Plane lattice paths avoiding a quadrant

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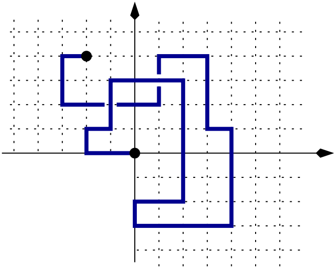
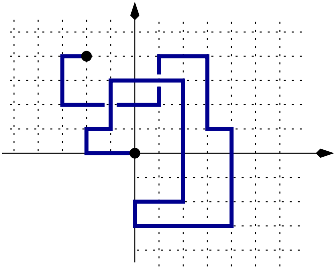


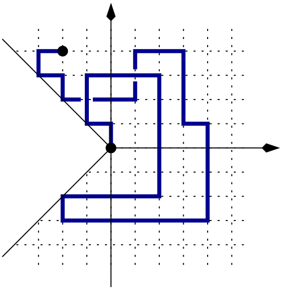
ArXiv:1511.02111

Two problems

Question: How many n-step walks on the square lattice, starting from $(0, 0)$, avoiding a given quadrant?

Two natural choices for the quadrant:





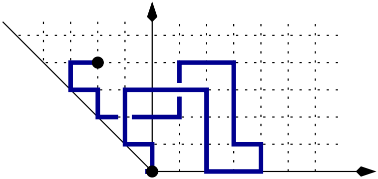
Counting walks in (rational) cones

Take a starting point p_0 in $\mathbb{Z}^d$, a (finite) step set $\mathcal{S} \subset \mathbb{Z}^d$ and a cone $\mathcal{C}$.

Questions

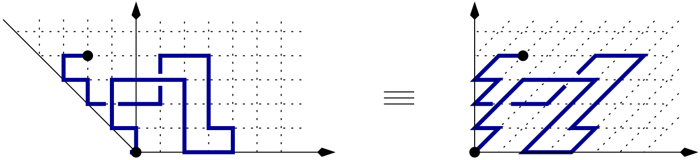
- What is the number $c(n)$ of n-step walks starting at p_0, taking their steps in $\mathcal{S}$ and contained in $\mathcal{C}$?
- For $\boldsymbol{i} = (i_1, \ldots, i_d) \in \mathcal{C}$, what is the number $c(\boldsymbol{i}; n)$ of such walks that end at $\boldsymbol{i}$?

Example: Gessel's walks



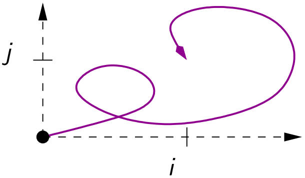
I. Counting walks in the first quadrant

2D walks in a convex cone ⇒ walks in the first quadrant



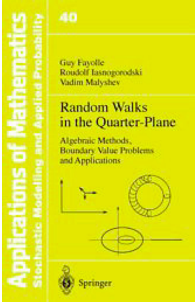
• The quarter plane, with $p_0 = (0, 0)$, and steps $\mathcal{S} \subset \mathbb{Z}^2$:

$$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n) x^i y^j t^n = ?$$



Quadrant walks at ALEA

- Alea 2002 : mbm
- Les journées jaunes, 2004 : Guy Fayolle
- [...]
- Alea 2011 : mbm-Kilian Raschel (cours), Marni Mishna
- Alea 2012 : Steve Melczer, Kilian Raschel
- Alea 2013 : Alin Bostan, Steve Melczer, Marni Mishna



A hierarchy of formal power series

- The formal power series $C(t)$ is rational if it can be written

$$C(t) = P(t)/Q(t)$$

where $P(t)$ and $Q(t)$ are polynomials in t.

