The number of corner polyhedra graphs

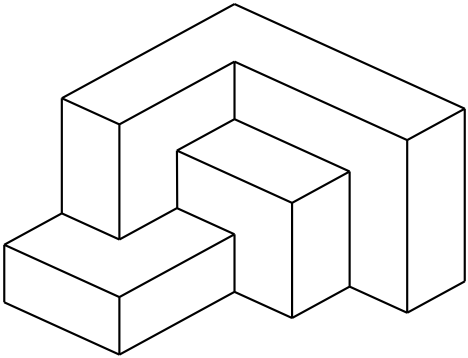
Common work with Dominique Poulalhon and Gilles Schaeffer

Clément Dervieux

IRIF, Université Paris-Diderot

5 mars 2016

Introduction : corner polyhedra



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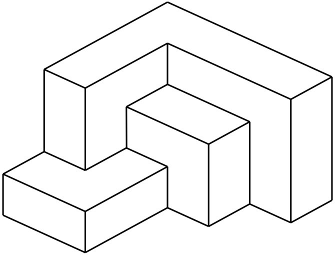

A corner polyhedron $\mathcal{P}$

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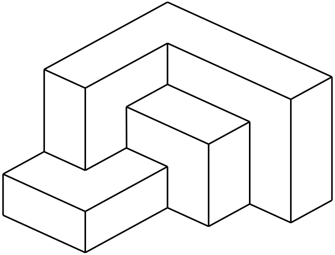

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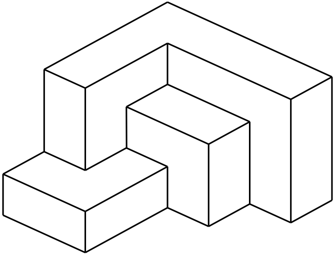

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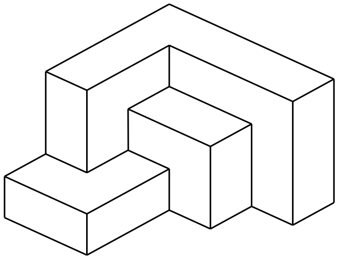

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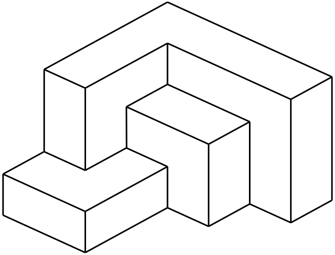

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- all vertices of $\mathcal{P}$ but v_0 are visible from infinity in the direction $(1, 1, 1)$.

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The skeleton of $\mathcal{P}$

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What kind of planar map ?

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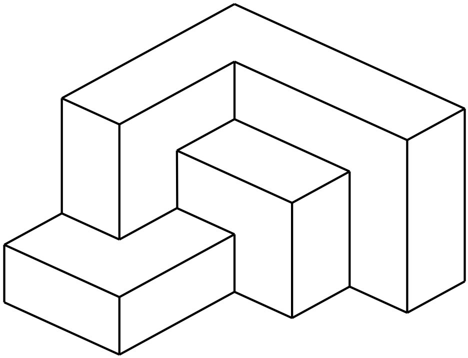

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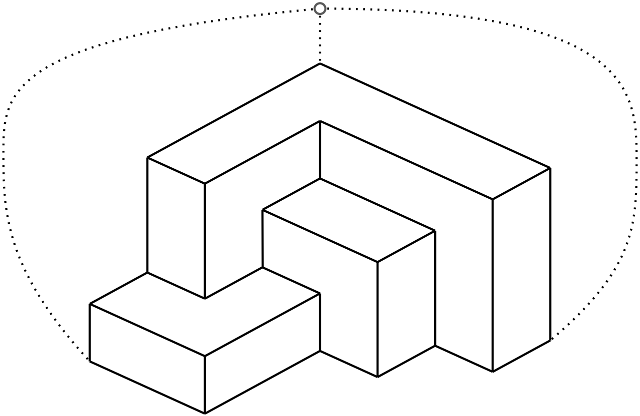

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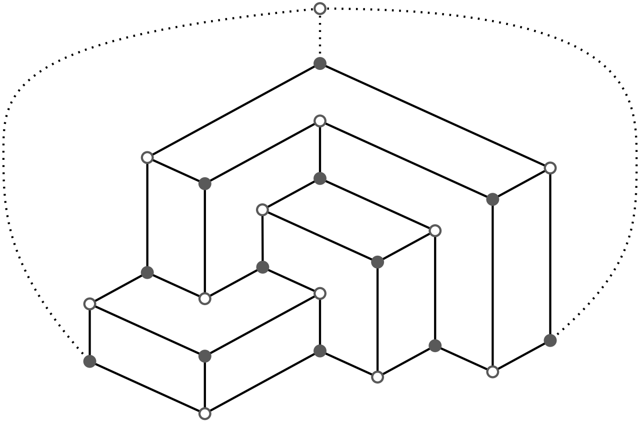

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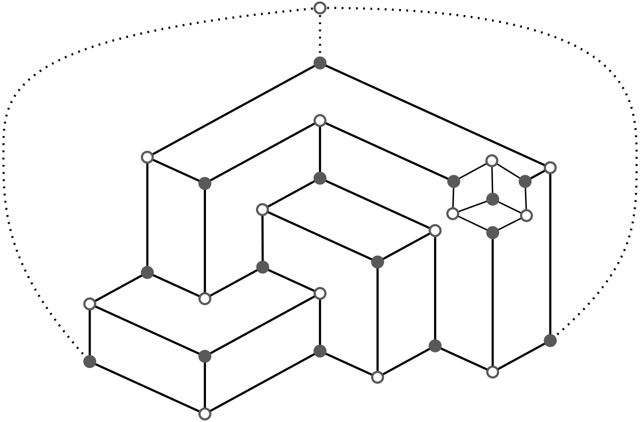

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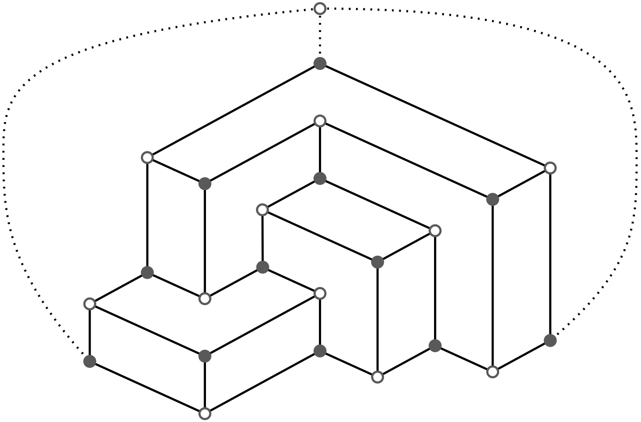

