+r} \quad +u_{2b+r}x + u_{3b+r}x^2 \quad +u_{4b+r}x^3 + \cdots$$

$$T_{b,r}S^bu(x) = u_{b+r} \quad +u_{2b+r}x^2 + u_{3b+r}x^3 \quad +u_{4b+r}x^3 + \cdots$$

$$S^bu(x) = u_b \qquad +u_{b+1}x + u_{b+2}x^2 \quad +u_{b+3}x^3 + \cdots$$

Algebraic machinery: Basic functional properties

$$ST_{b,r} = T_{b,r}S^b, \qquad \text{the same, but}\ldots$$

Proposition

The sections of a rational function are rational functions.

Proof

$f \in \mathbb{K}(x)$, $S^* f \in \mathcal{F}$ with $\dim \mathcal{F} < \infty$,
$g = T_{b,r} f$, $S^k g = T_{b,r} S^{bk} f \in T_{b,r}\mathcal{F}$ with $\dim T_{b,r}\mathcal{F} < \infty$ $\square$

motto : a subspace left stable by the operator(s)

Algebraic machinery: Basic functional properties

$$T_{b,r}(fMg) = (T_{b,r}f)g$$

$$\sum_{0 \le r < b} x^r M T_{b,r} = 1$$

Algebraic machinery: Basic functional properties

$T_{b,r}(f(x)g(x^b)) = (T_{b,r}f(x))g(x)$ useful for products

$\sum_{0 \le r < b} x^r T_{b,r} f(x^b) = f(x)$ It is possible to rebuild a function from its sections.

Example

$$T_{2,0}\frac{1+3x}{x^3\,(1+2x)} = \frac{1}{x(1-4x)}, \qquad T_{2,1}\frac{1+3x}{x^3\,(1+2x)} = \frac{1-6x}{x^2(1-4x)},$$

$$1 \times \frac{1}{x^2(1-4x^2)} + x \times \frac{1-6x^2}{x^4(1-4x^2)} = \frac{1+3x}{x^3\,(1+2x)} \qquad b=2$$

$$1 \times T_{2,0}f(x^2) + xT_{2,1}f(x^2) = f(x)$$

△

Definition of DAC recurrences

Definition of DAC recurrences

Definition

A (linear) Mahler equation is an equation

$$\ell_0(x)u(x) + \ell_1(x)u(x^b) + \cdots + \ell_d(x)u(x^{b^d}) = v(x)$$

where $\ell_0(x)$, $\ell_1(x)$, …, $\ell_d(x)$ and $v(x)$ are polynomials in $\mathbb{K}[x]$.

$$L(x,M) = \ell_0(x)+\ell_1(x)M+\cdots+\ell_d(x)M^d, \qquad L(x,M)u(x) = v(x)$$

motto : a subspace left stable by the operator(s)

Definition of DAC recurrences

Definition

A divide-and-conquer recurrence is the translation in terms of sequence of a Mahler equation.

Definition of DAC recurrences

Definition

$u_\nu = 0$ if $\nu \notin \mathbb{N}_{\geq 0}$

Example

$$(x+2x^2)u(x) - (1+x)u(x^2) + u(x^4) = 0, \qquad b = 2$$

$$\begin{aligned}
u_{m-1} + 2u_{m-2} \quad & -u_{\frac{m}{2}} - u_{\frac{m-1}{2}} \quad && +u_{\frac{m}{4}} = 0, && m \geq 0 \\
u_9 + 2u_8 \quad & -u_5 - u_{\frac{9}{2}} && +u_{\frac{5}{2}} = 0, && m = 10 \\
u_{10} + 2u_9 \quad & -u_{\frac{11}{2}} - u_5 && +u_{\frac{11}{4}} = 0, && m = 11 \\
u_{11} + 2u_{10} \quad & -u_6 - u_{\frac{11}{2}} && +u_3 = 0, && m = 12 \\
u_{12} + 2u_{11} \quad & -u_{\frac{13}{2}} - u_6 && +u_{\frac{13}{4}} = 0, && m = 13
\end{aligned}$$

△

Definition of DAC recurrences

Example

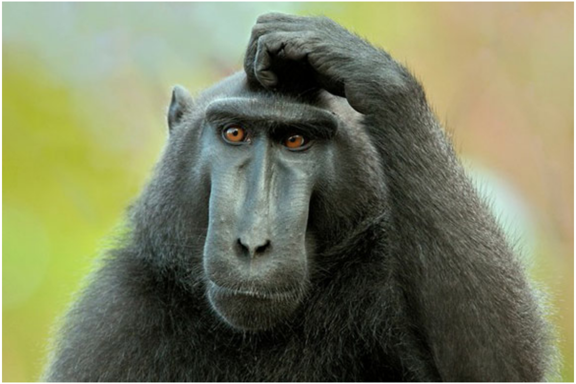
$$\begin{aligned}
&(x + 2x^2)u(x) - (1 + x)u(x^2) + u(x^4) = 0, && b = 2 \\
&u_{m-1} + 2u_{m-2} \quad -u_{\frac{m}{2}} - u_{\frac{m-1}{2}} \quad +u_{\frac{m}{4}} = 0, && m \geq 0 \\
&u_9 + 2u_8 \quad -u_5 = 0, && m = 10 \\
&u_{10} + 2u_9 \quad - u_5 = 0, && m = 11 \\
&u_{11} + 2u_{10} \quad -u_6 \quad +u_3 = 0, && m = 12 \\
&u_{12} + 2u_{11} \quad - u_6 = 0, && m = 13
\end{aligned}$$

$\triangle$

fractional type

Comparison of types

Three types for the same thing, that's a lot!



Comparison of types: Generating functions

- reference type = fractional type

$$t_m = u_{\frac{m-s}{b^k}} \qquad t(x) = x^s u(x^{b^k})$$

Comparison of types: Generating functions

- floor and ceil type

$$t_m = u_{\lfloor \frac{n+s}{b} \rfloor} \qquad t(x) = x^{-s}(1 + x + \cdots + x^{b-1})u(x^b)$$

$$-x^{-s}(1 + x + \cdots + x^{b-1}) \sum_{n=0}^{q-1} u_n x^{bn}$$

$$-x^{-r} \sum_{i=0}^{r-1} x^i u_q$$

$$s = bq + r,\ |r| < b,\ \mathrm{sgn}(r) = \mathrm{sgn}(s)$$

ceil ad libitum

symmetrical Euclidean division

$$\left\lceil \frac{n}{b} \right\rceil = \left\lfloor \frac{n+b-1}{b} \right\rfloor$$

corrective term $= 0$ for $-\infty < s \leq 0$

Comparison of types: Generating functions

- by case type

$$t_m = u_{bk+s} \qquad t(x) = \qquad x^{-q} T_{b,r} u(x) - x^{-q} \sum_{j=0}^{q-1} u_{bj+r} x^j$$

$s = bq + r$, $0 \le r < b$

natural Euclidean division

corrective term $= 0$ for $-\infty < s < b$

Comparison of types: Generating functions

neglecting details:

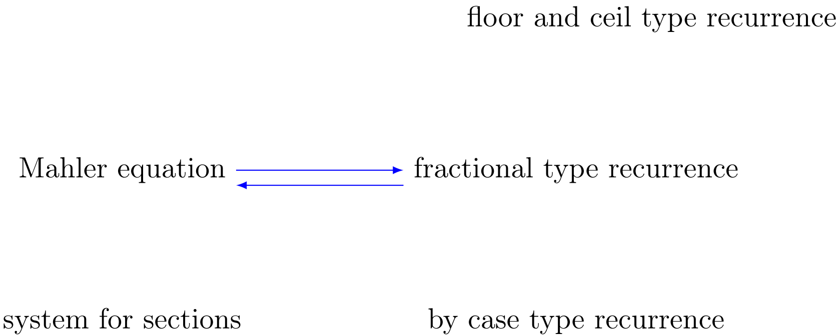
$$t_m = u_{\frac{m-s}{b^k}} \qquad t(x) = x^s u(x^{b^k})$$

$$t_m = u_{\lfloor \frac{n+s}{b} \rfloor} \qquad t(x) = x^{-s}(1 + x + \cdots + x^{b-1})u(x^b)$$

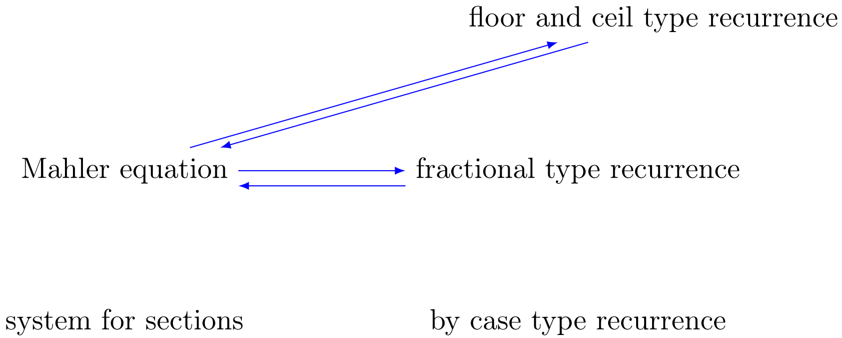
$$t_m = u_{bk+s} \qquad t(x) = x^{-q} T_{b,r} u(x)$$

