A simple explicit bijection between $(n, 2)$ Gog and Magog trapezoids

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What are Gog and Magog?

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In the mathematical world, these are combinatorial objects known to be in bijection with other fundamental objects.

Alternating sign matrices

Definition

An alternating sign matrix of size n is an $n \times n$ matrix with entries in $\{-1, 0, 1\}$ such that, on each fixed row or column, the nonzero entries start and end by 1 and alternate between 1 and -1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gog

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

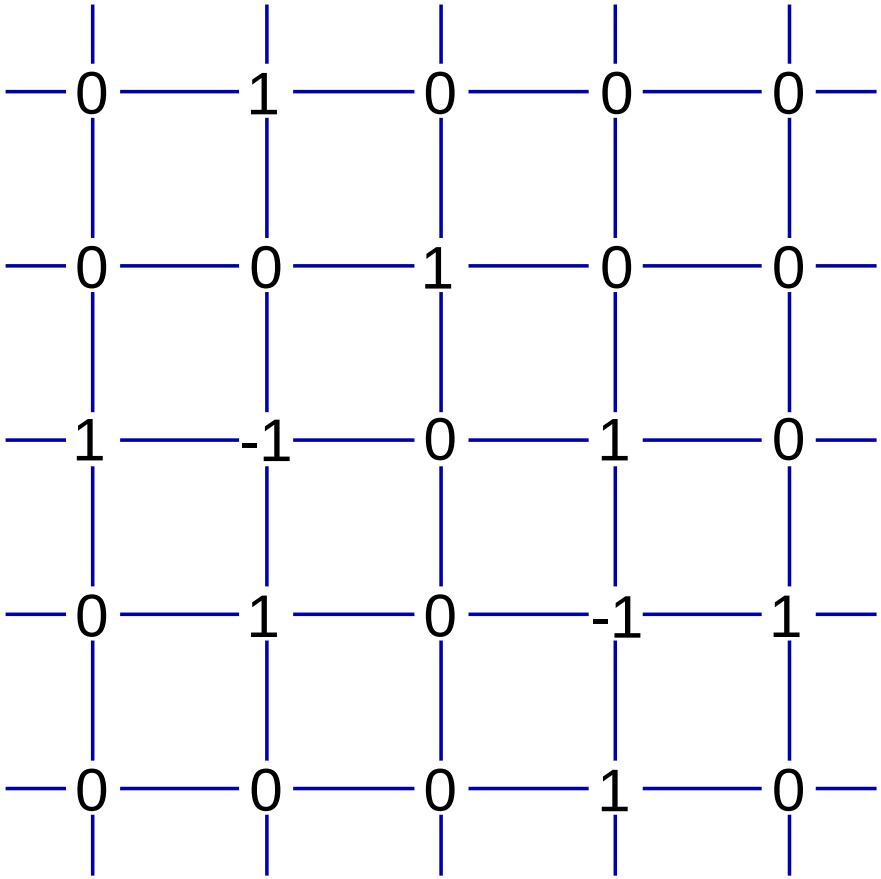