- The formal power series $C(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, C(t)) = 0.$$

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• The formal power series $C(t)$ is D-finite (holonomic) if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_k(t)C^{(k)}(t) + \cdots + P_0(t)C(t) = 0.$$

◦ Nice and effective closure properties + asymptotics of the coefficients

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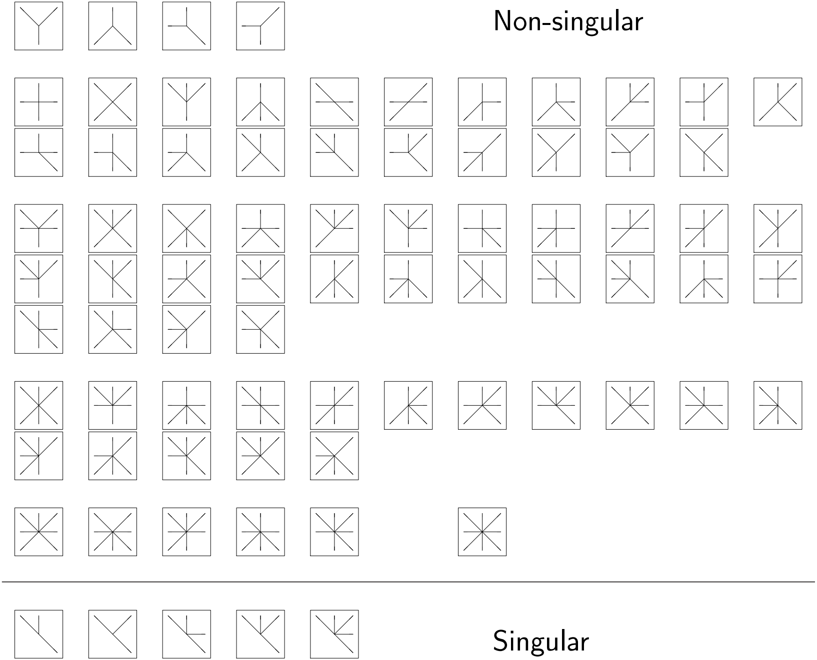
∘ Nice and effective closure properties + asymptotics of the coefficients
∘ Extension to several variables (D-finite: one DE per variable)

Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\}^2 \setminus \{(0,0)\} \Rightarrow 2^8 = 256$ step sets (or: models)

- However, some models are equivalent to a half-space problem (hence algebraic) and/or to another model (diagonal symmetry).

- One is left with 79 interesting distinct models [mbm-Mishna 09].


Non-singular
Singular

Classification of quadrant walks with small steps

Theorem (for the 79 interesting models)

The series $Q(x, y; t)$ is D-finite if and only if a certain group G associated with $\mathcal{S}$ is finite.

[mbm-Mishna 10], [Bostan-Kauers 10]	D-finite
[Kurkova-Raschel 12]	non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13]	singular non-D-finite

$\Rightarrow$ Classification of the corresponding functional equations, e.g.

$$\left(xy - t(x^2y + x + y + xy^2)\right) Q(x, y; t) = xy - tyQ(0, y; t) - txQ(x, 0; t)$$

The group of the model

To a step set $\mathcal{S}$, associate the step polynomial:

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j .$$

The group is generated by two birational transformations of (x, y) that leave $S(x, y)$ unchanged.

Example. For the square lattice,

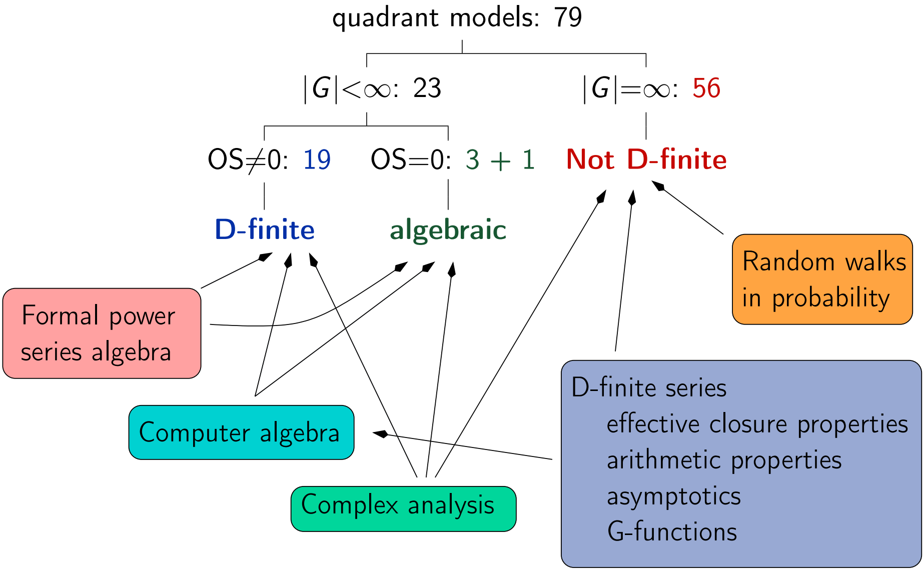
$$S(x, y) = \frac{1}{x} + x + y + \frac{1}{y} = \bar{x} + x + y + \bar{y}$$