- fractional type .... Mahler operator
- floor and ceil type .... Mahler operator
- by case type .... section operators

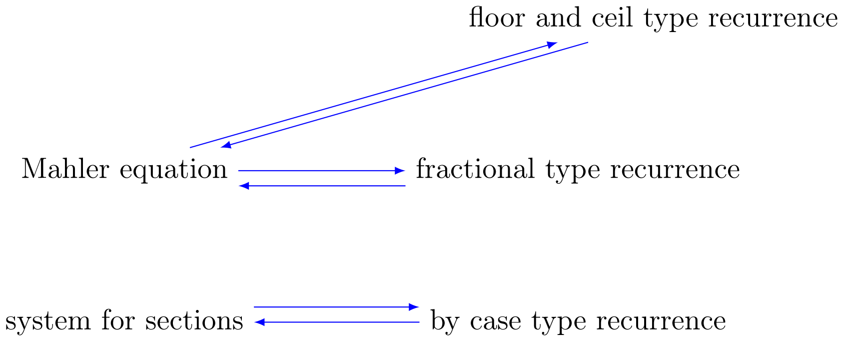
Comparison of types: Some links



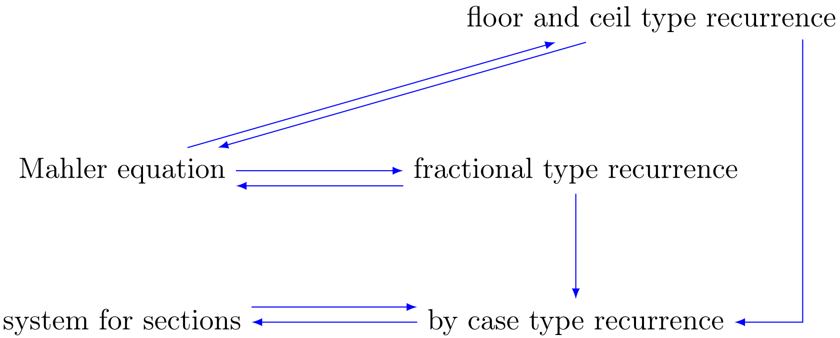
Comparison of types: Some links



Comparison of types: Some links



Comparison of types: Some links



Comparison of types: Mahler and sections

Theorem

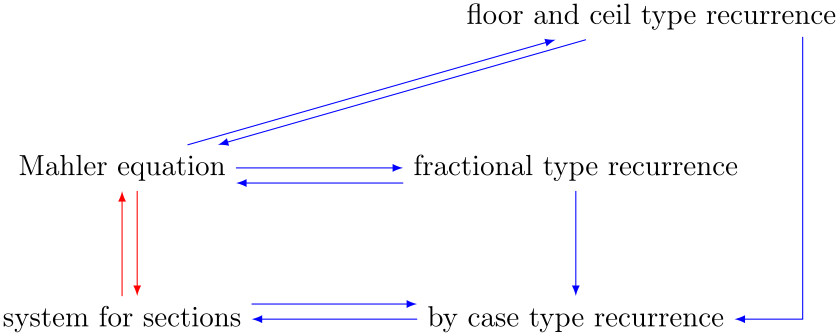
If u is a formal series which is a solution of a non trivial Mahler equation, then, under the action of the section operators, it generates a finite dimensional $\mathbb{K}(x)$-space.
Conversely, if the iterated sections of a formal series u remain in a finite dimensional $\mathbb{K}(x)$-space, then u is a solution a non trivial Mahler equation.

variation on

Gilles Christol, Teturo Kamae, Michel Mendès France, and Gérard Rauzy.
Suites algébriques, automates et substitutions.
Bull. Soc. Math. France, 108(4):401–419, 1980.

motto : a subspace left stable by the operator(s)

Comparison of types: All links



strongly connected graph

Comparison of types: Types equivalence

Theorem

For a sequence (u_n) with support in $\mathbb{N}_{\geq 0}$ and for its generating function $u(x)$, with a given integer $b \geq 2$,

- *a fractional type recurrence,*
- *a floor and ceil type recurrence,*
- *a by case type equation,*
- *a Mahler equation,*
- *a system about the sections,*

all have the same expressiveness.

Anatoli Karatsuba

Anatoli Karatsuba

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1) \qquad n \geq 2,$$

with $u(0) = 0$, $u(1) = 1$

Anatoli Karatsuba

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1) \qquad n \geq 2,$$

$$\text{with } u(0) = 0,\ u(1) = 1$$

$$xu(x) - (1+x)(2+x)u(x^2) = -x^2 + 4\frac{x^3}{(1-x)^2}$$

Anatoli Karatsuba

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1) \qquad n \geq 2,$$

$$\text{with } u(0) = 0,\ u(1) = 1$$

$$xu(x) - (1+x)(2+x)u(x^2) = -x^2 + 4\frac{x^3}{(1-x)^2}$$

$$u_{m-1} - (2u_{\frac{m}{2}} + 3u_{\frac{m-1}{2}} + u_{\frac{m-2}{2}}) = 4(m-1)$$

Anatoli Karatsuba

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1) \qquad n \geq 2,$$

$$\text{with } u(0) = 0,\ u(1) = 1$$

$$xu(x) - (1+x)(2+x)u(x^2) = -x^2 + 4\frac{x^3}{(1-x)^2}$$

$$u_{m-1} - (2u_{\frac{m}{2}} + 3u_{\frac{m-1}{2}} + u_{\frac{m-2}{2}}) = 4(m-1)$$

$$u_{2k-1} = 2u_k + u_{k-1} + 8k - 4, \qquad k \geq 2,$$

$$u_{2k} = 3u_k + 8k, \qquad k \geq 1$$

Anatoli Karatsuba

$$u(x) = \frac{(1+x)(2+x)}{x} u(x^2) - x + 4\frac{x^2}{(1-x)^2}$$

$$T_{2,0}u(x) = 3u(x) + \frac{4x+4x^2}{(1-x)^2} \qquad T_{2,1}u(x) = \frac{2+x}{x} u(x) - \frac{1-10x+x^2}{(1-x)^2}$$

Anatoli Karatsuba

$$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1)$$

$$xu(x) - (1+x)(2+x)u(x^2) = -x^2 + 4\frac{x^3}{(1-x)^2}$$

$$u_{m-1} - (2u_{\frac{m}{2}} + 3u_{\frac{m-1}{2}} + u_{\frac{m-2}{2}}) = 4(m-1)$$

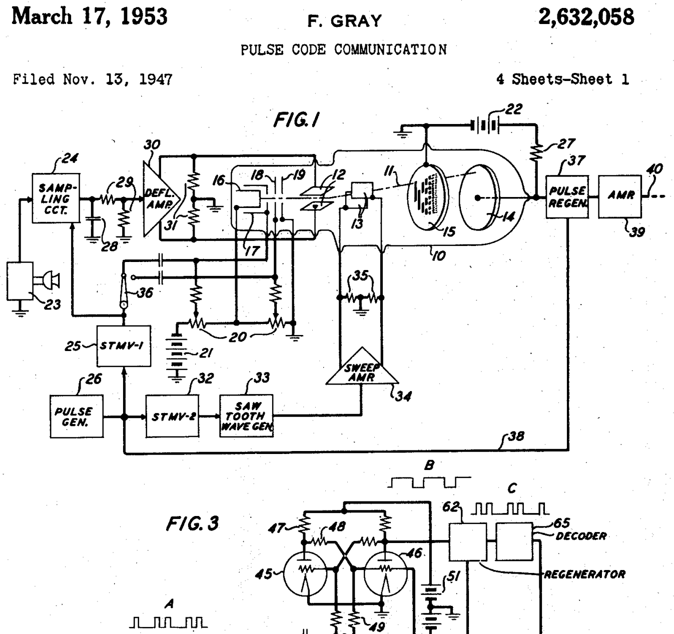
$$T_{2,0}u(x) = 3u(x) + \frac{4x + 4x^2}{(1-x)^2}$$

$$T_{2,1}u(x) = \frac{2+x}{x}u(x) - \frac{1 - 10x + x^2}{(1-x)^2}$$

$$u_{2k-1} = 2u_k + u_{k-1} + 8k - 4$$

$$u_{2k} = 3u_k + 8k$$

Frank Gray


March 17, 1953
F. GRAY
2,632,058
PULSE CODE COMMUNICATION
Filed Nov. 13, 1947
4 Sheets-Sheet 1
FIG. 1
SAMP-LING CCT.
DEFL. AMP.
PULSE REGEN.
AMR
STMV-1
PULSE GEN.
STMV-2
SAW TOOTH WAVE GEN.
SWEEP AMR
FIG. 3
DECODER
REGENERATOR

Frank Gray

$$
\begin{aligned}
u_{4n} &= 2u_{2n}, & u_0 = 0 \\
u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1} \\
u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1} \\
u_{4n+3} &= 2u_{2n+1}
\end{aligned}
$$