Alternating sign matrices

Gog

0	1	0	0	0
0	0	1	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0

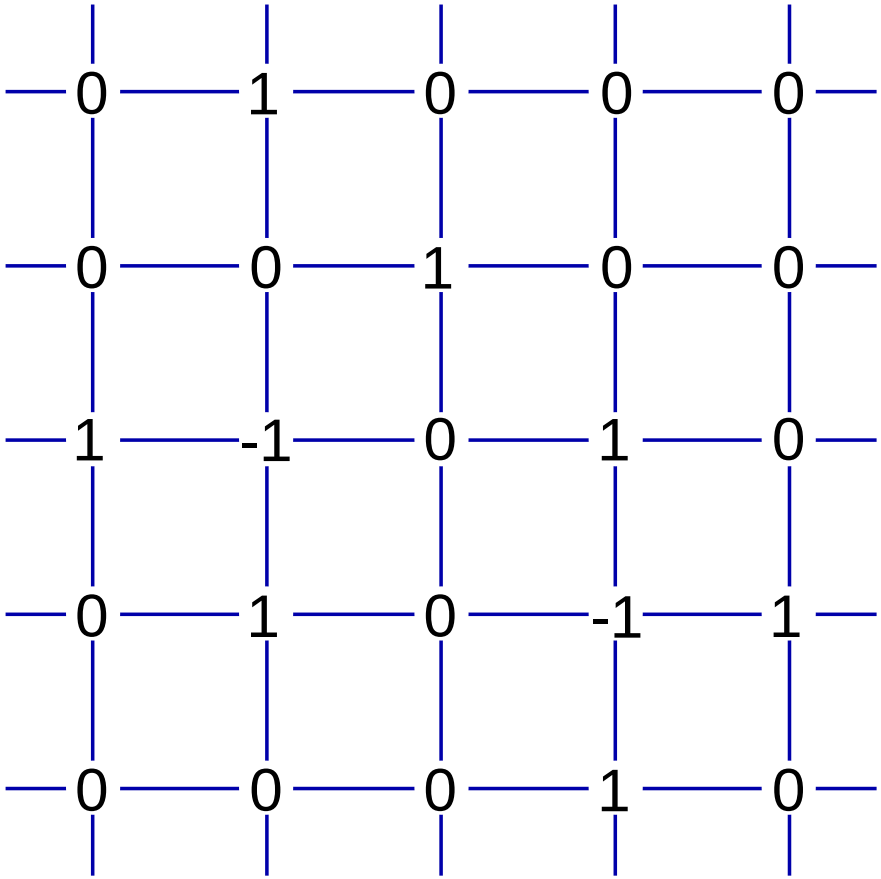
6-vertex model

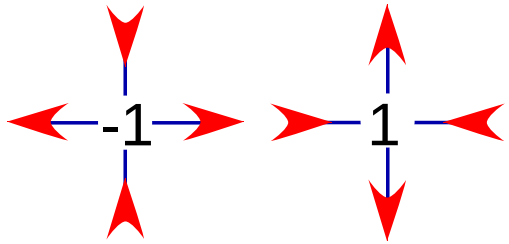
Gog



6-vertex model