and

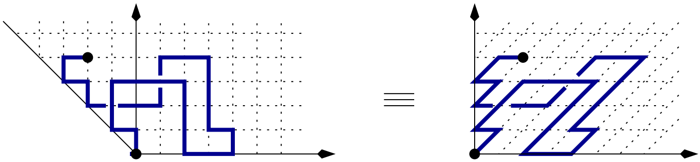
$$\Phi : (x, y) \mapsto (\bar{x}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{y}) .$$

These two involutions generate a group of order 4.

Classification of quadrant walks with small steps



One hard case: Gessel's walks



Conjecture: [Gessel, around 2000]

$$g_{0,0}(2n) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

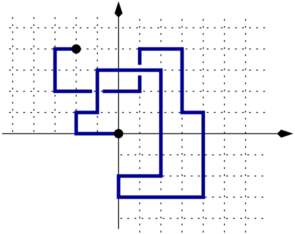
• Proved in 2009 using computer algebra [Kauers, Koutschan & Zeilberger 09]

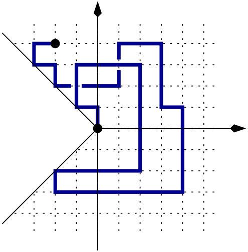
• In fact, $Q(x, y; t)$ is algebraic! (degree 72...) [Bostan & Kauers 10]

• Two other proofs, one analytic, one "elementary"
[Bostan, Kurkova and Raschel 13(a)], [mbm 15(a)]

II. Walks avoiding a quadrant

2D walks in a non-convex cone ⇒ walks avoiding the negative quadrant





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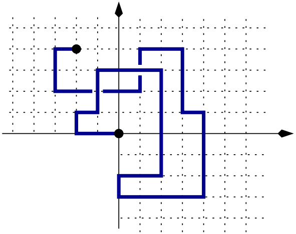
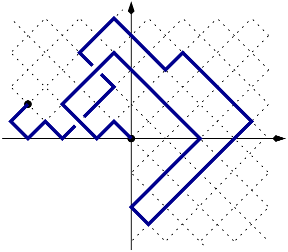
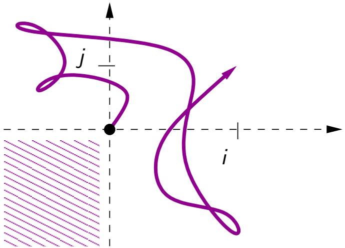






• Walks avoiding the negative quadrant, with $p_0 = (0,0)$, and steps $\mathcal{S} \subset \mathbb{Z}^2$

$$C(x,y;t) = \sum_{i,j,n \geq 0} c(i,j;n) x^i y^j t^n = \;?$$

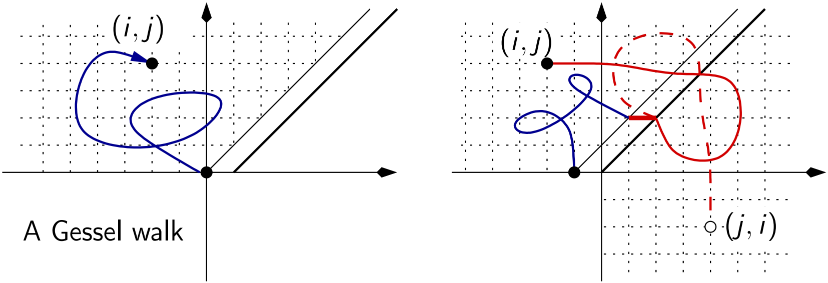
Is there a nice story to tell for walks avoiding a quadrant?