Frank Gray

$$\begin{aligned} u_{4n} &= 2u_{2n}, & u_0 &= 0 \\ u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1} \\ u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1} \\ u_{4n+3} &= 2u_{2n+1} \end{aligned}$$

$$\begin{aligned} T_{2,0}u(x) &= 2T_{2,0}u(x) \\ T_{4,1}u(x) &= -4u(x) + 3T_{2,0}u(x) + T_{2,1}u(x) \\ T_{4,2}u(x) &= -4u(x) + T_{2,0}u(x) + 3T_{2,1}u(x) \\ T_{4,3}u(x) &= 2T_{2,1}u(x) \end{aligned}$$

Frank Gray

$$
\begin{aligned}
u_{4n} &= 2u_{2n}, & u_0 = 0 \\
u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1} \\
u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1} \\
u_{4n+3} &= 2u_{2n+1}
\end{aligned}
$$

$$
\begin{aligned}
T_{2,0}u(x) &= 2T_{2,0}u(x) \\
T_{4,1}u(x) &= -4u(x) + 3T_{2,0}u(x) + T_{2,1}u(x) \\
T_{4,2}u(x) &= -4u(x) + T_{2,0}u(x) + 3T_{2,1}u(x) \\
T_{4,3}u(x) &= 2T_{2,1}u(x)
\end{aligned}
$$

$$v_1(x) = u(x),\ v_2(x) = T_{2,0}u(x),\ v_3(x) = T_{2,1}u(x)$$

Frank Gray

$$T_{2,0}u(x) = 2T_{2,0}u(x)$$
$$T_{4,1}u(x) = -4u(x) + 3T_{2,0}u(x) + T_{2,1}u(x)$$
$$T_{4,2}u(x) = -4u(x) + T_{2,0}u(x) + 3T_{2,1}u(x)$$
$$T_{4,3}u(x) = 2T_{2,1}u(x)$$

$$v_1(x) = u(x),\ v_2(x) = T_{2,0}u(x),\ v_3(x) = T_{2,1}u(x)$$

$$T_{2,0}v_1(x) = v_2(x)$$
$$T_{2,0}v_2(x) = 2v_2(x)$$
$$T_{2,0}v_3(x) = -4v_1(x) + 3v_2(x) + v_3(x)$$
$$T_{2,1}v_1(x) = v_3(x)$$
$$T_{2,1}v_2(x) = -4v_1(x) + v_2(x) + 3v_3(x)$$
$$T_{2,1}v_3(x) = 2v_3(x)$$

Frank Gray

$$v_1(x) = u(x),\ v_2(x) = T_{2,0}u(x),\ v_3(x) = T_{2,1}u(x)$$

$$\begin{aligned}
T_{2,0}v_1(x) &= v_2(x) \\
T_{2,0}v_2(x) &= 2v_2(x) \\
T_{2,0}v_3(x) &= -4v_1(x) + 3v_2(x) + v_3(x) \\
T_{2,1}v_1(x) &= v_3(x) \\
T_{2,1}v_2(x) &= -4v_1(x) + v_2(x) + 3v_3(x) \\
T_{2,1}v_3(x) &= 2v_3(x)
\end{aligned}$$

$$A_0 = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Frank Gray

$$v_1(x) = u(x),\ v_2(x) = T_{2,0}u(x),\ v_3(x) = T_{2,1}u(x)$$

$$A_0 = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$v(x) = T_{2,0}v(x^2) + xT_{2,1}v(x^2)$$

$$\begin{aligned} \left[\begin{array}{ccc} v_1(x) & v_2(x) & v_3(x) \end{array}\right] &= \left[\begin{array}{ccc} v_1\left(x^2\right) & v_2\left(x^2\right) & v_3\left(x^2\right) \end{array}\right] \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ &+ x \left[\begin{array}{ccc} v_1\left(x^2\right) & v_2\left(x^2\right) & v_3\left(x^2\right) \end{array}\right] \begin{bmatrix} 0 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} \end{aligned}$$

Frank Gray

$$v(x) = T_{2,0}v(x^2) + xT_{2,1}v(x^2)$$

$$\begin{aligned}\left[\begin{array}{ccc} v_1(x) & v_2(x) & v_3(x) \end{array}\right] &= \left[\begin{array}{ccc} v_1(x^2) & v_2(x^2) & v_3(x^2) \end{array}\right] \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad + x \left[\begin{array}{ccc} v_1(x^2) & v_2(x^2) & v_3(x^2) \end{array}\right] \begin{bmatrix} 0 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}\end{aligned}$$

$$V(x) = \left[\begin{array}{ccc} v_1(x) & v_2(x) & v_3(x) \end{array}\right], \qquad A(x) = A_0 + xA_1,$$

$$V(x) = V(x^2)A(x), \qquad u(x) = V(x)C$$

Frank Gray

$$V(x) = V(x^2)A(x), \qquad u(x) = V(x)C$$

$$u(x) = V(x)C$$
$$u(x) = V(x^2)A(x)C$$
$$u(x) = V(x^4)A(x^2)A(x)C$$
$$u(x) = V(x^8)A(x^4)A(x^2)A(x)C$$

$$u(x^8) = V(x^8)C$$
$$u(x^4) = V(x^8)A(x^4)C$$
$$u(x^2) = V(x^8)A(x^4)A(x^2)C$$
$$u(x) = V(x^8)A(x^4)A(x^2)A(x)C$$

Frank Gray

$$\begin{aligned} u(x^8) &= V(x^8)C \\ u(x^4) &= V(x^8)A(x^4)C \\ u(x^2) &= V(x^8)A(x^4)A(x^2)C \\ u(x) &= V(x^8)A(x^4)A(x^2)A(x)C \end{aligned}$$

4 column vectors in dimension 3

Frank Gray

$$A(x) = \begin{bmatrix} 0 & -4\,x & -4 \\ 1 & 2+x & 3 \\ x & 3\,x & 1+2\,x \end{bmatrix}$$

$$\Gamma(x) = \begin{bmatrix} 1 & 0 & -4x^2-4x^4 & -4x-12x^2-8x^3-8x^4-12x^5-4x^6 \\ 0 & 1 & 2+3x^2+x^4 & 4+5x+7x^2+6x^3+2x^4+3x^5+x^6 \\ 0 & x^4 & x^2+3x^4+2x^6 & x+3x^2+2x^3+6x^4+7x^5+5x^6+4x^7 \end{bmatrix}$$

$$K(x) = \begin{bmatrix} 0 \\ 2\,\dfrac{(1+x)(1+x^4)}{x} \\ -\dfrac{(1+x)(1+2x+2x^3+x^4)}{x(1+x^2)} \\ 1 \end{bmatrix}$$

Frank Gray

$$K(x) = \begin{bmatrix} 0 \\ 2\,\dfrac{(1+x)(1+x^4)}{x} \\ -\dfrac{(1+x)(1+2x+2x^3+x^4)}{x(1+x^2)} \\ 1 \end{bmatrix}$$

$$x(1+x^2)u(x)-(1+x)(1+2x+2x^3+x^4)u(x^2)+2(1+x)(1+x^2)(1+x^4)u(x^4)=0$$

Frank Gray

to summarize:

from

$$\begin{aligned} T_{2,0}u(x) &= 2T_{2,0}u(x) \\ T_{4,1}u(x) &= -4u(x) + 3T_{2,0}u(x) + T_{2,1}u(x) \\ T_{4,2}u(x) &= -4u(x) + T_{2,0}u(x) + 3T_{2,1}u(x) \\ T_{4,3}u(x) &= 2T_{2,1}u(x) \end{aligned}$$

to

$$\begin{aligned} &x(1+x^2)u(x) \\ &\qquad - (1+x)(1+2x+2x^3+x^4)u(x^2) \\ &\qquad\qquad + 2(1+x)(1+x^2)(1+x^4)u(x^4) = 0 \end{aligned}$$

Frank Gray

$$x(1+x^2)u(x) - (1+x)(1+2x+2x^3+x^4)u(x^2) + 2(1+x)(1+x^2)(1+x^4)u(x^4) = 0$$

$$u_n + u_{n-2} - \left(u_{\lfloor \frac{n+1}{2} \rfloor} + 2u_{\lfloor \frac{n}{2} \rfloor} + 2u_{\lfloor \frac{n-2}{2} \rfloor} + u_{\lfloor \frac{n-3}{2} \rfloor}\right) + 2\left(u_{\lfloor \frac{n+1}{4} \rfloor} + u_{\lfloor \frac{n-3}{4} \rfloor}\right) = 0$$

Frank Gray

to summarize:

from

$$\begin{aligned} u_{4n} &= 2u_{2n}, & u_0 = 0 \\ u_{4n+1} &= -4u_n + 3u_{2n} + u_{2n+1} \\ u_{4n+2} &= -4u_n + u_{2n} + 3u_{2n+1} \\ u_{4n+3} &= 2u_{2n+1} \end{aligned}$$

to

$$u_n + u_{n-2} \qquad u_0 = 0,\ u_1 = 1,\ u_2 = 3$$
$$- \left(u_{\lfloor \frac{n+1}{2} \rfloor} + 2u_{\lfloor \frac{n}{2} \rfloor} + 2u_{\lfloor \frac{n-2}{2} \rfloor} + u_{\lfloor \frac{n-3}{2} \rfloor} \right)$$
$$+ 2 \left(u_{\lfloor \frac{n+1}{4} \rfloor} + u_{\lfloor \frac{n-3}{4} \rfloor} \right) = 0$$