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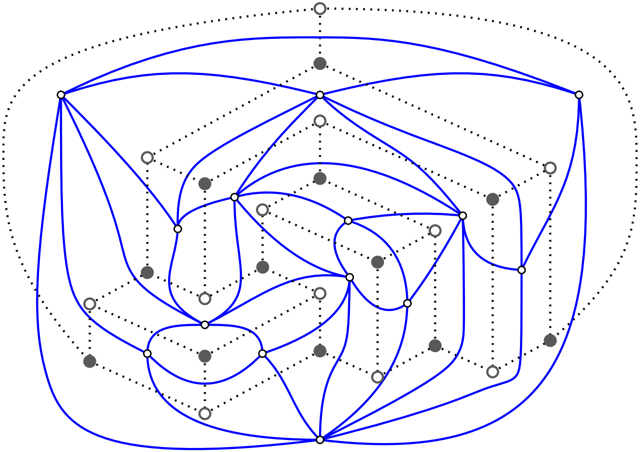

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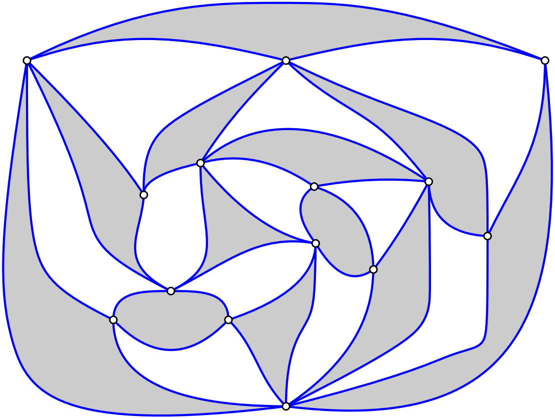
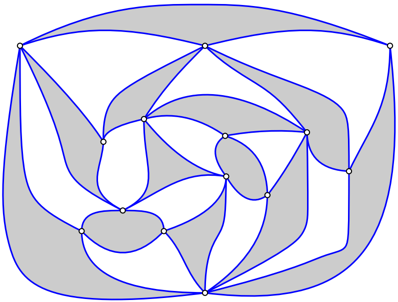


Eulerian triangulations

Eulerian triangulation



Eulerian triangulations

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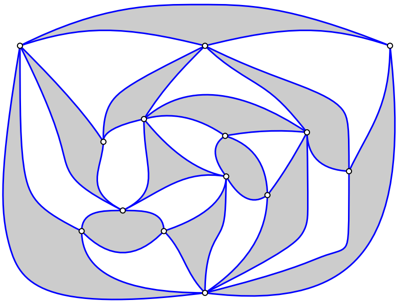

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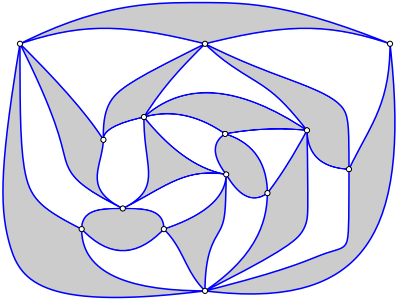

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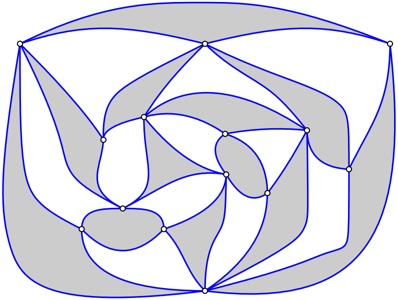

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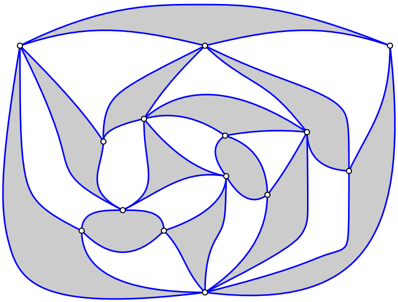

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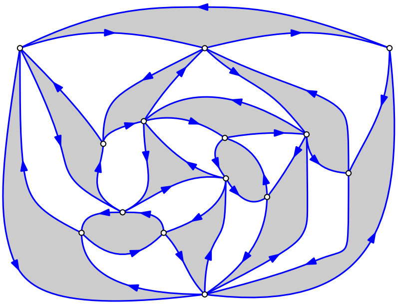

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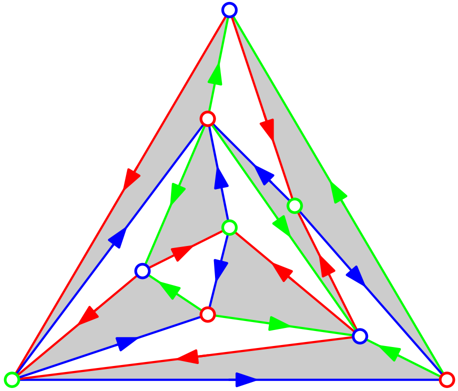
- the first terms of $E^s(y)$ are

$$E^s(y) = y + y^4 + 3y^6 + 7y^7 + 15y^8 + 63y^9 + O(y^{10})$$

Eulerian triangulations

Eulerian orientation

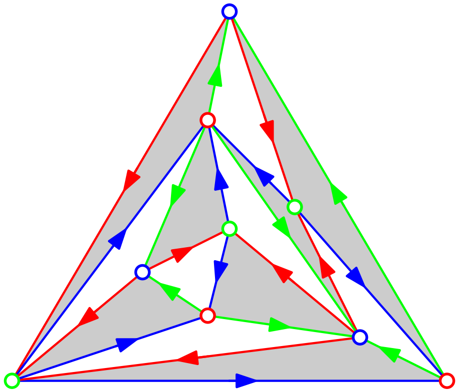
- partition of the **vertices**, of the **edges**



Eulerian triangulations

Eulerian orientation

- partition of the **vertices**, of the **edges**
- two oriented paths from a to b : **same length** modulo 3



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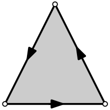

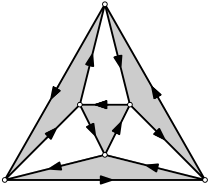

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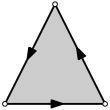

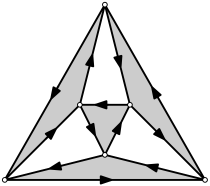

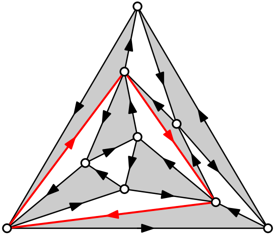

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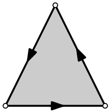

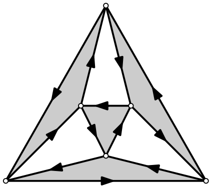

Theorem (Eppstein and Mumford (2014))

A rooted planar map is the skeleton of some corner polyhedron if and only if its dual map is a corner triangulation.