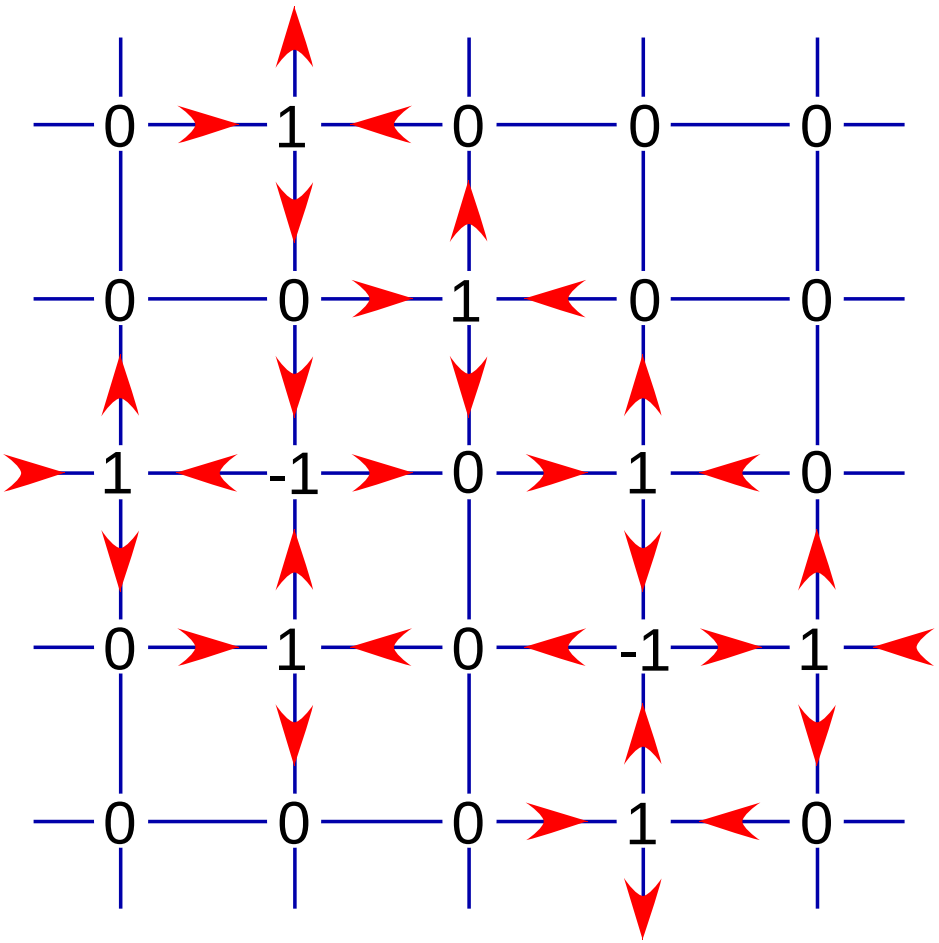
Gog

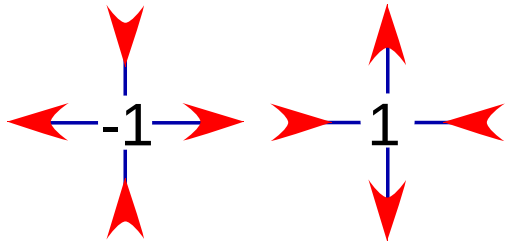




6-vertex model

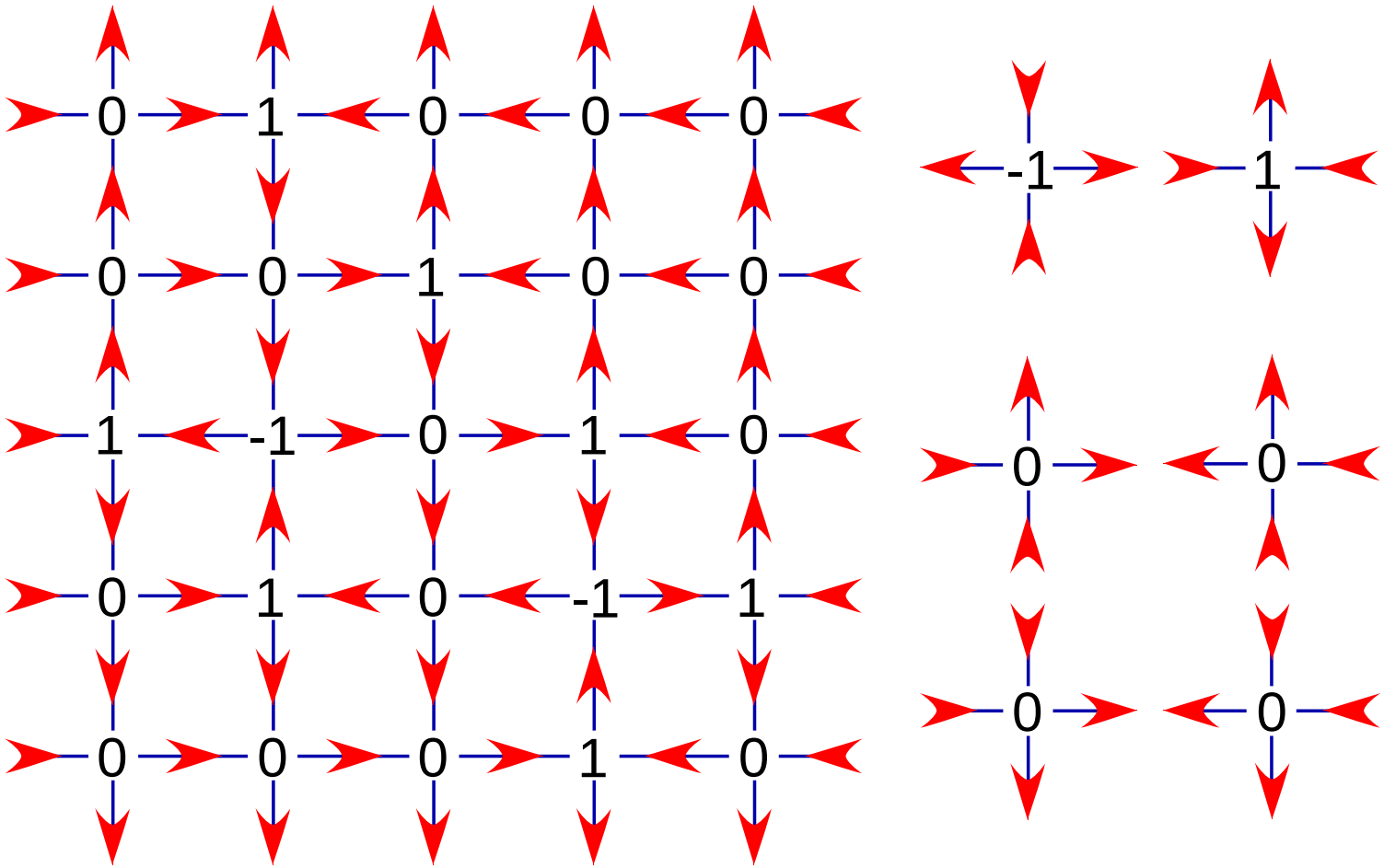
Gog





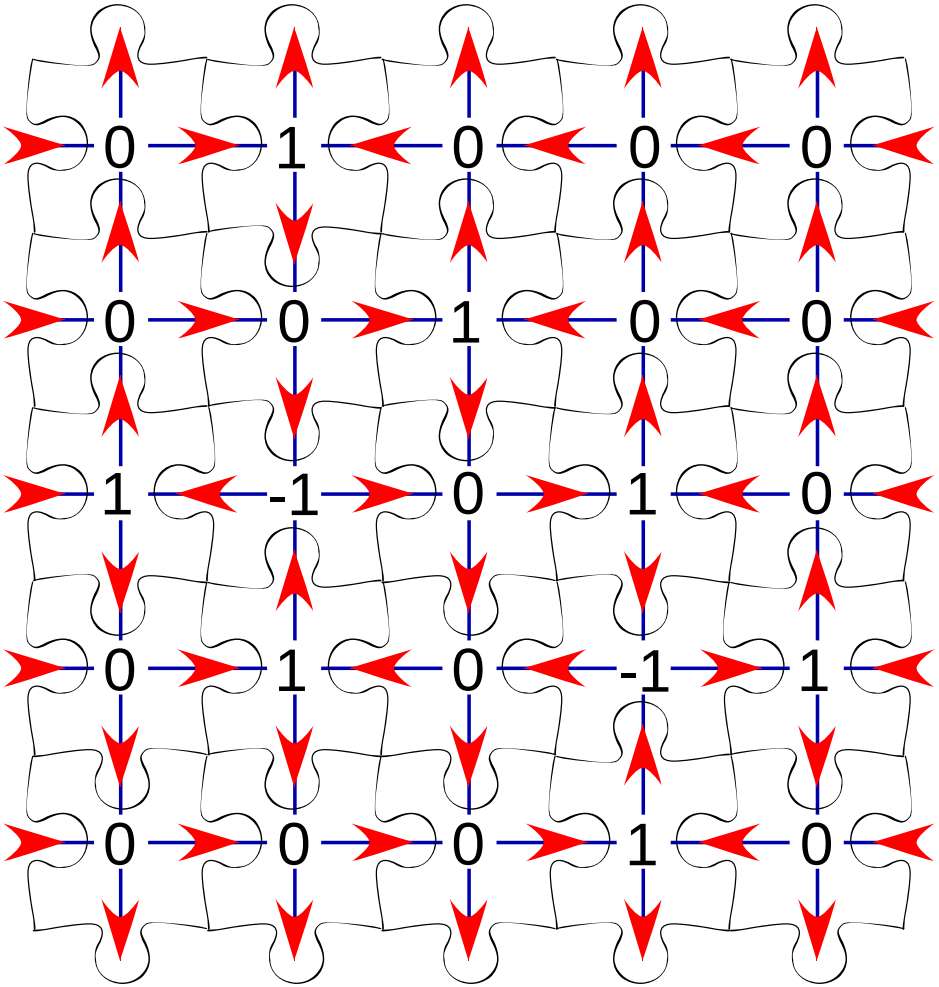
6-vertex model

Gog



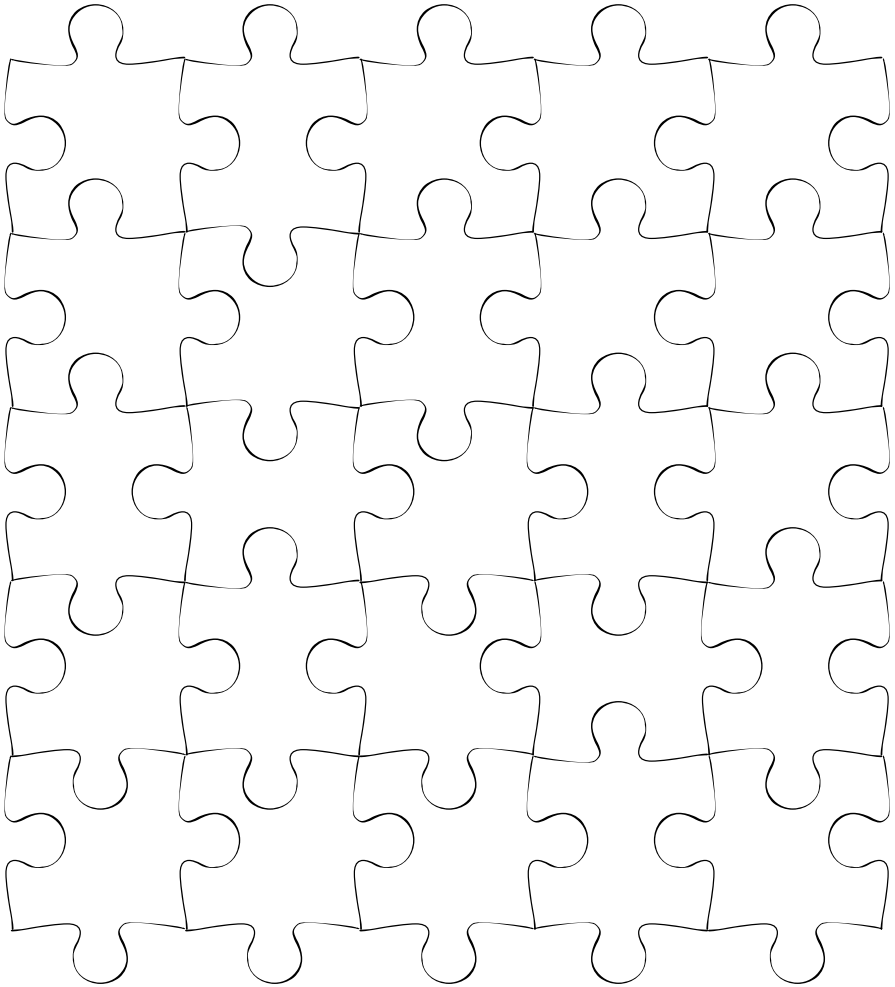