Frank Gray

$$u_n + u_{n-2}$$
$$-\left(u_{\lfloor \frac{n+1}{2} \rfloor} + 2u_{\lfloor \frac{n}{2} \rfloor} + 2u_{\lfloor \frac{n-2}{2} \rfloor} + u_{\lfloor \frac{n-3}{2} \rfloor}\right)$$
$$+2\left(u_{\lfloor \frac{n+1}{4} \rfloor} + u_{\lfloor \frac{n-3}{4} \rfloor}\right) = 0$$

$$x(1+x^2)u(x)$$
$$-(1+x)(1+2x+2x^3+x^4)u(x^2)$$
$$+2(1+x)(1+x^2)(1+x^4)u(x^4) = 0$$

$$u_{n-3} + u_{n-1}$$
$$-u_{\frac{n-5}{2}} - 3u_{\frac{n-4}{2}} - 2u_{\frac{n-3}{2}} - 2u_{\frac{n-2}{2}}$$
$$-3u_{\frac{n-1}{2}} - u_{\frac{n}{2}}$$
$$+2u_{\frac{n-7}{4}} + 2u_{\frac{n-6}{4}} + 2u_{\frac{n-5}{4}} + 2u_{\frac{n-4}{4}}$$
$$+2u_{\frac{n-3}{4}} + 2u_{\frac{n-2}{4}} + 2u_{\frac{n-1}{4}} + 2u_{\frac{n}{4}} = 0$$

$$T_{2,0}u(x) = 2T_{2,0}u(x)$$
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$$T_{4,2}u(x) = -4u(x) + T_{2,0}u(x) + 3T_{2,1}u(x)$$
$$T_{4,3}u(x) = 2T_{2,1}u(x)$$

$$u_{4n} = 2u_{2n}$$
$$u_{4n+1} = -4u_n + 3u_{2n} + u_{2n+1}$$
$$u_{4n+2} = -4u_n + u_{2n} + 3u_{2n+1}$$
$$u_{4n+3} = 2u_{2n+1}$$

Rational sequence

Rational sequence

Definition

A formal series (or a sequence) is rational wrt a numeration system with radix b, or is b-rational, if under the action of the section operators it generates a finite dimensionial $\mathbb{K}$-vector space.

Jean-Paul Allouche and Jeffrey Shallit.
The ring of k-regular sequences.
Theoret. Comput. Sci., 98(2):163–197, 1992.

Rational sequence

Proposition

A b-rational series satisfies a non trivial Mahler equation for the radix b.

Rational sequence

Proposition

A formal series $u(x)$ which satisfies a Mahler equation ($\omega \in \mathbb{N}_{\geq 0}$)

$$x^{\omega} u(x) = c_1(x) u(x^b) + \cdots + c_d(x) u(x^{b^d}),$$

with polynomial coefficients, is b-rational.

Rational sequence

Proposition

A formal series $u(x)$ which satisfies a Mahler equation ($\omega \in \mathbb{N}_{\geq 0}$)

$$x^{\omega}u(x) = c_1(x)u(x^b) + \cdots + c_d(x)u(x^{b^d}),$$

with polynomial coefficients, is b-rational.

Proposition

A sequence (u_n) which satisfies a fractional type recurrence

$$u_n = \sum_{k=1}^{d} \sum_{\ell=-s}^{s} c_{k,\ell} u_{\frac{n-\ell}{b^k}}$$

is b-rational.

Rational sequence

Proposition

A formal series $u(x)$ which satisfies a Mahler equation ($\omega \in \mathbb{N}_{\geq 0}$)

$$x^{\omega} u(x) = c_1(x) u(x^b) + \cdots + c_d(x) u(x^{b^d}),$$

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Proposition

A sequence (u_n) which satisfies a fractional type recurrence

$$u_n = \sum_{k=1}^{d} \sum_{\ell=-s}^{s} c_{k,\ell} u_{\frac{n-\ell}{b^k}}$$

the true DAC recurrences!

is b-rational.

Linear representation

Theorem

The Nth coefficient of a b-rational series $u(x)$ is expressed as

$$u_N = T_{b,r_\ell} \cdots T_{b,r_0} u(0)$$

if $N = (r_\ell \ldots r_0)_b$.

Linear representation

Theorem

The Nth coefficient of a b-rational series $u(x)$ is expressed as

$$u_N = T_{b,r_\ell} \cdots T_{b,r_0} u(0)$$

if $N = (r_\ell \ldots r_0)_b$.

$$10 = (1010)_2 \qquad u(x) = u_0 + u_1 x + u_2 x^2 + \cdots$$

$$T_{2,0} u(x) = u_0 + u_2 x + u_4 x^2 + \cdots$$

$$T_{2,1} T_{2,0} u(x) = u_2 + u_6 x + u_{10} x^2 + \cdots$$

$$T_{2,0} T_{2,1} T_{2,0} u(x) = u_2 + u_{10} x + u_{18} x^2 + \cdots$$

$$T_{2,1} T_{2,0} T_{2,1} T_{2,0} u(x) = u_{10} + u_{26} x + u_{42} x^2 + \cdots$$

$$T_{2,1} T_{2,0} T_{2,1} T_{2,0} u(0) = u_{10}$$

Linear representation

Definition

A linear representation of a b-rational series $u(x)$ or sequence (u_n) is a triplet (L, A, C) made from

- a row vector L (initial values);
- a family of square matrices $(A_r)_{0\leq r<b}$ (action);
- a column vector C (coordinates),

with the same size and coefficients in $\mathbb{K}$, such that

$$u_N = LA_{r_\ell} \cdots A_{r_0}C$$

when

$$N = (r_\ell \dots r_0)_b.$$

Linear representation

$$u_N = LA_{r_\ell} \cdots A_{r_0} C$$

when

$$N = (r_\ell \ldots r_0)_b$$

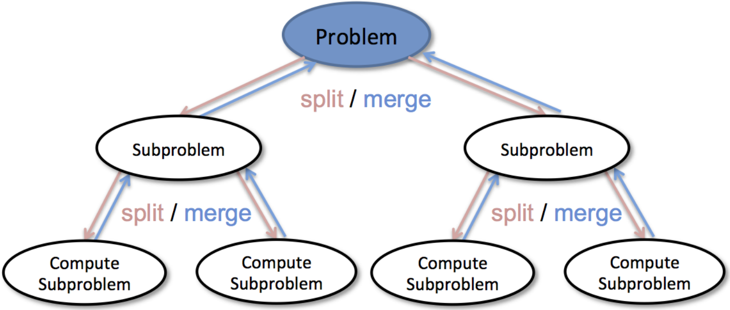
for the Gray code:

$$L = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

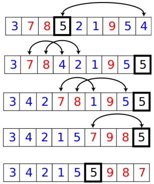
$$A_0 = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Bestiary

Divide and conquer algorithms

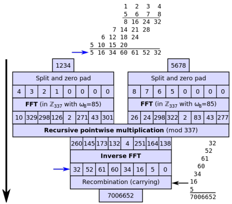


Divide and conquer algorithms



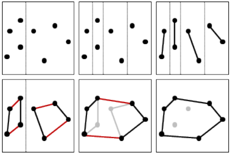
binary search
extrema
kth smallest element
mergesort
quicksort

sorting



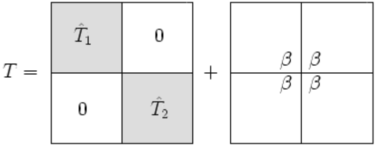
binary powering
Karatsuba algorithm
Toom-Cook algorithm
Schönhage-Strassen algorithm
multipoint evaluation

algebraic computations



convex hull
nearest pair
Voronoï diagram
maxima in dim ≥ 2

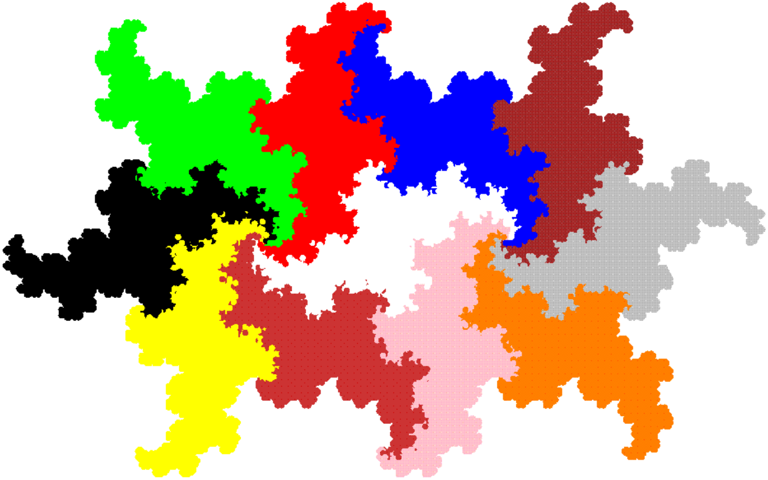
algorithmic geometry



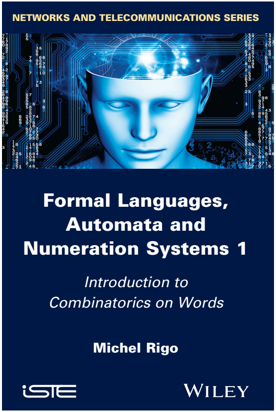
Strassen multiplication
triangular matrix inversion
fast Fourier transform
singular values decomposition
eigenvalues/vectors of symmetric tridiagonal matrices

matrix computations

Combinatorics on words



Combinatorics on words



The goal is that, after reading this book (or at least parts of this book), the reader should be able to fruitfully attend a conference or a seminar in the field.