Enumeration of corner triangulations

Let $E^c(z)$ be the generating function of corner triangulations.

Theorem (D., Poulalhon, Schaeffer (2015))

$E^c(z)$ satisfies the following equation :

$$E^c(z) = \frac{z}{1+z}\left(1 + \frac{zA(z) + z^2A(z)^2}{1+z} - \frac{z^2A(z)^2 + 2z^3A(z)^3}{(1+z)^2}\right).$$

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3. We need **more families** of planar maps for intermediate decompositions
4. They have some **properties** in common

A set of properties for some maps

Corner quasi-triangulations

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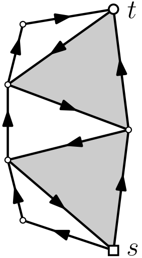
Almond triangulations



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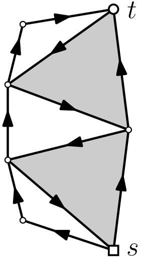

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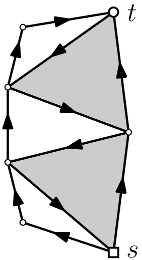

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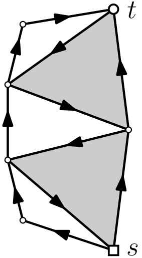

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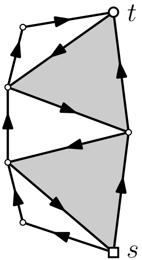

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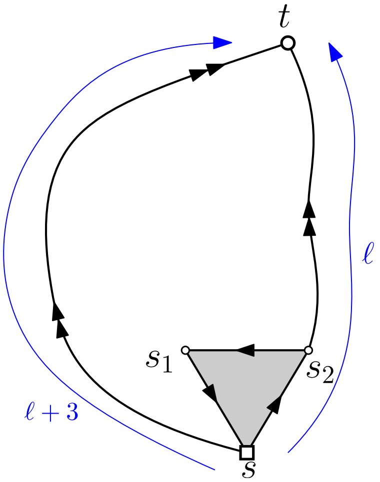
- a **left boundary** : clockwise, from s to t, length $\ell + 3$

Cutting an almond



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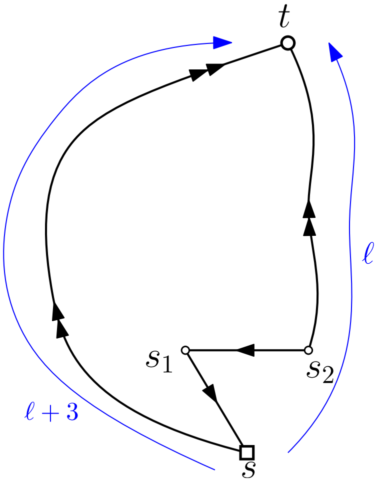

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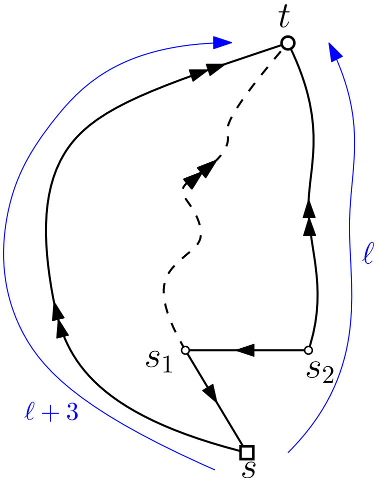

$\ell - 2 \quad ?$

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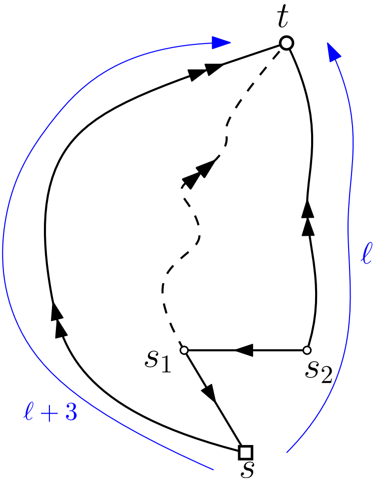

$\ell - 2$ ✗

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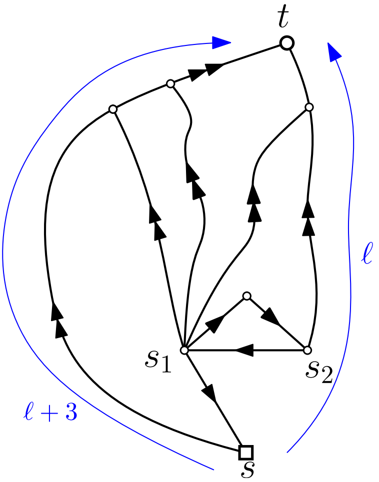

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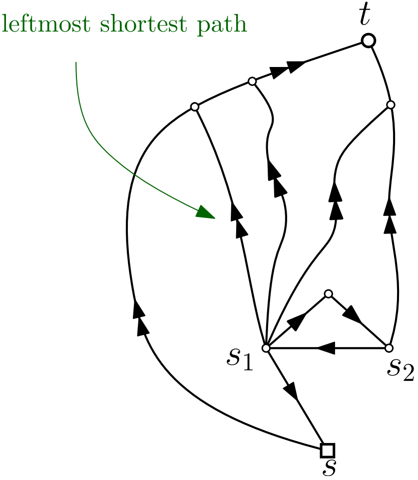

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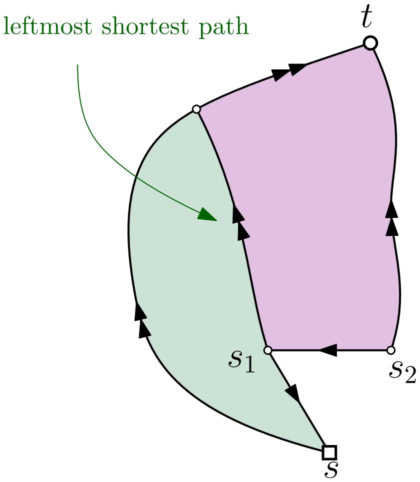