6-vertex model

Gog



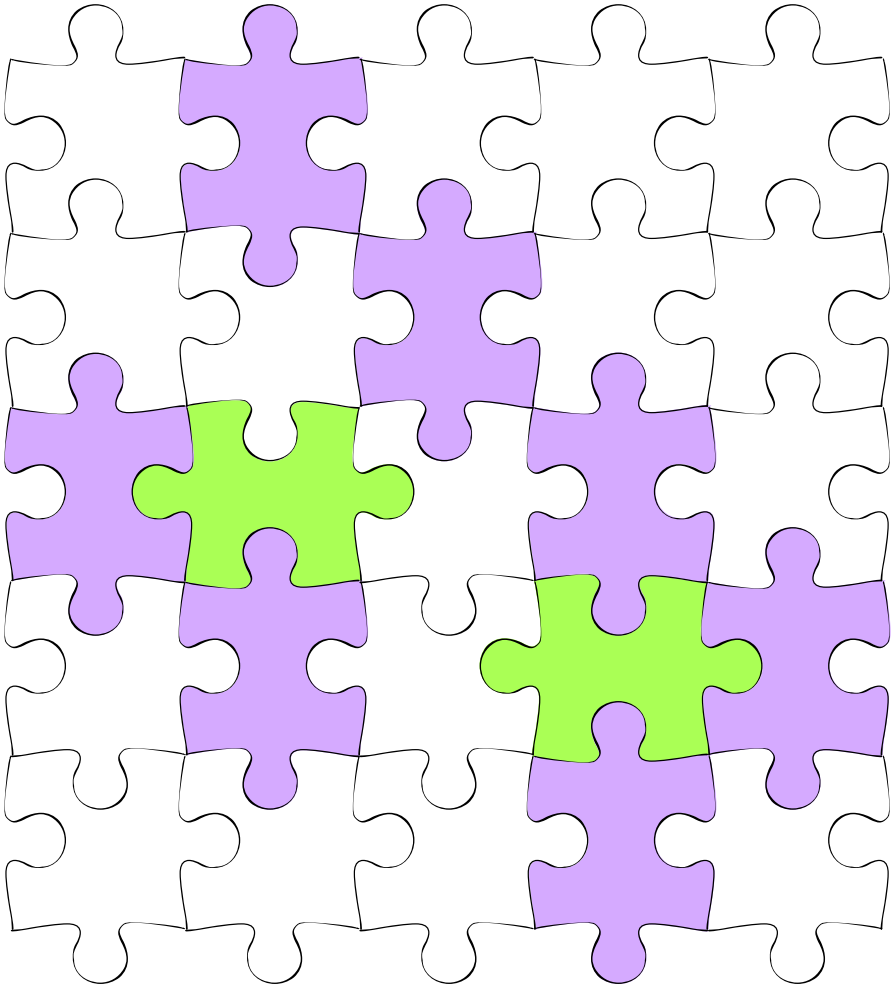
6-vertex model

Gog



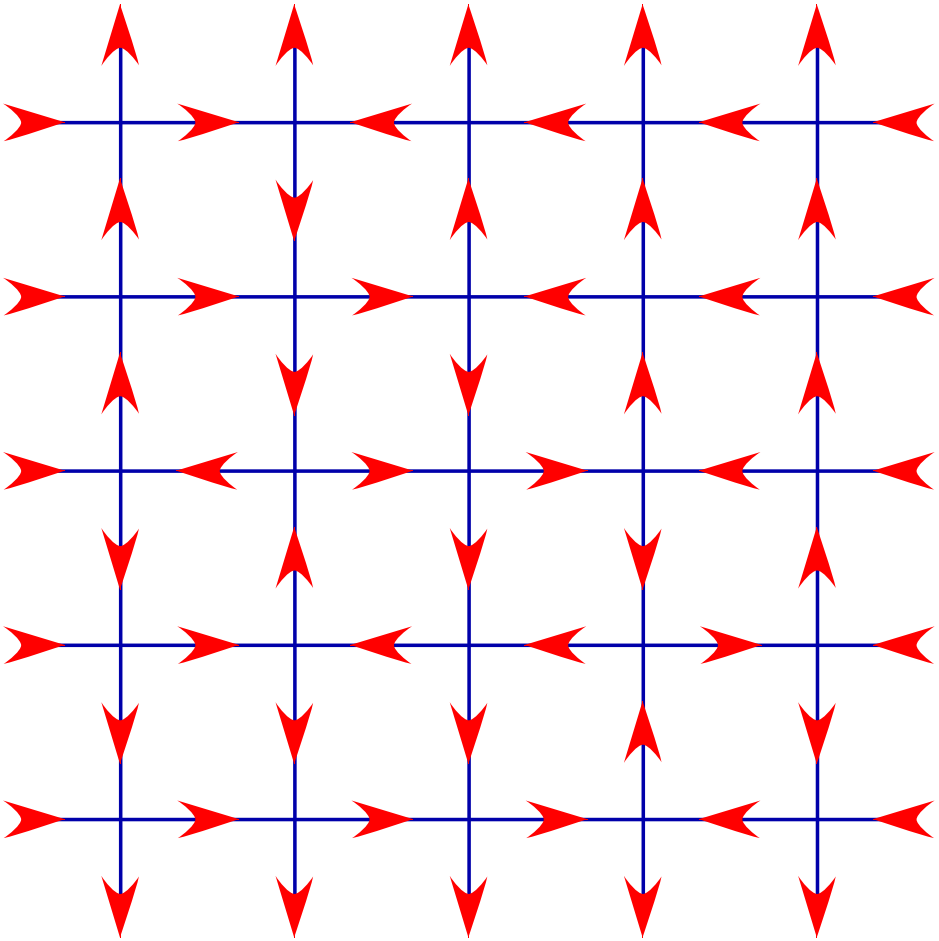
6-vertex model

Gog



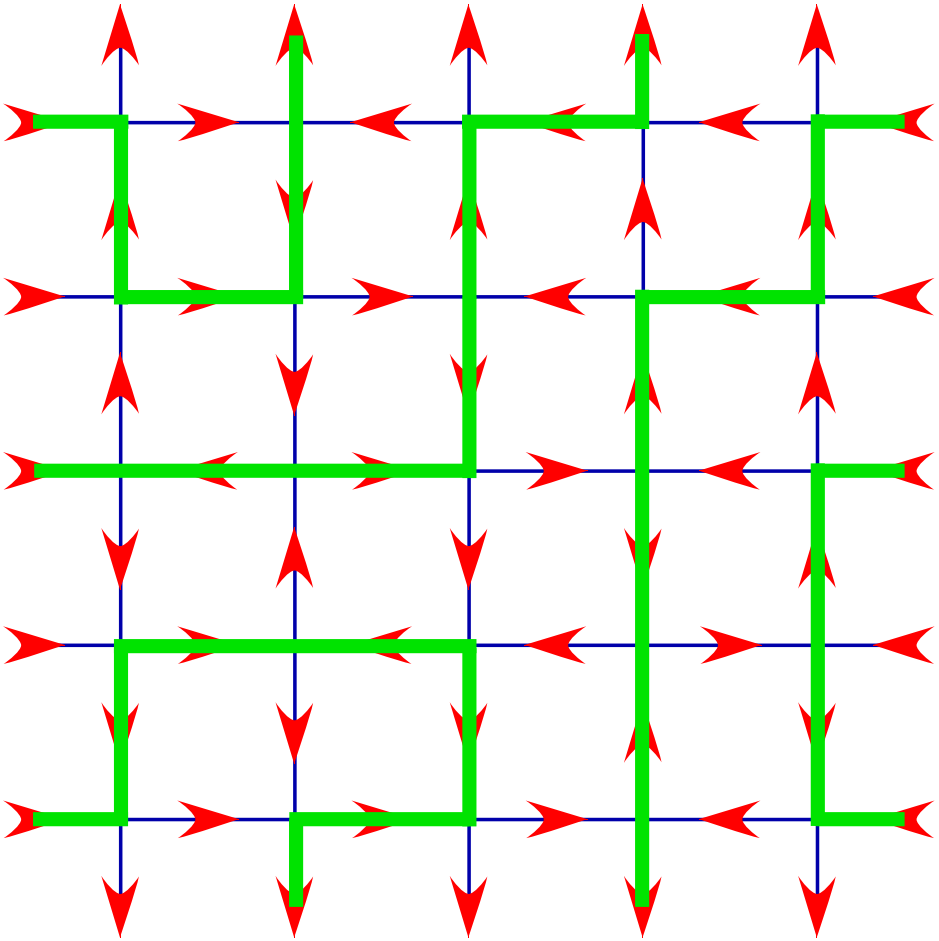
6-vertex model

Gog