Michel Rigo

Combinatorics on words

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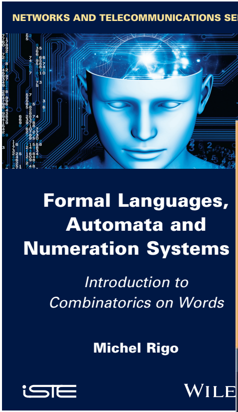
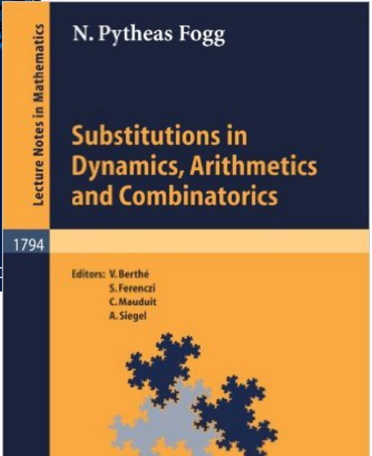




Combinatorics on words


Axel Thue

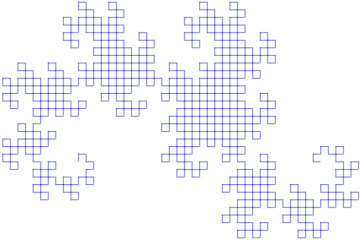

Marston Morse

$e_1(n)$ number of 1's in binary expansion of n
$e_{11}(n)$ Golay-Rudin-Shapiro sequence
$(-1)^{e_1(n)}$ Thue-Morse sequence
$(-1)^{e_1(3n)}$ Newman-Slater-Coquet
overlapping free words

patterns counting

Combinatorics on words

$$w_0 = \varepsilon$$
$$w_{k+1} = w_k 1 \overline{w}_k^R$$



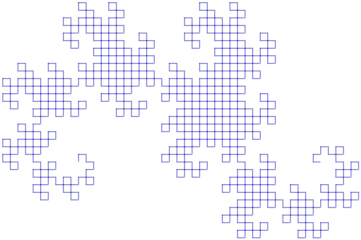
paperfolding sequence
Rauzy's fractal (irrelevant)

substitutions

Combinatorics on words

$$w_0 = \varepsilon$$
$$w_{k+1} = w_k 1 \overline{w}_k^R$$

$$a \longrightarrow ab$$
$$b \longrightarrow cb$$
$$c \longrightarrow ad$$
$$d \longrightarrow cd$$
$$a \to ab \to abcb \to abcbadcb \to \ldots$$
$$a := 1, \quad b := 1, \quad c := 0, \quad d := 0$$
$$w_\infty = 0010011000110110\ldots$$



paperfolding sequence
Rauzy's fractal (irrelevant)

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Combinatorics on words

$$w_0 = \varepsilon$$
$$w_{k+1} = w_k 1 \overline{w}_k^R$$

$$u_{4n} = 0$$
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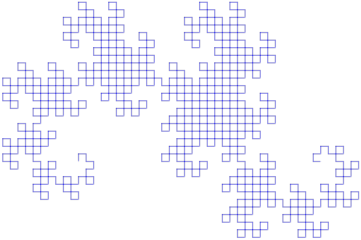

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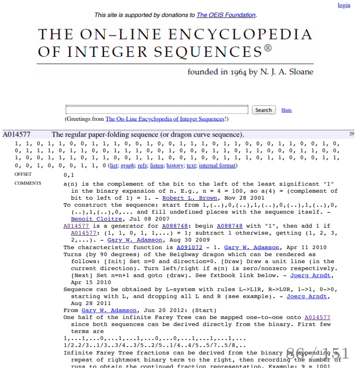
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THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A014577 The regular paper-folding sequence (or dragon curve sequence). 29

1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,1

COMMENTS
a(n) is the complement of the bit to the left of the least significant "1" in the binary expansion of n. E.g., n = 4 = 100, so a(4) = (complement of bit to left of 1) = 1. - Robert L. Brown, Nov 28 2001
To construct the sequence: start from 1,(..),0,(..),1,(..),0,(..),1,(..),0,(..),1,(..),0,... and fill undefined places with the sequence itself. - Benoit Cloitre, Jul 08 2007
A014577 is a generator for A088748: begin A088748 with "1", then add 1 if A014577: (1, 1, 0, 1, 1,...) = 1; subtract 1 otherwise, getting (1, 2, 3, 2,...). - Gary W. Adamson, Aug 30 2009
The characteristic function is A091072 - 1. Gary W. Adamson, Apr 11 2010
Turns (by 90 degrees) of the Heighway dragon which can be rendered as follows: [Init] Set n=0 and direction=0. [Draw] Draw a unit line (in the current direction). Turn left/right if a(n) is zero/nonzero respectively. [Next] Set n=n+1 and goto (draw). See fxtbook link below. - Joerg Arndt, Apr 15 2010
Sequence can be obtained by L-system with rules L->L1R, R->L0R, 1->1, 0->0, starting with L, and dropping all L and R (see example). - Joerg Arndt, Aug 28 2011
From Gary W. Adamson, Jun 20 2012: (Start)
One half of the infinite Farey Tree can be mapped one-to-one onto A014577 since both sequences can be derived directly from the binary. First few terms are
1,...1,...0,...1,...1,...0,...0,...1,...1,...1,...
1/2.2/3..1/3..3/4..3/5..2/5..1/4..4/5..5/7..5/8,..
Infinite Farey Tree fractions can be derived from the binary by appending a repeat of rightmost binary term to the right, then recording the number of runs to obtain the continued fraction representation. Example: 9 = 1001

Combinatorics on words

$$w_0 = \varepsilon$$
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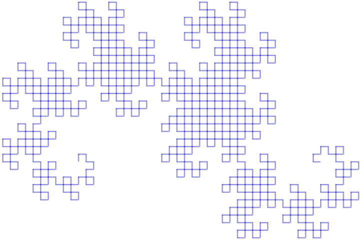
$$d \longrightarrow cd$$
$$a \to ab \to abcb \to abcbadcb \to \dots$$

$a := 1,$ … $= 0$

$w_\infty = 001$…



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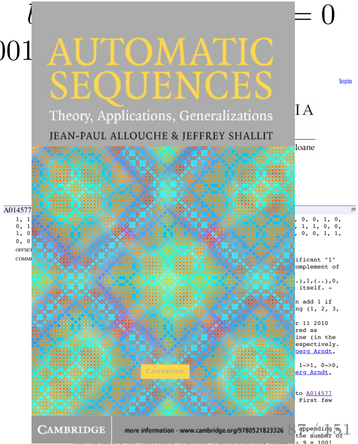
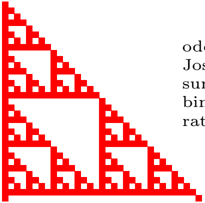


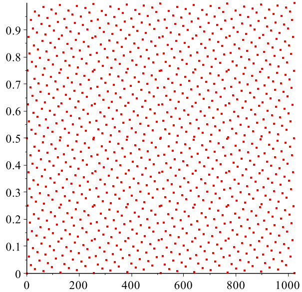
Number theory



odd binomial coefficients
Josephus problem
sums of three squares
binary partitions
rational approximation (Stern-Brocot)

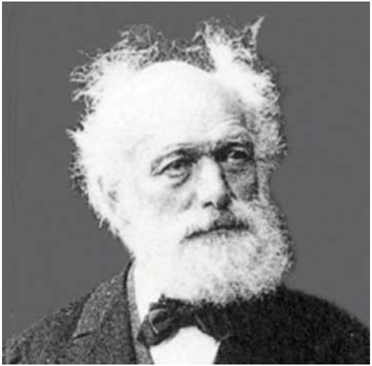
elementary number theory

algebraic series modulo p
discrepancy
transcendency

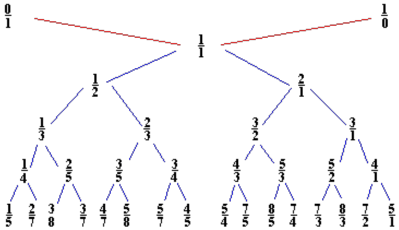


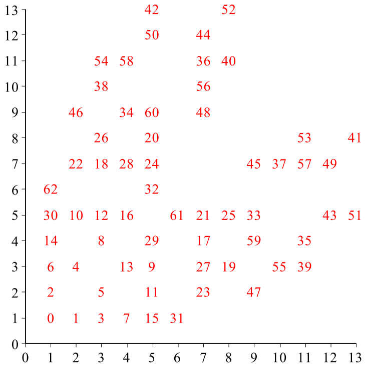
sophisticated number theory

Moritz Stern, Achille Brocot



Moritz Stern, Achille Brocot





$$u_0 = 0,\ u_1 = 1, \qquad u_{2n} = u_n, \quad u_{2n+1} = u_n + u_{n+1}$$

$$n \in \mathbb{N}_{>0} \longmapsto r_n = \frac{u_{n+2}}{u_{n+1}} \in \mathbb{Q}_{>0} \quad \text{one-t-one}$$

Part II

Analysis

Overview of Part II

Slaves bound

Slaves bound

source

Jon Louis Bentley, Dorothea Haken, and James B. Saxe.
A general method for solving divide-and-conquer recurrences.
SIGACT News, 12(3):36–44, September 1980.

a good version:

Alin Bostan, Frédéric Chyzak, Marc Giusti, Romain Lebreton, Grégoire Lecerf, Bruno Salvy, and Éric Schost.
Algorithmes Efficaces en Calcul Formel.
Version provisoire disponible à l'url
`http://specfun.inria.fr/chyzak/mpri/poly.pdf`, 2016.

