Increasing Diamonds

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Outline

- Introduction
- Asymptotic study of the diamonds
- Random Generation
- Conclusion

Introduction

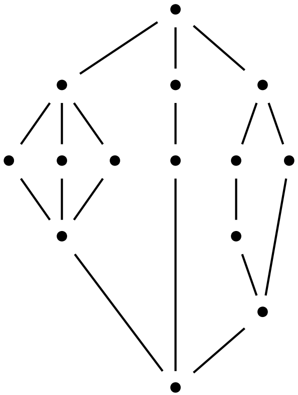
Motivations

- Combinatorial study of concurrents programs (seen as discrete structures)
- Quantitative study of the *combinatorial explosion* phenomena: the large number of possible runs (seen as increasing labellings)

Approach: Analytic Combinatorics

- symbolic method to modelize (Greene's “box” operators)
- singularity analysis to obtain asymptotics of the number of increasing labellings
- based on previous work on increasing trees of [F. Bergeron, P. Flajolet and B. Salvy '92]

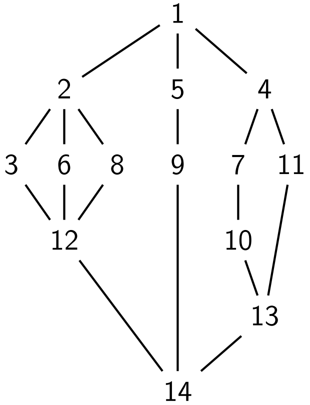
Combinatorial specifications



Skeleton

$$\mathcal{S} = \mathcal{Z} + \mathcal{Z} \cdot G(\mathcal{S}) \cdot \mathcal{Z}$$

Combinatorial specifications



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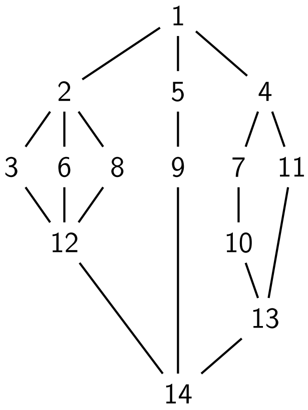
Increasing labellings

$$\mathcal{I} = \mathcal{Z} + \mathcal{Z}^{\square} \star G(\mathcal{I}) \star \mathcal{Z}^{\blacksquare}$$

Combinatorial specifications



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Increasing labellings

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Differential equation

$$\begin{cases} I'' = G(I) \\ I(0) = 0 \\ I'(0) = 1 \end{cases}$$

Easy case: non-plane diamonds

We start with the differential equation: $A''(z) = e^{A(z)}$

We can solve it: $A'(z) = \tan z + \sec z$

The poles are the $(2k + \frac{1}{2})\pi$

Using the residue theorem we get:

$$a_n = \frac{2^{n+1}\ (n-1)!}{\pi^n} \sum_{j=-\infty}^{+\infty} \frac{1}{(1+4j)^n}.$$

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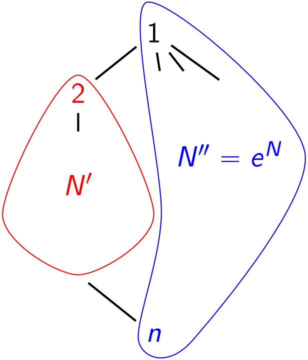
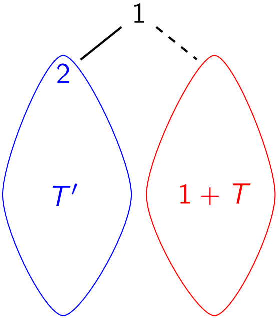
$$(a_n)_{n\geqslant 1} = \{1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, \dots\}$$

Known in OEIS to count the number of number of increasing unary-binary trees on n vertices.

Bijection

Non-plane diamonds	Increasing unary-binary trees
$\mathcal{A} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathsf{Set}(\mathcal{A}) \star \mathcal{Z}^{\blacksquare}$	$\mathcal{T} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{T} + \mathsf{Set}_{=2}(\mathcal{T}))$
$N'''(z) = N'(z) \cdot N''(z)$	$T''(z) = (1 + T(z)) \cdot T'(z)$





Thanks to A. Bacher, G. Collet and C. Mailler (and ALEA Network)

Elliptic cases

Weierstrass's case

$F'' = P(F)$ where P is a polynomial of degree 2, then:

$$F(z) = K\wp(z - \rho; \omega_1, \omega_2)$$

with $\rho = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{1 + 2\int_0^t P(v)\mathrm{d}v}}$ and K a constant.

Weierstrass's elliptic function

$\wp$ is defined periodically over a lattice that contains one double pole in a corner of each cell:

$$\wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(k,l)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z + k\omega_1 + l\omega_2)^2} - \frac{1}{(k\omega_1 + l\omega_2)^2} \right)$$

Elliptic cases

Jacobi's case

$F'' = P(F)$ where P is a polynomial of degree 3, then

let $g^2 = \dfrac{\beta - \delta}{\alpha - \delta} \cdot \dfrac{F - \sqrt{2}\,\alpha}{F - \sqrt{2}\,\beta}$ with α, β and δ well chosen then

$g'(z) = M\sqrt{(1 - z^2)(1 - \ell^2 z^2)}$ and so

$$g(z) = \mathrm{sn}(Mz; \ell)$$

Jacobi's elliptic sinus function

sn is defined periodically over a lattice that contains two simple poles in each cell and a zero in a corner.

