Refinement and generalisation of Siladić's theorem

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Outline

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Integer partitions

Definition

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Generating functions

Let n, k be positive integers. Let $Q(n, k)$ denote the number of partitions of n into k distinct parts. Then

$$1 + \sum_{n\geq 1} \sum_{k\geq 1} Q(n,k) z^k q^n = (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots$$
$$= \prod_{n\geq 1} (1+zq^n).$$

Let $p(n, k)$ denote the number of partitions of n into k parts. Then

$$1 + \sum_{n\geq 1} \sum_{k\geq 1} p(n,k) z^k q^n = \prod_{n\geq 1} \left(1 + zq^n + z^2 q^{2n} + \cdots\right)$$
$$= \prod_{n\geq 1} \frac{1}{(1-zq^n)}.$$

Partition identities

Theorem (Euler 1748)

For every integer n, the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

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Proof.

$$\prod_{n\geq 1}(1+q^n) = \prod_{n\geq 1}\frac{(1+q^n)(1-q^n)}{1-q^n} = \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^n} = \prod_{n\geq 1}\frac{1}{1-q^{2n-1}}.$$

□

The Rogers-Ramanujan identities

Theorem (Rogers 1894, Rogers-Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

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Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

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Rogers-Ramanujan type identity: "for all n, the number of partitions of n satisfying some difference conditions is equal to the number of partitions of n satisfying some congruence conditions."

Outline

1. Introduction
2. Schur's theorem and the method of weighted words
3. Siladić's theorem

Schur's theorem

Theorem (Schur 1926)

For any positive integer n, let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3 and $B(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_m$ of n such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_i \equiv 1,2 \mod 3, \\ 4 & \text{if } \lambda_i \equiv 0 \mod 3. \end{cases}$$

Then $A(n) = B(n)$.

Example

The partitions counted by $A(10)$ are 10, $8+2$, $7+2+1$ and $5+4+1$.
The partitions counted by $B(10)$ are 10, $9+1$, $8+2$ and $7+3$.
There are 4 partitions in both cases.

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  Let $A(n,k)$ denote the number of partitions of n into k distinct parts congruent to 1 or 2 modulo 3. Let $B(n,k)$ denote the number of partitions of n, satisfying the difference conditions of Schur's theorem, such that $k = \#\{\text{parts} \equiv 1,2 \mod 3\} + 2\#\{\text{parts} \equiv 0 \mod 3\}$. Then for all $k, n \in \mathbb{N}$, $A(n,k) = B(n,k)$.

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- Bijections : Bressoud (1980), Bessenrodt (1991)
- The method of weighted words : Alladi-Gordon (1993)
  $\rightarrow$ further refinement
  $\rightarrow$ generalisation

The method of weighted words

The principle of method of weighted words is the following :

- Assign a color to each part according to its value modulo 3 :
  color c : $0 \mod 3$,
  color a : $1 \mod 3$,
  color b : $2 \mod 3$.

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- Assign a color to each part according to its value modulo 3 :
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- Order the colors

$$c < a < b,$$

such that the corresponding ordering of the positive integers in three colors a, b, c,

$$1_c < 1_a < 1_b < 2_c < 2_a < 2_b < 3_c < 3_a < 3_b < \cdots ,$$

becomes the natural ordering of integers

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < \cdots ,$$

under the transformations

$$k_c \mapsto 3k-3, \, k_a \mapsto 3k-2, \, k_b \mapsto 3k-1.$$

The method of weighted words

- Find difference conditions on the non-dilated colored integers

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 \text{ if } color(\lambda_i) = c \text{ or } color(\lambda_i) < color(\lambda_{i+1}), \\ 1 \text{ otherwise,} \end{cases}$$

such that after the same transformations

$$k_c \mapsto 3k-3,\ k_a \mapsto 3k-2,\ k_b \mapsto 3k-1,$$

they become the difference conditions of Schur's theorem

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- Find conditions on the colors such that the generating function for partitions with difference conditions equals

$$\prod_{k \geq 1} (1+aq^k)(1+bq^k).$$

Let $S(u, v, w, n)$ denote the number of partitions of n with u parts colored a, v parts colored b and w parts colored c, satisfying the difference conditions

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with no part 1_c.

Its generating function is

$$\sum_{u,v,w,n \geq 0} S(u, v, w, n) a^u b^v c^w q^n = \sum_{r,s,t \geq 0} q^{\binom{r+s+t}{2}} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(c)^t q^t q^{\binom{t+1}{2}}}{(q)_t}.$$

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partition into r parts of color a,
partition into s parts of color b,
partition into t distinct parts ≥ 2 of color c,
$q^{\binom{r+s+t}{2}}$: staircase of size $r+s+t$.