Slaves bound

Theorem

Let (c_n) *be s.t.* $0 \leq c_n \leq \begin{cases} ac_{\lceil \frac{n}{b} \rceil} + t_n, & \text{if } n \geq n_0 \geq b, \\ \kappa & \text{otherwise,} \end{cases}$ *with*

- $b \geq 2$ *is an integer;*
- $a > 0$ *is a real number;*
- $\kappa \geq$ *is a real number;*
- t *a toll function*
  - *non decreasing,*
  - *such that* $a't_n \leq t_{bn} \leq a''t_n$ *for some constants* $a'' \geq a' > 1$,

then

$$c_n \underset{n\to\infty}{=} \begin{cases} O(t_n) & a' > a, \\ O(t_n \log n) & \text{if } a' = a, \\ O(n^{\alpha-\alpha'} t_n) & \text{if } a' < a \end{cases}$$

with $\alpha = \log_b a$, $\alpha' = \log_b a'$

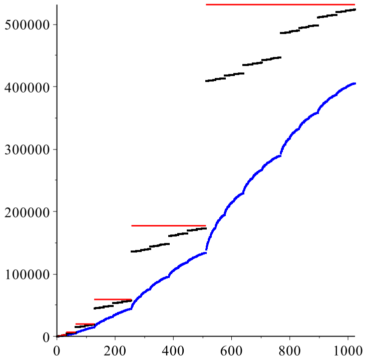
Slaves bound

Karatsuba

$u_n = 2u_{\lceil \frac{n}{2} \rceil} + u_{\lfloor \frac{n}{2} \rfloor} + 4(n-1)$

$v_n \leq 3v_{\lceil \frac{n}{2} \rceil} + 4n$

$w_n = 9 \cdot 3^{\lceil \log_2 n \rceil}$



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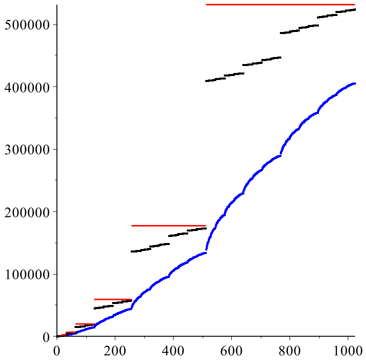
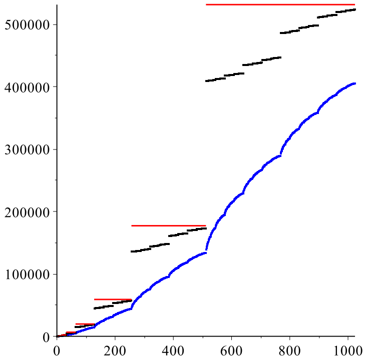


$b = 2,\ a = 3,\ a' = 2,\ \alpha = \log_2 3,\ \alpha' = \log_2 2 = 1,\ v_n = O(n^{\log_2 3})$

Slaves bound



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We want to catch the oscillations!

Goal

$$u(x) = \sum_{n\geq 0} u_n x^n = \prod_{k\geq 0} \frac{1}{1-\rho x^{2^k}}$$

$\rho > 1$

$$u_n = u(1/\rho^2)\rho^n + O(\rho^{n/2})$$

$\rho = 1$

$$\log u_{2n} = \log u_{2n+1} = \frac{1}{2\log 2}\log^2 \frac{n}{\log n} + \left(\frac{1}{2} + \frac{1}{\log 2} + \frac{\log\log 2}{\log 2}\right)\log n + O(\log\log n)$$

$\rho < 1$

$$\sum_{n=1}^{N} u_n = \varphi(\log_2 n) N^{\alpha} + O(N^{\alpha - 1/2 + \varepsilon})$$

$$\alpha = \log_2 \frac{1}{1-\rho}$$

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$\rho < 1$ $$\sum_{n=1}^{N} u_n = \varphi(\log_2 n) N^\alpha + O(N^{\alpha - 1/2 + \varepsilon})$$

$$\alpha = \log_2 \frac{1}{1-\rho}$$

$$u_n = \rho u_{n-1} + u_{\frac{n}{2}}$$

Goal

We want to study the asymptotic behavior of true divide and conquer sequences, that is b-rational sequences.

Some tools

Integers and words

$b \geq 2$, $\mathcal{Z} = \{0, 1, \ldots, b-1\}$

generating series	formal series
$u(x) = \sum_{n \geq 0} u_n x^n$	$s = \sum_{w \in \mathcal{Z}^*} s_w \, w$
$T_{b,r} u(x) = \sum_{k \geq 0} u_{bk+r} x^k$	$sr^{-1} = \sum_{w = w'r} s_w \, w'$

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$$n = (w)_b \in \mathbb{N} \longrightarrow w \in \mathcal{Z}^* \longrightarrow s_w = u_n$$

maps composition

Integers and words

We do not use the words which begins with some zeroes.

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Definition

A linear representation (L, A, C) is insensitive to the leftmost zeroes, or zero-insensitive, if it satisfies $LA_0 = L$.

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A linear representation (L, A, C) is insensitive to the leftmost zeroes, or zero-insensitive, if it satisfies $LA_0 = L$.

Concretely, we always use zero-insensitive linear representations.

Extraction of classical rational sequences

sequence integers whose b-ary expansions have a regular expression e.g. $2^k = (10^k)_2$, $2^k - 1 = (1^k)_2$

Stern-Brocot sequence

$$L = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_{2^k-1} = LA_1^k C = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1-k & -k \\ k & 1+k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = k+1$$

1																		
1	2																	
1	3	2	3															
1	4	3	5	2	5	3	4											
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5			
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	...

Extraction of classical rational sequences

Stern-Brocot sequence

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$$\sum_{n=2^k}^{2^{k+1}-1} u_n = LA_1(A_0+A_1)^k C = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (3^k-1)/2 & 3^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3^k$$

1																		
1	2																	
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1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5			
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A mere idea

$u(x)$ b-rational series

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$\delta(x) = (1 - x)u(x)$ b-rational

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linear representation for $\delta(x)$, insensitive to the leftmost zeroes

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A mere idea

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linear representation for $\delta(x)$, insensitive to the leftmost zeroes

$$u_N = \sum_{n=0}^{N} \delta_n = \sum_{n \leq N} L A_w C, \qquad (w)_b = n$$

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \leq x}} A_w C \qquad 0 \leq x \leq 1$$

A mere idea

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$$\delta_0 = LA_0C$$
$$\delta_1 = LA_1C$$
$$\delta_2 = LA_1A_0C$$
$$\delta_3 = LA_1A_1C$$
$$\delta_4 = LA_1A_0A_0C$$
$$\delta_5 = LA_1A_0A_1C$$

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$$\begin{aligned}
\delta_0 &= L A_0 C \\
\delta_1 &= L A_1 C \\
\delta_2 &= L A_1 A_0 C \\
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$$\begin{aligned} u_5 = {} & L A_0 A_0 A_0 C \\ & + L A_0 A_0 A_1 C \\ & + L A_0 A_1 A_0 C \\ & + L A_0 A_1 A_1 C \\ & + L A_1 A_0 A_0 C \\ & + L A_1 A_0 A_1 C \end{aligned}$$

A mere idea

$$u_N = \sum_{n=0}^{N} \delta_n = \sum_{n \le N} L A_w C, \qquad (w)_b = n$$

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \le x}} A_w C \qquad 0 \le x \le 1$$

$$\begin{aligned} u_5 &= L A_0 A_0 A_0 C \\ &+ L A_0 A_0 A_1 C \\ &+ L A_0 A_1 A_0 C \\ &+ L A_0 A_1 A_1 C \\ &+ L A_1 A_0 A_0 C \\ &+ L A_1 A_0 A_1 C \end{aligned}$$

$$\begin{aligned} L S_3(5/8) &= L A_0 A_0 A_0 C \\ &+ L A_0 A_0 A_1 C \\ &+ L A_0 A_1 A_0 C \\ &+ L A_0 A_1 A_1 C \\ &+ L A_1 A_0 A_0 C \\ &+ L A_1 A_0 A_1 C \end{aligned}$$

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$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \le x}} A_w C \qquad 0 \le x \le 1$$

Proposition

Let (L, A, C) be a insensitive to the leftmost zeroes linear representation for the sequence (δ_n) of backward differences of a b-rational sequence (u_n). Then

$$u_N = L S_{K+1}(b^{\{\log_b N\}-1}),$$

with $K = \lfloor \log_b N \rfloor$ and $\{t\} = t - \lfloor t \rfloor$.