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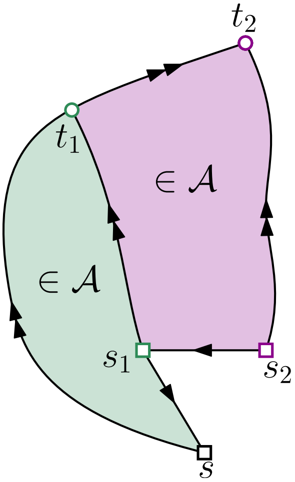
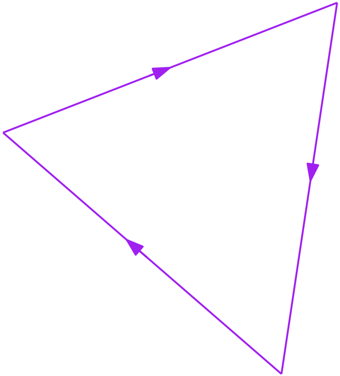


Gluing two almonds : no clockwise separating triangle



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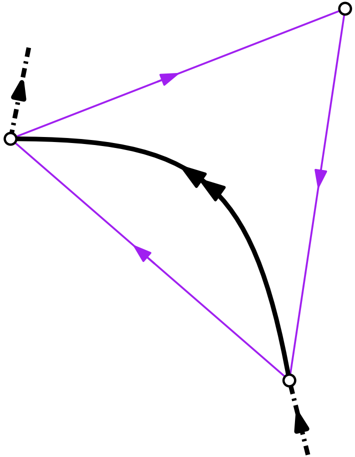

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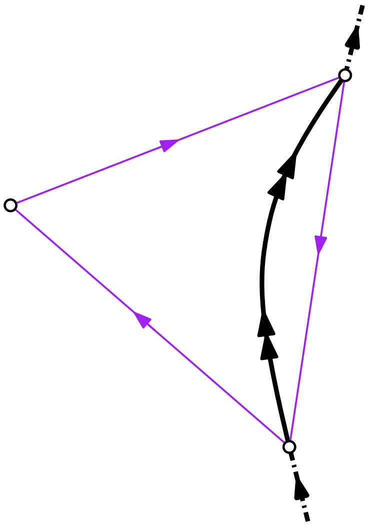
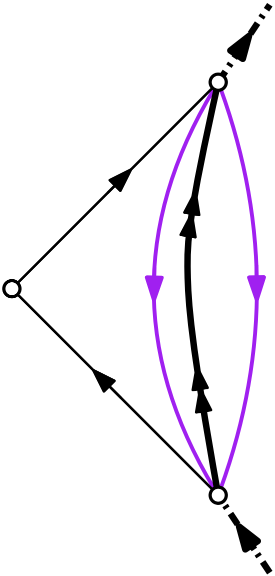


Gluing two almonds : no double edge



Almond triangulations

Let $A(z)$ be the generating function of the almonds.

Theorem

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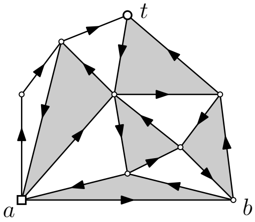
$$A(z) = 1 + zA(z)^2$$

Slices



A slice of height $\ell \geq 1$

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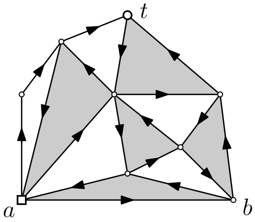

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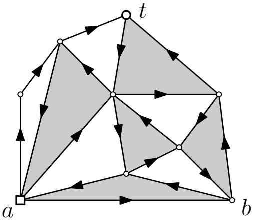

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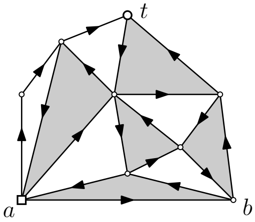


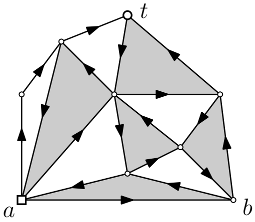
A slice of height $\ell \geq 1$

- a **root edge** (a, b)
- an **apex** t

The boundary is divided into :

- the root edge

Slices



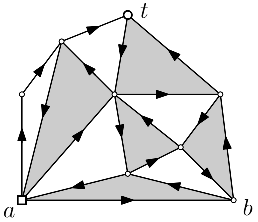
A slice of height $\ell \geq 1$

- a **root edge** (a, b)
- an **apex** t

The boundary is divided into :

- the root edge
- a **left boundary**, shortest path from a to t

Slices



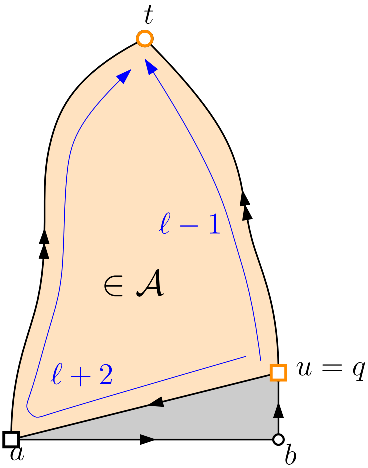
A slice of height $\ell \geq 1$

- a **root edge** (a, b)
- an **apex** t

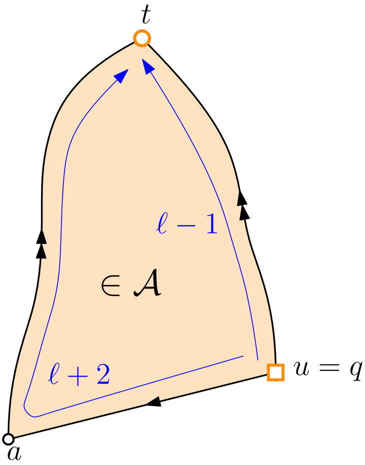
The boundary is divided into :

- the root edge
- a **left boundary**, shortest path from a to t
- a **right boundary**, unique shortest path from b to t, of length ℓ