Elliptic cases: binary and ternary diamonds

Weierstrass case: binary diamonds

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{B} \star \mathcal{B}) \star \mathcal{Z}^{\blacksquare} \qquad B'' = 1 + B^2$$

$$b_n = 6\frac{(n+1)!}{\rho^{n+2}} \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{\left(1 + \frac{k\omega_1}{\rho} + \frac{l\omega_2}{\rho}\right)^{n+2}} \underset{n\to\infty}{\sim} 6\frac{(n+1)!}{\rho^{n+2}}$$

Jacobi's case: ternary diamonds

$$\mathcal{T} = \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{T} \star \mathcal{T} \star \mathcal{T}) \star \mathcal{Z}^{\blacksquare} \qquad T'' = 1 + T^3$$

$$t_n = \frac{\sqrt{2}\; n!}{\rho^{n+1}} \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{\left(1 + C_{k,l}\right)^{n+1}} - \frac{1}{\left(2 + C_{k,l}\right)^{n+1}} \underset{n\to\infty}{\sim} 6\sqrt{2}\frac{(n+1)!}{\rho^{n+1}}$$

with $C_{k,l} = \frac{3k}{2} + i\frac{\sqrt{3}}{2}(k+2l)$

More general cases

Asymptotics results

- Diamonds of fixed arity ($G \in \mathbb{Z}[X]$ and $deg(G) = m$):

$$f_n = n! \left(\frac{\sqrt{2(m+1)}}{(m-1)\sqrt{b_m}} \right)^{\frac{2}{m-1}} \frac{n^{-\frac{m-3}{m-1}}}{\Gamma(\frac{2}{m-1})} \rho^{-n-\frac{2}{m-1}} \left(1 + \mathcal{O}\left(n^{-\frac{4}{m-1}}\right)\right)$$

- Plane general diamonds ($G = \text{Seq}$):

$$f_n = \frac{n! \rho^{1-n}}{n^2 \sqrt{2 \log n}} \left(\sum_{0 \leqslant k < K} \frac{P_k(\log \log n)}{(\log n)^k} + \mathcal{O}\left(\frac{(\log \log n)^K}{(\log n)^K} \right) \right)$$

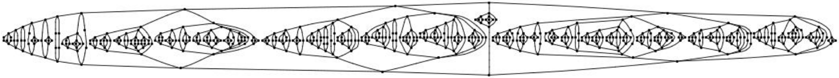
Sequence A032035 in OEIS which also enumerates increasing rooted (2,3)-cacti with $n-1$ nodes

Random Generation of the skeletons

Boltzmann method

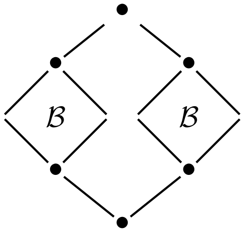
- Straightforward use of standard techniques
  [P. Duchon, P. Flajolet, G. Louchard & G. Schaeffer '04]
- a bit of tricks to draw an object from $\mathcal{F}$ from $\Gamma\mathcal{F}''$
  [O. Bodini, O. Roussel & M. Soria '12] and [O. Bodini '10]

$\Rightarrow$ Boltzmann generator using **only uniform random variable** to draw object such that $\mathcal{F}'' = \phi(\mathcal{F})$



Random Generation of the increasing labellings

$\mathcal{B} \quad = \quad \bullet \quad |$

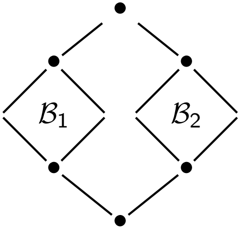


Random Generation of the increasing labellings

diamond	$\Rightarrow$	increasing labelling
$\bullet$	$\Rightarrow$	return (1)

Random Generation of the increasing labellings

diamond $\Rightarrow$ increasing labelling



$\Rightarrow$

$\mathbf{x} :=$ draw_inc_lbl$(\mathcal{B}_1)$
$\mathbf{y} :=$ draw_inc_lbl$(\mathcal{B}_2)$
$\mathbf{t} :=$ shuffle$(\mathbf{x}, \mathbf{y})$ $\quad |\mathbf{t}| = |\mathbf{x}| + |\mathbf{y}|$
return $(1,\ \mathbf{t} + \mathbb{1}, |\mathbf{t}| + 1)$

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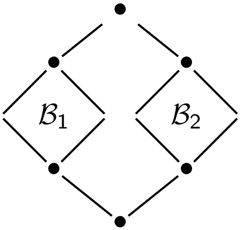
diamond $\Rightarrow$ increasing labelling



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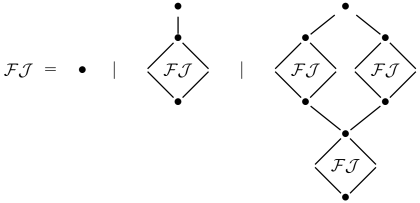
Average complexity

- The average complexity of `draw_inc_lbl` in memory writings is $\mathcal{O}(n\sqrt{n})$
- The average number of random bits needed during the generation is $\mathcal{O}(n^{3/2} \log n)$

Current work

- study of the average of some parameters (width, depth, root's degree ...) of the increasingly labelled structures
- study of a bit more realistic model, from a concurrency point of view:



- more efficient algorithms for the random generation of increasing labellings

Open question

- for the elliptic cases, how to do for showing the periodicity of the solutions directly from the differential equation ?
- is this periodic behaviour still present for higher degree of polynomial ?