A mere idea

$$u_N = \sum_{n=0}^{N} \delta_n = \sum_{n \leq N} L A_w C, \qquad (w)_b = n$$

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \leq x}} A_w C \qquad 0 \leq x \leq 1$$

Proposition

The sequence $S_K(x)$ *satisfies*

$$S_{K+1}(x) = \sum_{r_1 < x_1} A_{r_1} Q^K C + A_{x_1} S_K(bx - x_1),$$

for $x = (0.x_1 x_2 \ldots)_b$ *in* $[0, 1[$.

Joint spectral radius

$$u_N = LA_{r_\ell} \cdots A_{r_0} C \qquad \text{for } N = (r_\ell \dots r_0)_b$$

$$|u_N| \leq \|L\| \|A_{r_\ell}\| \cdots \|A_{r_0}\| \|C\|$$

Joint spectral radius

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$$|u_N| \le \|L\| \|A_{r_\ell}\| \cdots \|A_{r_0}\| \|C\|$$
$$\le \|L\| \|C\| a^{\ell+1} = \|L\| \|C\| a^{\lfloor \log_b N \rfloor} \le K N^{\log_b a}$$

Joint spectral radius

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Proposition

A b-rational sequence has a growth order at most polynomial.

Joint spectral radius

Proposition

Let $A = (A_z)_{z \in \mathcal{Z}}$ *be a finite family of square matrices. The sequence*

$$\hat{\rho}_\ell(A) = \max_{w \in \mathcal{Z}^\ell} \|A_w\|^{1/\ell},$$

converges towards

$$\hat{\rho}(A) = \lim_{\ell \to +\infty} \hat{\rho}_\ell(A) = \inf_\ell \hat{\rho}_\ell(A).$$

Moreover the limit is independent of the used multiplicative norm. It is the joint spectral radius of A*.*

Joint spectral radius

Proposition

If (L, A, C) *is a linear representation for a b-rational sequence* (u_n)*, then for all* $\varepsilon > 0$

$$u_N \underset{N \to +\infty}{=} O\left(N^{\log_b \hat{\rho}(A) + \varepsilon}\right)$$

Joint spectral radius

Karatsuba

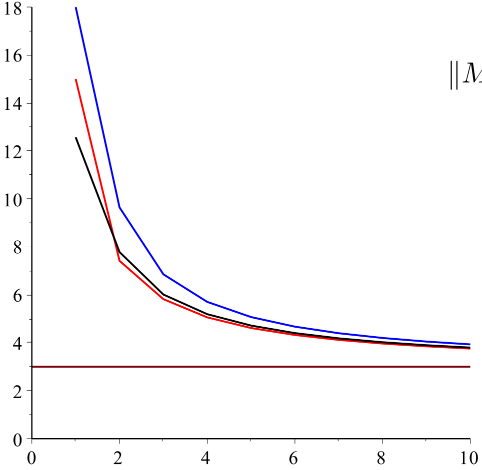
$$A_0 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 4 & 10 & 1 & 2 & 1 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ -1 & 4 & 2 & 1 & 0 \\ 10 & 4 & 0 & 1 & 2 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\|M\|_1 = \max_j \sum_i |M_{i,j}|, \qquad \|M\|_\infty = \max_i \sum_j |M_{i,j}|,$$

$$\|M\|_F = \left(\sum_{i,j} |M_{i,j}|^2 \right)^{1/2}$$

Joint spectral radius

$$\|M\|_1 = \max_j \sum_i |M_{i,j}|, \quad \|M\|_\infty = \max_i \sum_j |M_{i,j}|,$$



$$\|M\|_F = \left(\sum_{i,j} |M_{i,j}|^2 \right)^{1/2}$$

$$\hat{\rho}_{10}(A) \simeq 3.76 \geq \hat{\rho}(A)$$

Joint spectral radius

Proposition

If the matrices of $A = (A_z)_{z \in \mathcal{Z}}$ *can be simultaneously block-triangulated,*

$$P^{-1} A_z P = \begin{pmatrix} B_z & C_z \\ 0 & D_z \end{pmatrix}, \qquad z \in \mathcal{Z},$$

then the joint spectral radius of A *is*

$$\hat{\rho}(A) = \max(\hat{\rho}(B), \hat{\rho}(D)).$$

Joint spectral radius

$$A_0 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 4 & 10 & 1 & 2 & 1 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} B_0 & 0 \\ C_0 & D_0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ -1 & 4 & 2 & 1 & 0 \\ 10 & 4 & 0 & 1 & 2 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ C_1 & D_1 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},$$

$$P = \begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}, \quad P^{-1} B_0 P = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad P^{-1} B_1 P = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

$\hat{\rho}(B) = 3, \; \hat{\rho}(D) = 2 \qquad \hat{\rho}(A) = \max(3, 2) = 3$

Joint spectral radius

Consequence:

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \le x}} A_w C \qquad 0 \le x \le 1$$

$$S_{K+1}(x) = \sum_{r_1 < x_1} A_{r_1} Q^K C + A_{x_1} S_K(bx - x_1),$$

Proposition

Let V be an eigenvector of $Q = A_0 + \cdots + A_{b-1}$ for an eigenvalue $\rho\omega$ with $|\omega| = 1$ and $\rho \le \hat{\rho}(A)$. Then

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \le x}} A_w V$$

is $O(r^K)$ uniformly wrt x for $r > \hat{\rho}(A) \ge \rho$.

Dilation equations

coin tossing

$(T_n)_{n\geq 1}$ i.i.d. with $\mathbf{P}(T=0)=p_0$, $\mathbf{P}(T=1)=p_1$

$p_0+p_1=1$, $0<p_0, p_1<1$

$$X=\sum_{n\geq 1}\frac{T_n}{2^n}$$

distribution function $F(x)$

Dilation equations

coin tossing

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$p_0+p_1=1$, $0<p_0$, $p_1<1$

$$X=\sum_{n\geq 1}\frac{T_n}{2^n}$$

distribution function $F(x)$

$0\leq x<1/2$

$$F(x)=\mathbf{P}(X\leq x)=\mathbf{P}(T_1=0,\sum_{n\geq 2}\frac{T_n}{2^{n-1}}\leq 2x)=p_0F(2x)$$

$1/2\leq x\leq 1$

$$F(x)=\mathbf{P}(X\leq x)=\mathbf{P}(T_1=0)+\mathbf{P}(T_1=1,\sum_{n\geq 2}\frac{T_n}{2^{n-1}}\leq 2x-1)$$
$$=p_0+p_1F(2x-1),$$

Dilation equations

$$
\begin{array}{ll}
0 \leq x < 1/2 & F(x) = p_0 F(2x) \\
1/2 \leq x \leq 1 & F(x) = p_0 + p_1 F(2x-1)
\end{array}
$$

Dilation equations

$$\begin{array}{ll} 0 \le x < 1/2 & F(x) = p_0 F(2x) \\ 1/2 \le x \le 1 & F(x) = p_0 + p_1 F(2x-1) \end{array}$$

$$F(x) = p_0 F(2x) + p_1 F(2x-1)$$

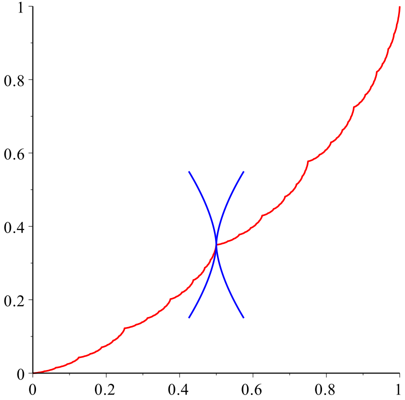
$$F(x) = 0 \quad \text{for } x \le 0 \qquad F(x) = 1 \quad \text{for } x \ge 1$$

dilation equation
two-scale difference equation

Dilation equations

$$F(x) = p_0 F(2x) + p_1 F(2x - 1)$$

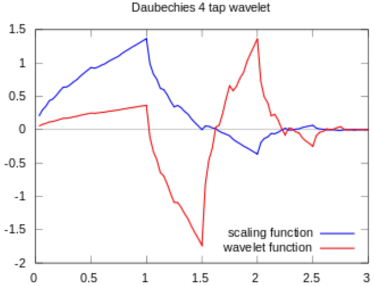
$$F(x) = 0 \quad \text{for } x \le 0 \qquad F(x) = 1 \quad \text{for } x \ge 1$$



cascade algorithm   Hölder with exponent $\log_2 1/\max(p_0, p_1)$

Dilation equations

in wavelet theory (Daubechies)



in interpolation scheme (Dubuc, Deslauriers)

Dilation equations

multidimensional version

Proposition

Under the hypothesis $\rho > \hat{\rho}(A)$, *the dilation equation*

$$\rho\omega F(x) = \sum_{0 \le r < b} A_r F(bx - r)$$

with boundary conditions

$$F(x) = 0 \quad \textit{for } x \le 0, \qquad F(x) = V \quad \textit{for } x \ge 1,$$

where V *is an eigenvector for* $Q = A_0 + \cdots + A_{b-1}$ *and the eigenvalue* $\rho\omega$, $|\omega| = 1$, *has a unique continuous solution from* $\mathbb{R}$ *into* $\mathbb{C}^d$. *Moreover this solution is Hölder with exponent* $\log_b(\rho/r)$ *for* $r > \hat{\rho}(A)$.