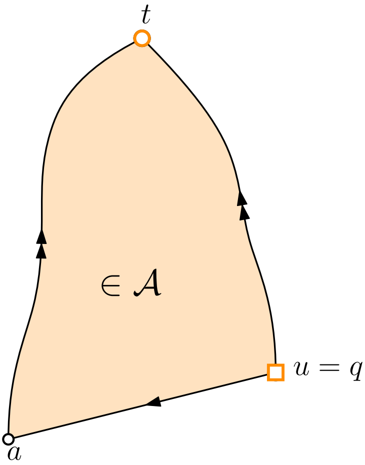
Cutting a slice



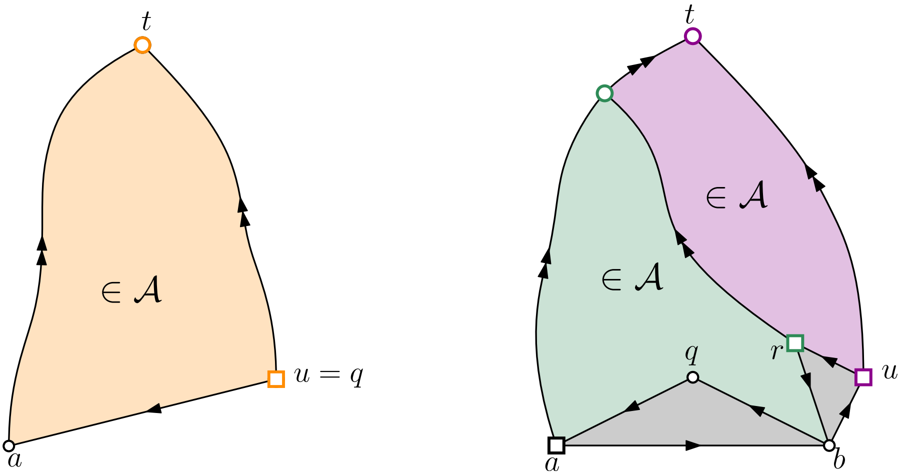
Cutting a slice



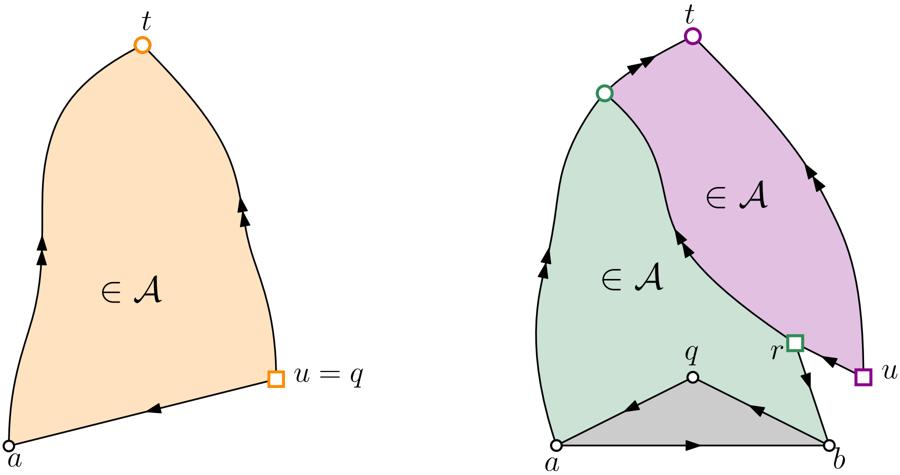
Cutting a slice



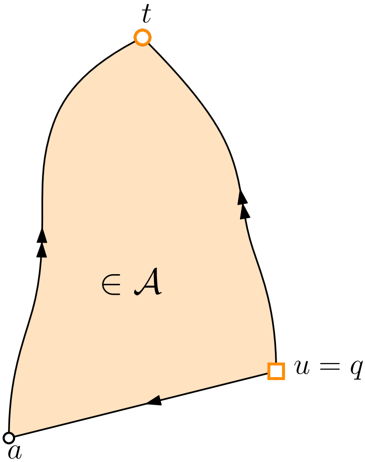
Cutting a slice

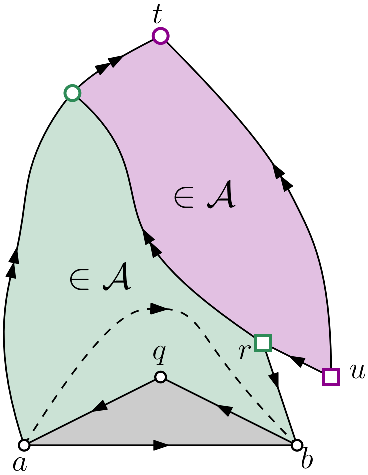


Cutting a slice

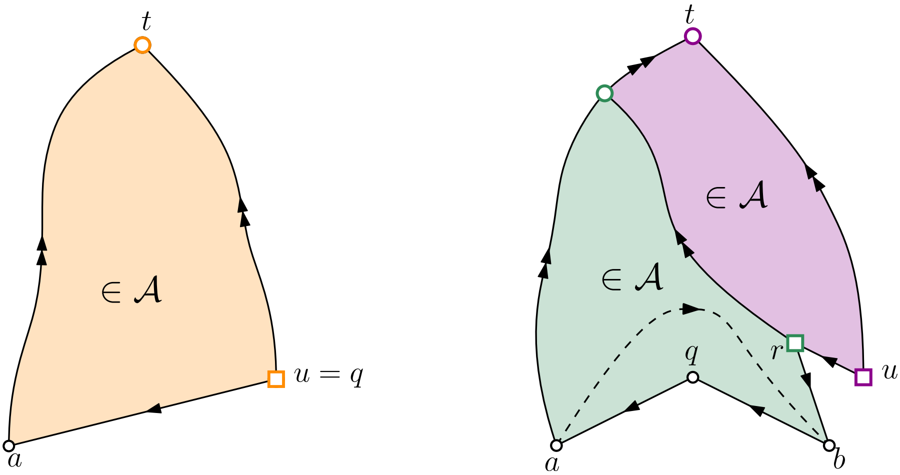


Cutting a slice

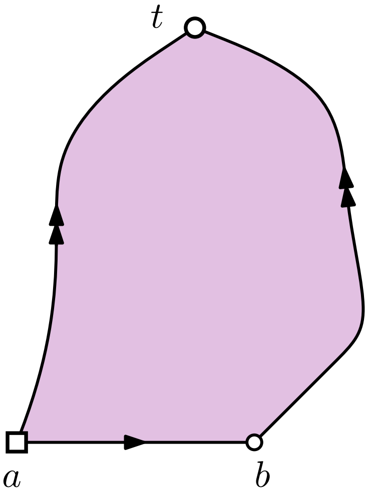




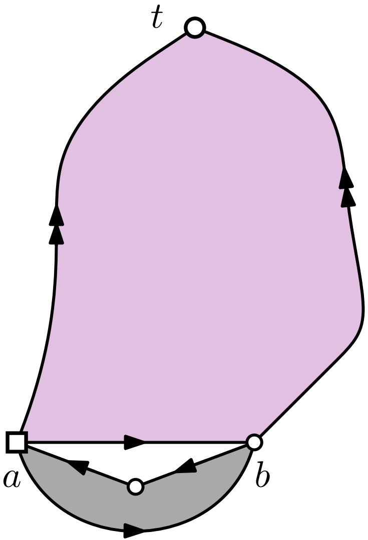
Cutting a slice



Slices or not slices



Slices or not slices



Slices

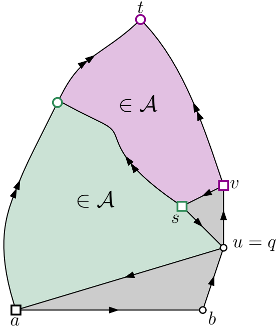
Let $S(z)$ be the generating function of the slices.