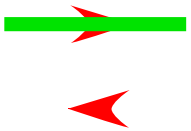


loop model

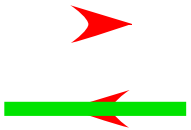
Gog



even coordinates

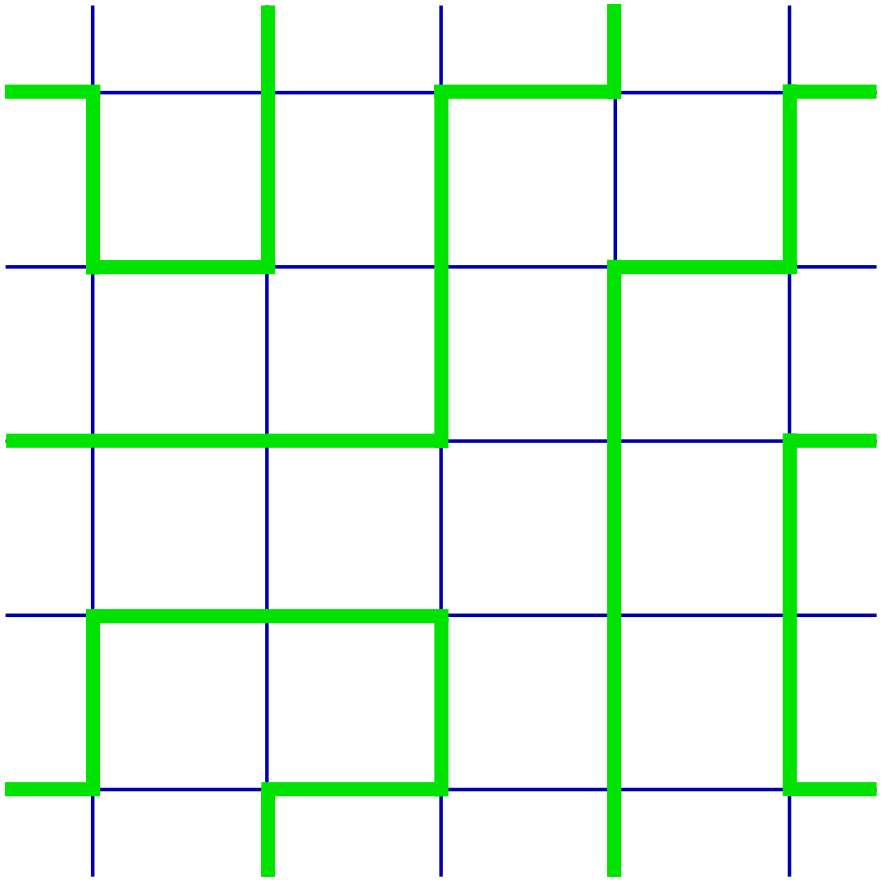


odd coordinates



loop model

Gog



loop model

Gog

0	1	0	0	0