Another motivation: a new approach to Gessel's walks

Idea: count Gessel's walks by the reflection principle

If $c(i, j; n)$ counts walks starting at $(-1, 0)$ and avoiding the negative quadrant, then for $j \geq 0$ and $i < j$,

$$c_{i,j}(n) - c_{j,i}(n) = g_{i+1,j}(n).$$



An *a posteriori* motivation

A hard (?) problem with nice numbers!

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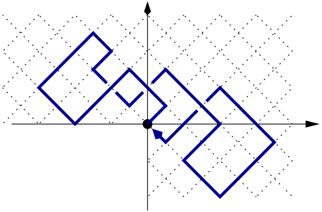
Example: the number of walks of length $2n$ on the diagonal square lattice starting and ending at $(0,0)$ and avoiding the negative quadrant is

$$c(0,0;2n) = \frac{16^n}{9}\left(3\,\frac{(1/2)_n^2}{(2)_n^2} + 8\,\frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2\,\frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n}\right)$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.



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Two known components: Quadrant walks (left) and Gessel walks (right).
Conjectured before 2007...
Asymptotically,

$$c(0,0;2n) \sim \frac{2^5}{3^2}\frac{\Gamma(2/3)}{\pi}\frac{4^{2n}}{(2n)^{5/3}}.$$

Results (square lattice, starting point $(0,0)$)

- Let $Q(x, y) \equiv Q(x, y; t)$ be the generating function of square lattice walks starting from $(0, 0)$ and *confined to the first quadrant*:

$$Q(x,y) = \sum_{i,j,n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n$$

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Theorem [mbm 15(a)]

The generating function $C(x,y;t) \equiv C(x,y)$ counting walks that avoid the negative quadrant is

$$A(x,y) + \frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right)$$

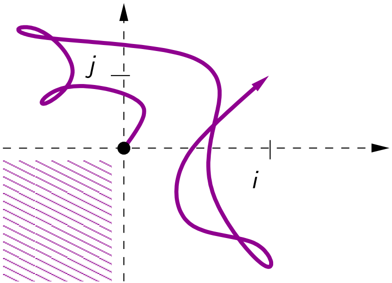
where $\bar{x} = 1/x$, $\bar{y} = 1/y$ and $A(x,y)$ is algebraic. This series satisfies

$$(1 - t(x+\bar{x}+y+\bar{y}))\, A(x,y) = (2+\bar{x}^2+\bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

where $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24, given explicitly.

III. Some ingredients of the solution

- a functional equation for $C(x, y)$
- relating $C(x, y)$ to the quadrant series $Q(x, y)$
- the ordinary kernel method: cancel the kernel
- ...
- polynomial equations with one catalytic variable have algebraic solutions [mbm-Jehanne 06]



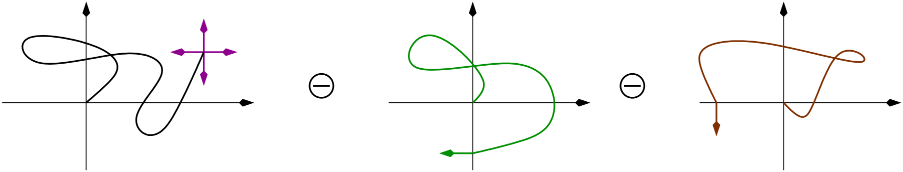
Walks avoiding a quadrant: a functional equation

Step by step construction:

$$C(x,y;t) \equiv C(x,y) = 1 + t(x+\bar{x}+y+\bar{y})C(x,y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x})$$

with

$$C_-(\bar{x}) = \sum_{i<0, n\geq 0} c(i,0;n)x^i t^n.$$



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or

$$\big(1 - t(x+\bar{x}+y+\bar{y})\big)C(x,y) = 1 - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}),$$

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or

$$\boxed{\big(1 - t(x+\bar{x}+y+\bar{y})\big)xyC(x,y) = xy - tyC_-(\bar{y}) - txC_-(\bar{x}).}$$

The polynomial $\big(1 - t(x+\bar{x}+y+\bar{y})\big)$ is the kernel of the equation.