Theorem

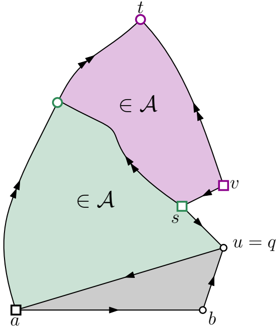
$S(z)$ satisfies the following equation :

$$(1 + z)S(z) = zA(z) + z^2 A(z)^2$$

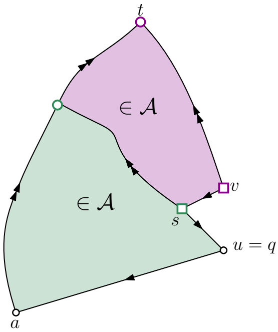
Cutting a slice of height at least 2



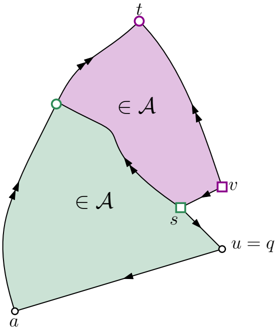
Cutting a slice of height at least 2



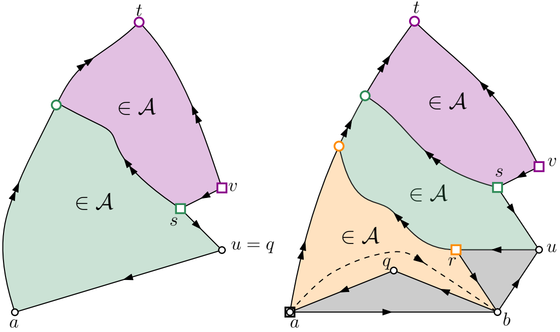
Cutting a slice of height at least 2



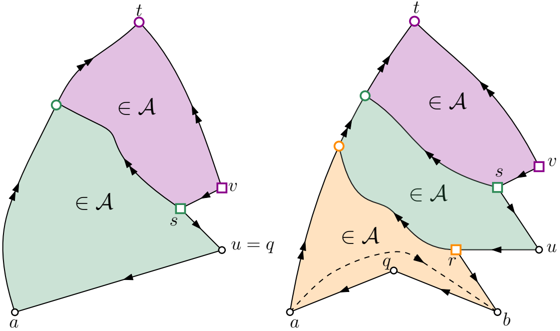
Cutting a slice of height at least 2



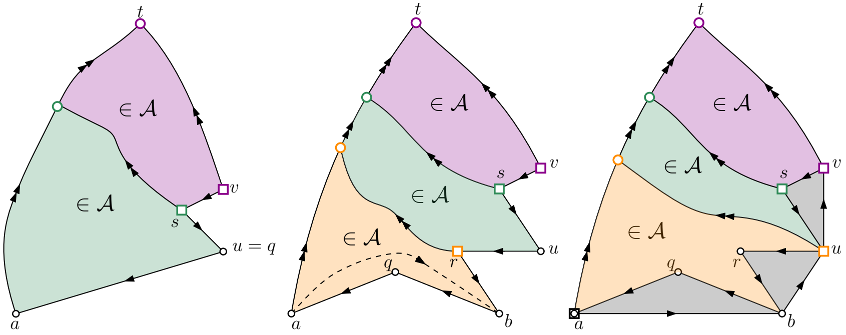
Cutting a slice of height at least 2



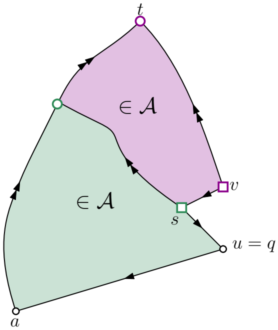
Cutting a slice of height at least 2



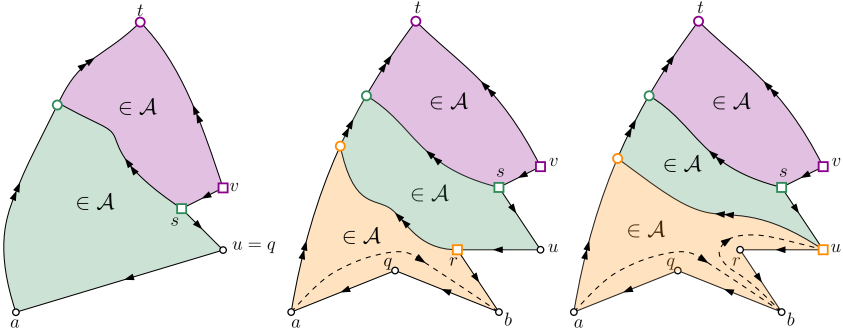
Cutting a slice of height at least 2



Cutting a slice of height at least 2



Cutting a slice of height at least 2



Slices of height at least 2

Let $S^+(z)$ be the generating function of the slices of height at least 2.

Theorem

$S^+(z)$ satisfies the following equation :

$$(1+z)^2 S^+(z) = z^2 A(z)^2 (1 + 2zA(z))$$

Bijective formula

- Corner triangulations

Bijective formula

- Corner triangulations
- Slices of height 1

Bijective formula

- Corner triangulations
- Slices of height 1

Theorem

The generating functions satisfy the following equation :

$$(1+z)E^c(z) = z + z(S(z) - S^+(z))$$

Bijective proof

Elements for bijective proof

Bijective proof

Elements for bijective proof

- corner triangulations $\mathfrak{c}$ with a **marked inner white triangle**.