The non-dilated theorem

By q-series calculations (q-binomial identity, q-Chu-Vandermonde identity), one sees that the generating function for $S(u, v, w, n)$ is an infinite product if and only if $c = ab$, and in that case it equals indeed $\prod_{n \geq 1} (1 + aq^n)(1 + bq^n)$.

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Non-dilated version of Schur's theorem (Alladi-Gordon 1993)

Let $S(u, v, n)$ denote the number of partitions of n with u parts colored a or ab and v parts colored b or ab such that there is no part 1_{ab}, satisfying the difference conditions. Then we have

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The dilation $q \to q^3, a \to aq^{-2}, b \to bq^{-1}$ gives Schur's theorem.

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The theorem

Theorem (Siladić 2005)

The number of partitions of an integer n into distinct odd parts equals the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n into parts different from 2 such that the difference between two consecutive parts is at least 5 (ie. $\lambda_i - \lambda_{i+1} \geq 5$) and

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16,$$
$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \mod 16,$$
$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \mod 16,$$
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Originally proved by studying representations of the twisted affine Lie algebra $A_2^{(2)}$.

Refinement of Siladić's theorem (D. 2013)

For $k, n \in \mathbb{N}$, let $C(k,n)$ denote the number of partitions of n into k distinct odd parts. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, let $D(k,n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that k equals the number of odd part plus twice the number of even parts, satisfying the following conditions:

1. $\forall i \geq 1, \lambda_i \neq 2$,
2. $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5$,
3. $\forall i \geq 1$,

$$\begin{aligned} \lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i \equiv 1,4 \mod 8, \\ \lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i \equiv 1,3,5,7 \mod 8, \\ \lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i \equiv 0,1,3,4,6,7 \mod 8, \\ \lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i \equiv 0,1,3,4,5,7 \mod 8. \end{aligned}$$

Then for all $k, n \in \mathbb{N}$, $C(k,n) = D(k,n)$.

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  - Let $d_N(k, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ counted by $D(k, n)$ such that the largest part λ_1 is at most N, and

    $$G_N(t, q) = 1 + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} d_N(k, n) t^k q^n.$$

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  - By induction, we show that for all $m\in\mathbb{N}^*$,
  $$G_{2m}(t,q)=(1+tq)G_{2m-3}(tq^2,q).$$

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  By a combinatorial reasoning, we establish eight q-difference equations satisfied by $G_N(t,q)$.
  - By induction, we show that for all $m \in \mathbb{N}^*$,
  $$G_{2m}(t,q) = (1+tq)G_{2m-3}(tq^2,q).$$
  - Letting $m \to \infty$ and iterating leads to
  $$\lim_{N\to\infty} G_N(t,q) = \prod_{k=0}^{\infty}\left(1+tq^{2k+1}\right).$$

The q-difference equations

For all $N \in \mathbb{N}^*$,

$$G_{8N}(t,q) = G_{8N-1}(t,q) + t^2q^{8N}G_{8N-7}(t,q),$$

$$G_{8N+1}(t,q) = G_{8N}(t,q) + tq^{8N+1}G_{8N-4}(t,q),$$

$$G_{8N+2}(t,q) = G_{8N+1}(t,q) + t^2q^{8N+2}G_{8N-7}(t,q),$$

$$G_{8N+3}(t,q) = G_{8N+2}(t,q) + tq^{8N+3}G_{8N-3}(t,q),$$

$$G_{8N+4}(t,q) = G_{8N+3}(t,q) + t^2q^{8N+4}G_{8N-3}(t,q) + t^3q^{16N+3}G_{8N-7}(t,q),$$

$$G_{8N+5}(t,q) = G_{8N+4}(t,q) + tq^{8N+5}G_{8N-3}(t,q) + t^2q^{16N+4}G_{8N-7}(t,q),$$

$$G_{8N+6}(t,q) = G_{8N+5}(t,q) + t^2q^{8N+6}G_{8N-3}(t,q) + t^3q^{16N+5}G_{8N-7}(t,q),$$

$$G_{8N+7}(t,q) = G_{8N+6}(t,q) + tq^{8N+7}G_{8N+1}(t,q).$$

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- Only five different colors remain at the end:
  color a : integers congruent to $1 \mod 4$,
  color b: integers congruent to $3 \mod 4$,
  color ab : integers congruent to $0 \mod 4$,
  color a^2 : integers congruent to $6 \mod 8$,
  color b^2 : integers congruent to $2 \mod 8$.