Where the series $A(x, y)$ comes from

- The equation:

$$\big(1 - t(x + \bar{x} + y + \bar{y})\big) xyC(x, y) = xy - tyC_-(\bar{y}) - txC_-(\bar{x})$$

Compare with the equation for quadrant walks:

$$\big(1 - t(x + \bar{x} + y + \bar{y})\big) xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

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- Playing with the group (...) leads us to write:

$$C(x, y) := A(x, y) + \frac{1}{3}\left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y})\right).$$

The series $A(x, y)$ then satisfies

$$(1 - t(x + \bar{x} + y + \bar{y}))xyA(x, y) = (2xy + \bar{x}y + x\bar{y})/3 - tyA_-(\bar{y}) - txA_-(\bar{x}),$$

with

$$A_-(\bar{x}) = \sum_{i<0, n\geq 0} a(i, 0; n)x^i t^n.$$

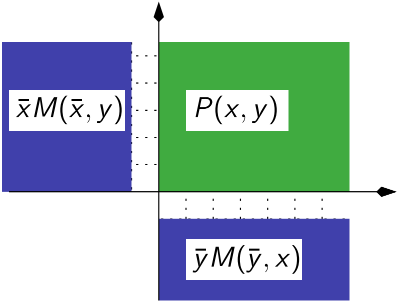
Back to series with *polynomial coefficients* in x and y

- The equation for $A(x,y)$ (with $K(x,y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x,y)A(x,y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x,y)$ into three parts:

$$A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x)$$



$\Rightarrow$ Equations for $P(x,y)$ and $M(x,y)$

The equation for $M(x, y)$ is (almost) a quadrant equation

- $M(x, y)$ is a series in t with coefficients in $\mathbb{Q}[x, y]$, satisfying

$$(1 - t(x + \bar{x} + y + \bar{y}))\,(2M(x, y) - M(0, y)) = \\ 2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0).$$

This looks (remotely) like a quadrant problem, cf.

$$\big(1 - t(x + \bar{x} + y + \bar{y})\big)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

Two main differences:

- $M(y, 0)$ is a new ingredient
- the r.h.s. cannot be decoupled into $F(x) + G(y)$

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More results

- Walks ending at a prescribed position
- Walks on the diagonal square lattice avoiding the negative cone
- Walks on the square lattice starting at $(-1, 0)$ (entirely algebraic!)
- The solution of Gessel's model in a $135°$ cone then comes for free

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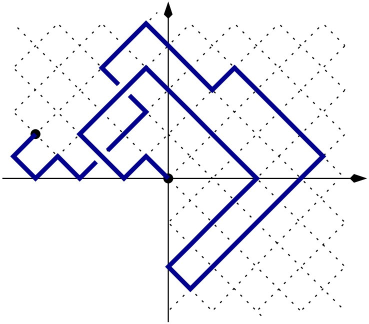

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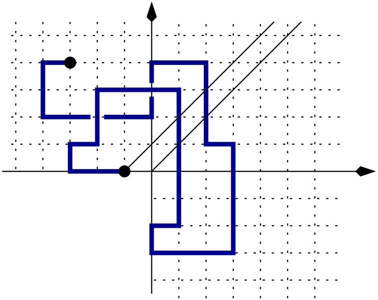

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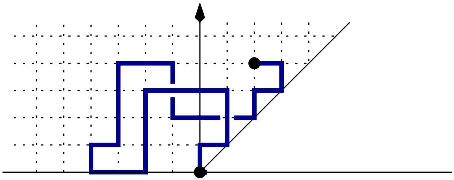

Perspectives: classification

Walks with small steps confined to the positive quadrant:

- models: 79
  - $|G|<\infty$: 23
    - OS$\neq$0: 19
      - **D-finite**
    - OS$=$0: $3+1$
      - **algebraic**
  - $|G|=\infty$: 56
    - **Not D-finite**