Bijective proof

Elements for bijective proof

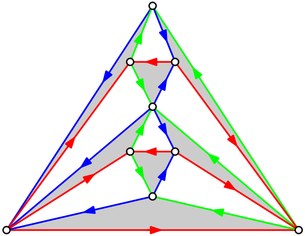
- corner triangulations $\mathfrak{c}$ with a **marked inner white triangle**.
- the lift of such a corner triangulation

Bijective proof

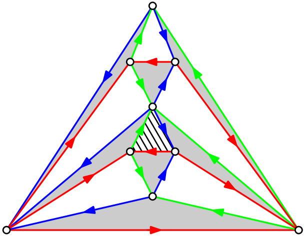
Elements for bijective proof

- corner triangulations $\mathfrak{c}$ with a **marked inner white triangle**.
- the lift of such a corner triangulation
- the fundamental domain of this lift

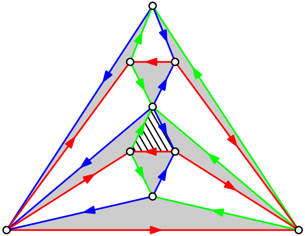
A corner triangulation with a marked inner white triangle



A corner triangulation with a marked inner white triangle

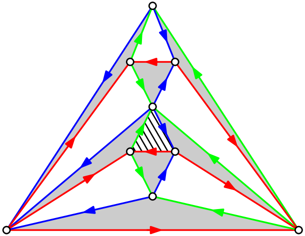


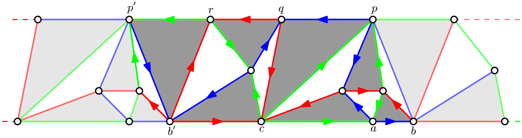
A corner triangulation with a marked inner white triangle



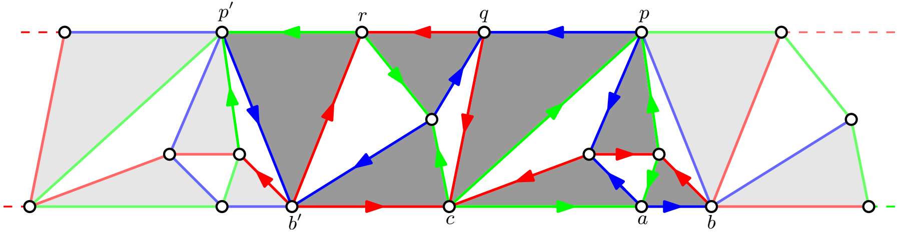
The lift of $\mathfrak{c}$ is its preimage under $\Psi : z \to \exp(2i\pi z)$.

A corner triangulation with a marked inner white triangle

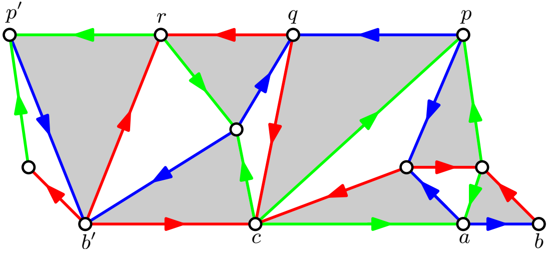




A lift

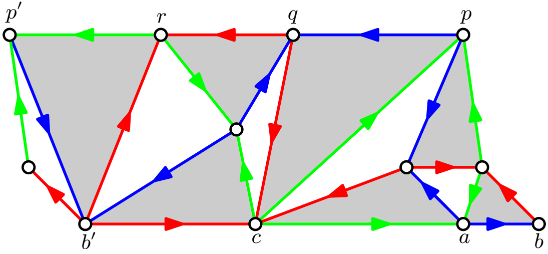


Definition of fundamental domains



A fundamental domain

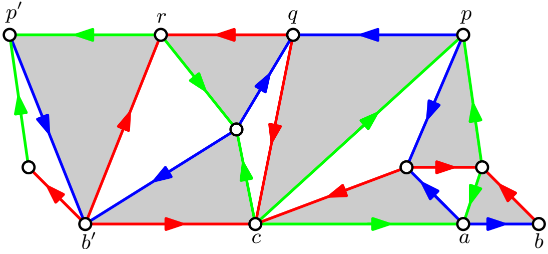
Definition of fundamental domains



A fundamental domain

- a **left boundary** of length ℓ

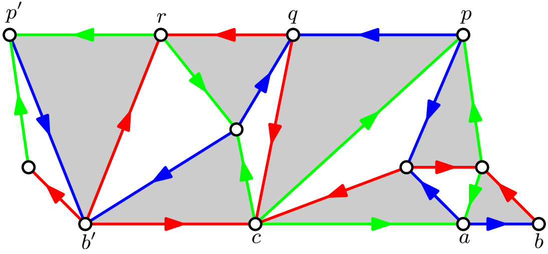
Definition of fundamental domains



A fundamental domain

- a **left boundary** of length ℓ
- a **lower boundary** of length 3, with vertices b', c, a, b, towards the right

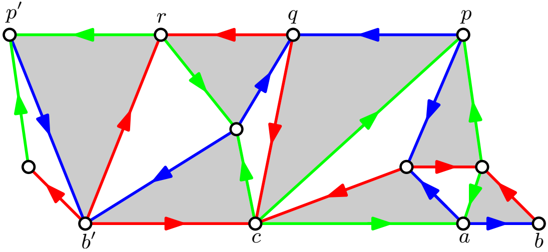
Definition of fundamental domains



A fundamental domain

- a **left boundary** of length ℓ
- a **lower boundary** of length 3, with vertices b', c, a, b, towards the right
- a **right boundary**, identical to the left boundary

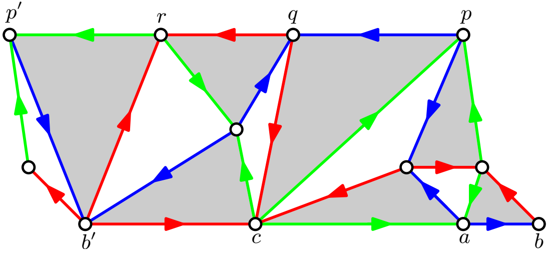
Definition of fundamental domains



A fundamental domain

- a **left boundary** of length ℓ
- a **lower boundary** of length 3, with vertices b', c, a, b, towards the right
- a **right boundary**, identical to the left boundary
- an **upper boundary** of length 3, with vertices p, q, r, p', towards the left

Definition of fundamental domains



A fundamental domain

- a **left boundary** of length ℓ
- a **lower boundary** of length 3, with vertices b', c, a, b, towards the right
- a **right boundary**, identical to the left boundary
- an **upper boundary** of length 3, with vertices p, q, r, p', towards the left

and with **no edges** (b, c), (a, b'), (r, p), (p', q).

Bijective proof

Let $E^c_\triangle$ be the generating function of corner triangulations with a marked inner white triangle, and let F be the one of fundamental domains.