Dilation equations

Consequence:

Proposition

Let V be an eigenvector for an eigenvalue $\rho\omega$, $|\omega| = 1$, $\rho > \hat{\rho}(A)$, of $Q = A_0 + \cdots + A_{b-1}$. Then

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_b \le x}} A_w V$$

satisfies

$$S_K(x) \underset{K\to\infty}{=} (\rho\omega)^K F(x) + O(r^K)$$

for $\rho > r > \hat{\rho}(A)$ uniformly wrt x.

Theorem

Theorem

Let (u_n) be a b-rational sequence and (L, A, C) a linear representation for the sequence of its backward differences. Then the sequence (u_n) has an asymptotic expansion which is a sum of terms

$$N^{\log_b \rho} \binom{\log_b N}{m} \times e^{i\vartheta \log_b N} \times \varphi(\log_b N).$$

In this writing, $\rho e^{i\vartheta}$ is an eigenvalue of $Q = A_0 + A_1 + \cdots + A_{b-1}$ with a modulus $\rho > \hat{\rho}(A)$. The integer m is bounded by the maxima size of the Jordan blocks related to $\rho e^{i\vartheta}$. The function $\varphi(t)$ is 1-periodic and Hölder with exponent $\log_b(\rho/r)$ for $\rho > r > \hat{\rho}(A)$. The error term is $O(N^{\log_b r})$ for $r > \hat{\rho}(A)$.

A worked example

Karatsuba!

$$xu(x) - (1+x)(2+x)u(x^2) = -x^2 + 4\frac{x^3}{(1-x)^2}$$

$$\delta(x) = \frac{(2+x)}{x}\delta(x^2) - \frac{x - 6x^2 + x^3}{1-x}.$$

basis

$$\delta(x),\ \frac{\delta(x)}{x},\ \frac{1}{1-x},\ \frac{x}{1-x},\ \frac{x^2}{1-x},\ \frac{x^3}{1-x}$$

A worked example: Linear representation

$$L = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},\ A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 5 & 1 & 1 & 0 & 0 \\ 5 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A worked example: Joint spectral radius

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 5 & 1 & 1 & 0 & 0 \\ 5 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\rho}(A) = 2$$

A worked example: Jordan reduction

$$Q = A_0 + A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ -1 & 4 & 2 & 1 & 0 & 0 \\ 10 & 4 & 0 & 1 & 2 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 24 & 0 & 0 \\ 8 & 0 & 0 & -24 & 0 & 0 \\ 48 & -96 & -96 & 120 & 24 & 179 \\ 16 & 0 & 96 & -96 & -48 & -334 \\ 0 & 0 & 0 & -24 & 24 & 131 \\ 0 & 0 & 0 & 0 & 0 & 24 \end{bmatrix}, \; Q' = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A worked example: Jordan reduction

$$L' = LP = \begin{bmatrix} 56 & -96 & -96 & 96 & 24 & 179 \end{bmatrix},$$

$$A'_0 = P^{-1}A_0P = \begin{bmatrix} 2 & 0 & 0 & -3 & 0 & 0 \\ 1/3 & 1 & 0 & -2 & 0 & 0 \\ 1/4 & 0 & 1 & 1/4 & -1/4 & -\frac{155}{96} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A'_1 = P^{-1}A_1P = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 \\ -1/3 & 1 & 0 & 2 & 0 & 0 \\ -1/4 & 0 & 0 & 3/4 & 1/4 & \frac{155}{96} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; C' = P^{-1}C = \begin{bmatrix} 1/8 \\ 1/12 \\ 1/24 \\ 1/24 \\ 1/24 \\ 0 \end{bmatrix}$$

$$\rho = 3, \quad \omega = 1, \quad V = E_2 + 6E_3 + 2E_4$$

A worked example: Dilation equation

$$3F(x) = A_0F(2x) + A_1F(2x-1).$$

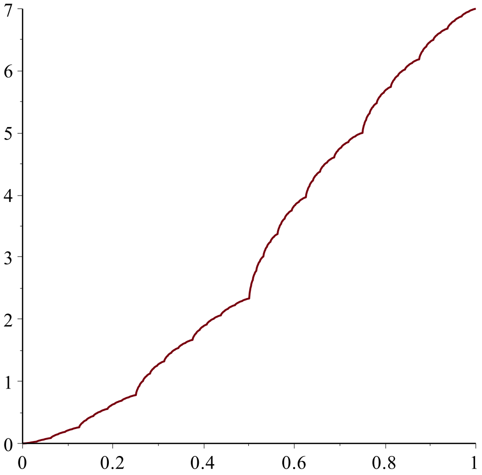
$$F(x) = 0 \quad \text{for } x \leq 0, \qquad F(x) = E_2 + 6E_3 + 2E_4 \quad \text{for } x \geq 1.$$

A worked example: Dilation equation

$$
\begin{aligned}
f_1(x) &= \frac{2}{3} f_1(2x) - f_4(2x), \\
f_2(x) &= \frac{1}{9} f_1(2x) + \frac{1}{3} f_2(2x) - \frac{2}{3} f_4(2x) + \frac{2}{3} f_1(2x-1) + \frac{1}{3} f_2(2x-1), \\
f_3(x) &= \frac{1}{12} f_1(2x) + \frac{1}{3} f_3(2x) + \frac{1}{12} f_4(2x) - \frac{1}{12} f_5(2x) - \frac{155}{288} f_6(2x) \\
&\qquad - \frac{1}{3} f_1(2x-1) + \frac{5}{3} f_2(2x-1) + \frac{1}{3} f_3(2x-1) + \frac{1}{3} f_4(2x-1), \\
f_4(x) &= \frac{1}{3} f_4(2x) + \frac{5}{3} f_1(2x-1) - \frac{1}{3} f_2(2x-1) + \frac{1}{3} f_5(2x-1) + \frac{1}{3} f_6(2x-1), \\
f_5(x) &= \frac{1}{3} f_6(2x), \\
f_6(x) &= 0.
\end{aligned}
$$

$$
f_j(x) = 0 \quad \text{for } x \le 0 \qquad\qquad
\begin{aligned}
f_1(x) &= 0 \\
f_2(x) &= 1 \\
f_3(x) &= 6 \\
f_4(x) &= 2 \\
f_5(x) &= 0 \\
f_6(x) &= 0
\end{aligned}
\quad \text{for } x \ge 1
$$

A worked example: Cascade algorithm

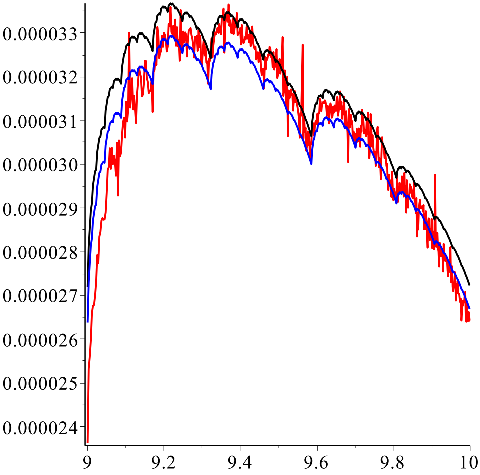


$$f(x) = LF(x) = f_2(x) + f_3(x)$$

A worked example: !

$$u_N \underset{N\to\infty}{=} N^{\log_2 3}\varphi(\log_2 N) + O(N^{1+\varepsilon})$$

$$\varphi(t) = 3^{1-\{t\}} f(2^{\{t\}-1}).$$



$\varphi(t)$

$u_N/N^{\log_2 3}$

normalized execution of the algorithm

What I did not speak about

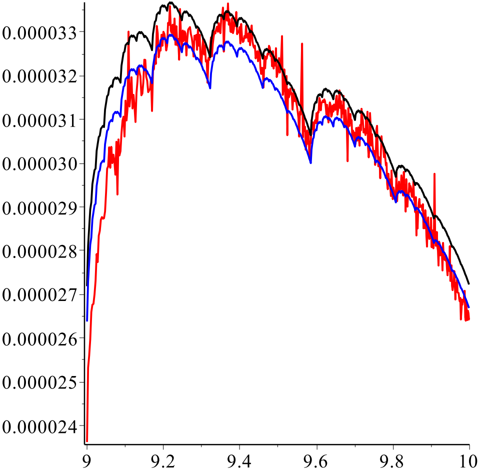
- analytic number theory

Michael Drmota and Peter J. Grabner.
Analysis of digital functions and applications.
In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 452–504. Cambridge Univ. Press, Cambridge, 2010.

- probability theory

Louis H.Y. Chen, Hsien-Kuei Hwang, and Vytas Zacharovas.
Distribution of the sum-of-digits function of random integers: a survey.
Probababilty Surveys, 11:177–236, 2014.

Thanks for your attention!



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