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Walks with small steps avoiding the negative quadrant:

- models: 74
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      - **D-finite?**
    - OS=0: 3 + 1
      - **algebraic?**
  - $|G|=\infty$: 51
    - **Not D-finite?**

Perspectives: classification

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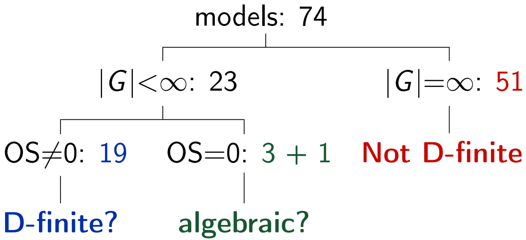

NEW: non-D-finiteness is proved (S. Mustapha)

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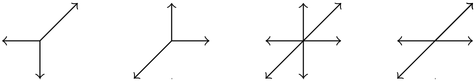
- $|G|=\infty$: 51 → **Not D-finite**

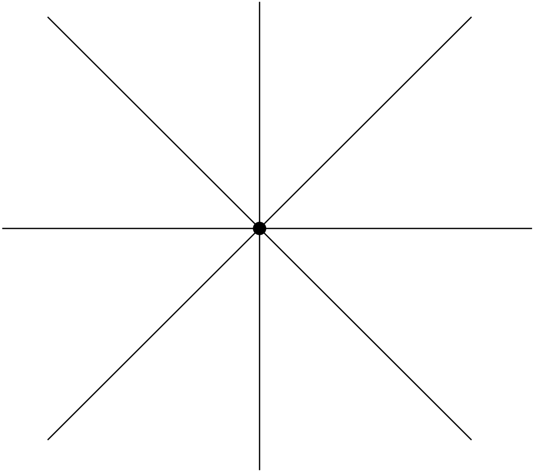
NEW: non-D-finiteness is proved (S. Mustapha)

Conjecture: the generating function of walks starting and ending at $(0,0)$ and avoiding the negative quadrant is algebraic for the four models for which $Q(x,y)$ (the quadrant GF) is algebraic:



Square lattice walks in a cone

Twelve problems: Count walks on the square lattice (steps NSEW) starting from $(0, 0)$ and confined to a cone bounded by one of the 8 half-lines of slope 0, ± 1 and ∞ starting from $(0, 0)$.



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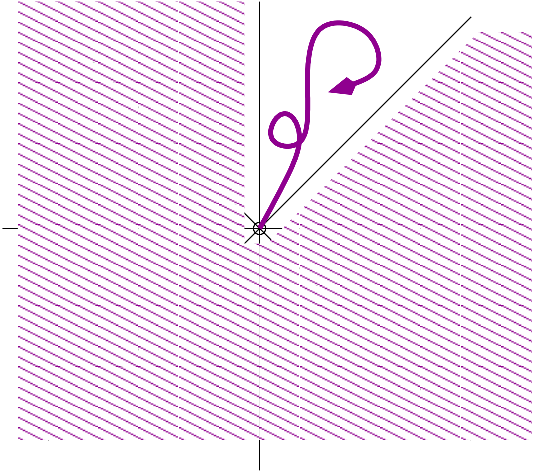

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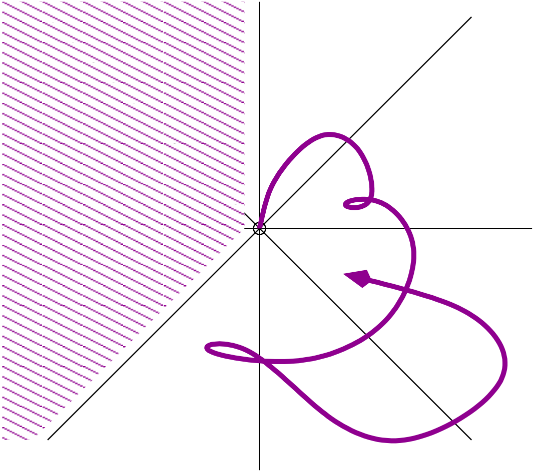

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