Theorem

The generating functions satisfy the following equation :

$$E^c_\triangle(z) = 3F(z)$$

Bijective proof

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Theorem

The generating functions satisfy the following equation :

$$(1+z)^2 F(z) = zS^+(z).$$

Bijective proof

We conclude the bijective proof with the formula :

Formula

$$(1+z)^2 \frac{\partial}{\partial z} \frac{E^c(z)}{z} = 3zS_+(z).$$

Conclusion

Theorem (D., Poulalhon, Schaeffer (2015))

$E^c(z)$ satisfies the following equation :

$$E^c(z) = \frac{z}{1+z}\left(1 + \frac{zA(z) + z^2A(z)^2}{1+z} - \frac{z^2A(z)^2 + 2z^3A(z)^3}{(1+z)^2}\right).$$

where $A(z)$ is the Catalan series.

Catalan decomposition : the cactus

Cactus

- a binary tree

Catalan decomposition : the cactus

Cactus

- a binary tree
- black and white triangles

Catalan decomposition : the cactus

Cactus

- a binary tree
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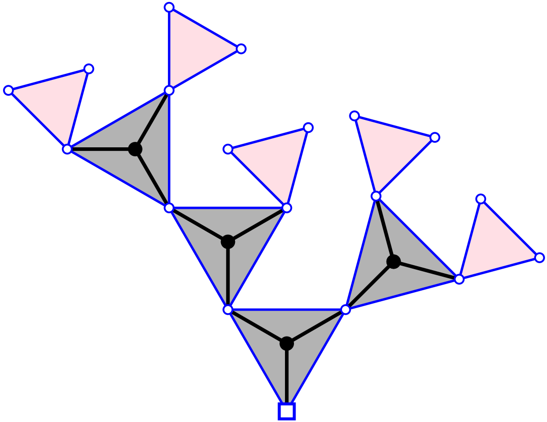
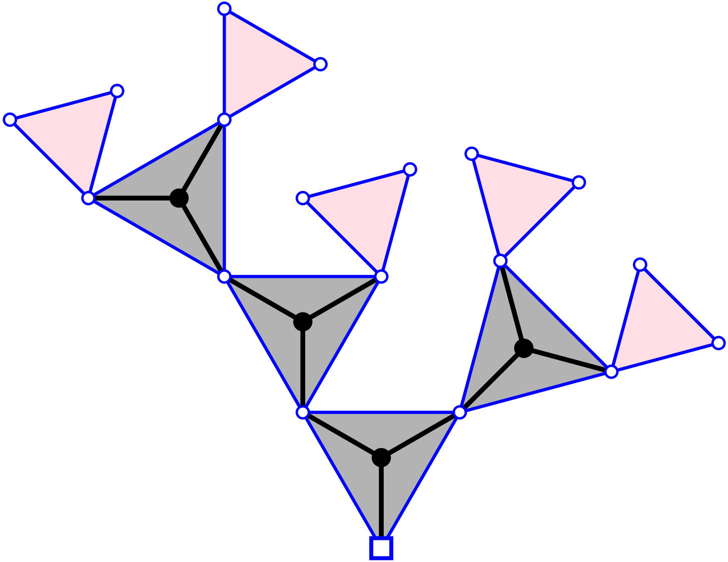
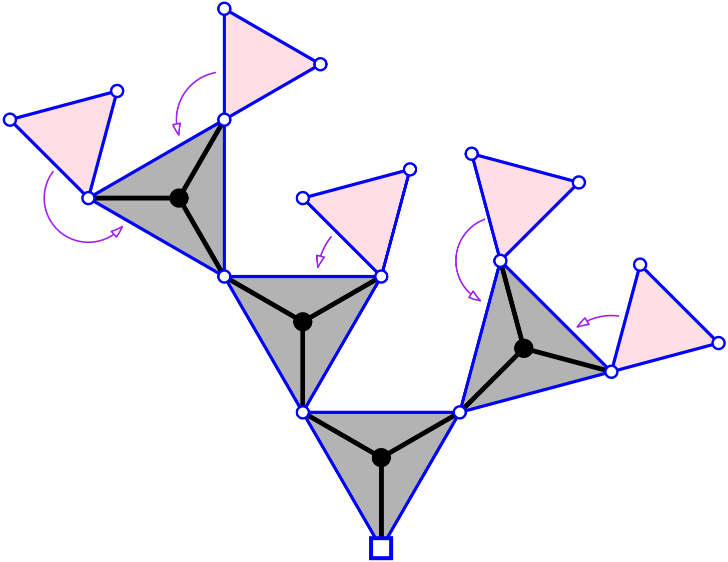


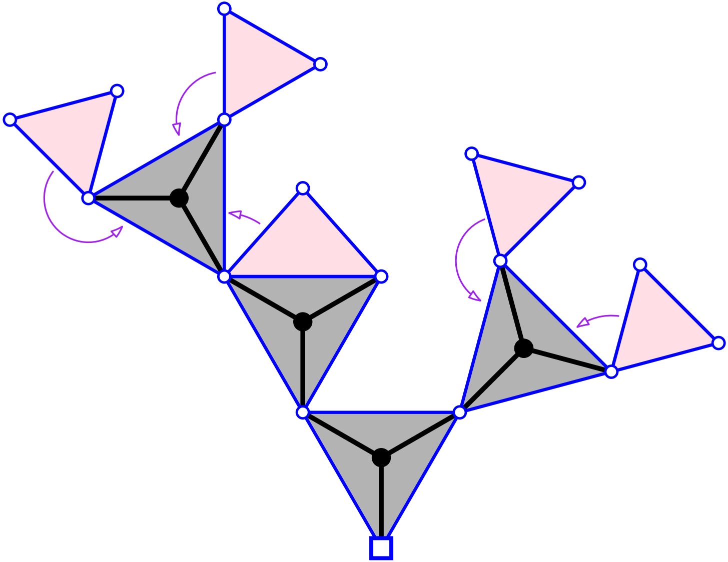
Catalan decomposition : the cactus



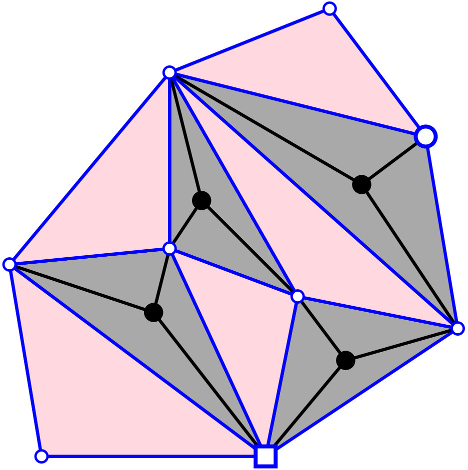
Catalan decomposition : the cactus



Catalan decomposition : the cactus



Catalan decomposition : the cactus



Merci pour votre attention !