We consider the following order on colored integers:

$$1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < \cdots$$

and difference conditions given by the matrix A (the entry (x,y) gives the minimal difference between λ_i of color x and λ_{i+1} of color y):

$$A = \begin{array}{c} \\ a \\ b \\ ab \\ a^2 \\ b^2 \end{array} \begin{array}{cccccccc} a_{odd} & b^2 & b_{odd} & ab_{even} & a_{even} & a^2 & b_{even} & ab_{odd} \end{array} \hspace{-0em}$$

$$A = \bordermatrix{ & a_{odd} & b^2 & b_{odd} & ab_{even} & a_{even} & a^2 & b_{even} & ab_{odd} \cr a & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \cr b & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \cr ab & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 \cr a^2 & 4 & 4 & 4 & 3 & 3 & 4 & 3 & 4 \cr b^2 & 2 & 4 & 4 & 3 & 3 & 2 & 3 & 2 }.$$

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Then the transformations

$$k_{ab} \mapsto 4k-4,\ k_a \mapsto 4k-3,\ k_b \mapsto 4k-1,\ k_{b^2} \mapsto 4k-2,\ k_{a^2} \mapsto 4k-6$$

give the conditions of Siladić's theorem.

The non-dilated theorem

Theorem (D. 2016)

Let $D(u, v, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n, with no part 1_{ab} or 1_{b^2}, satisfying the difference conditions given by the matrix A, such that u equals the number of parts a or ab plus twice the number of parts a^2 and v equals the number of parts b or ab plus twice the number of parts b^2.
Then for all $u, v, n \in \mathbb{N}$,

$$\sum D(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n)(1 + bq^n).$$

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Then for all $u, v, n \in \mathbb{N}$,

$$\sum D(u,v,n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n)(1 + bq^n).$$

The dilation $q \to q^4, a \to aq^{-3}, b \to bq^{-1}$ gives a refinement of Siladić's theorem.

If we keep the same order and difference conditions but do the dilation $q \to q^4, a \to aq^{-1}, b \to bq^{-3}$, we obtain a companion of Siladic's theorem.

Companion of Siladić's theorem (D. 2016)

The number of partitions of n into k distinct odd parts equals the number of partitions of n, where 2 is not a part, such that k equals the number of odd part plus twice the number of even parts, s.t.

$$\lambda_i - \lambda_{i+1} \begin{cases} = 5, 6, 8, 9 \text{ or } \geq 11 \text{ if } \lambda_i \equiv 0 \mod 8, \\ = 2 \text{ or } \geq 5 \text{ if } \lambda_i \equiv 1 \mod 8, \\ = 11 \text{ or } \geq 13 \text{ if } \lambda_i \equiv 2 \mod 8, \\ \geq 7 \text{ if } \lambda \equiv 3 \mod 8, \\ = 5 \text{ or } \geq 7 \text{ if } \lambda_i \equiv 4 \mod 8, \\ = 2, 3, 5, 6 \text{ or } \geq 8 \text{ if } \lambda_i \equiv 5 \mod 8, \\ = 3, 4, 6, 7 \text{ or } \geq 9 \text{ if } \lambda_i \equiv 6 \mod 8, \\ = 8 \text{ or } \geq 10 \text{ if } \lambda_i \equiv 7 \mod 8. \end{cases}$$

Back to Schur's theorem

The infinite product in the non-dilated version of Siladić's theorem is the same as the one in the non-dilated version of Schur's theorem.
With the dilations $q \to q^3, a \to aq^{-2}, b \to bq^{-1}$, the ordering of integers

$$1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < \cdots$$

becomes

$$0 < 1 < 1 < 2 < 3 < 4 < 5 < 5 < 6 < 7 < 7 < \cdots$$

So the integers congruent to $\pm 1 \mod 6$ can appear in two colours.
We obtain a new companion of Schur's theorem.

Back to Schur's theorem

Companion of Schur's theorem (D. 2016)

Let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1 modulo 3 and v distinct parts congruent to 2 modulo 3. Let $C(n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n such that only parts congruent to $\pm 1 \mod 6$ can be overlined, $\overline{1}$ is not a part, and such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 4 + \chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_i \equiv 1,2,3,5 \mod 6, \\ 5 + \chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_i \equiv 0,4 \mod 6, \\ 6 + \chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_i \equiv 1,5 \mod 6 \text{ and is overlined,} \end{cases}$$

where

$$\chi(\overline{\lambda_{i+1}}) = \begin{cases} = 1 \text{ if } \lambda_{i+1} \text{ is overlined,} \\ = 0 \text{ otherwise.} \end{cases}$$

Then $A(n) = C(n)$.

Thank you!