0	0	1	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0

Gog

Gog

0	1	0	0	0
0	0	1	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0

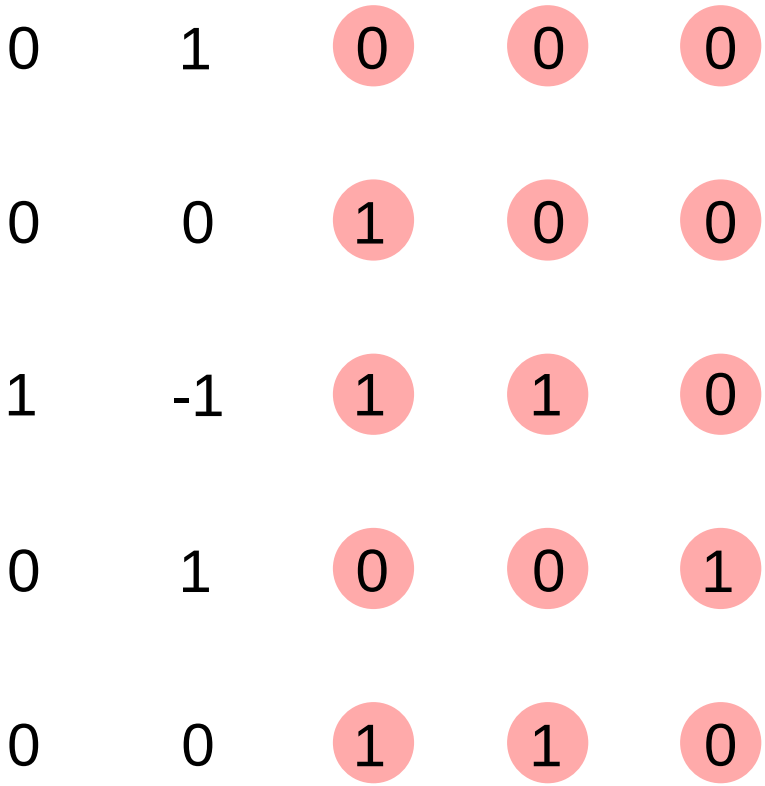
Gog

Gog

0	1	0	0	0
0	0	1	0	0
1	-1	0	1	0
0	1	0	0	1
0	0	0	1	0

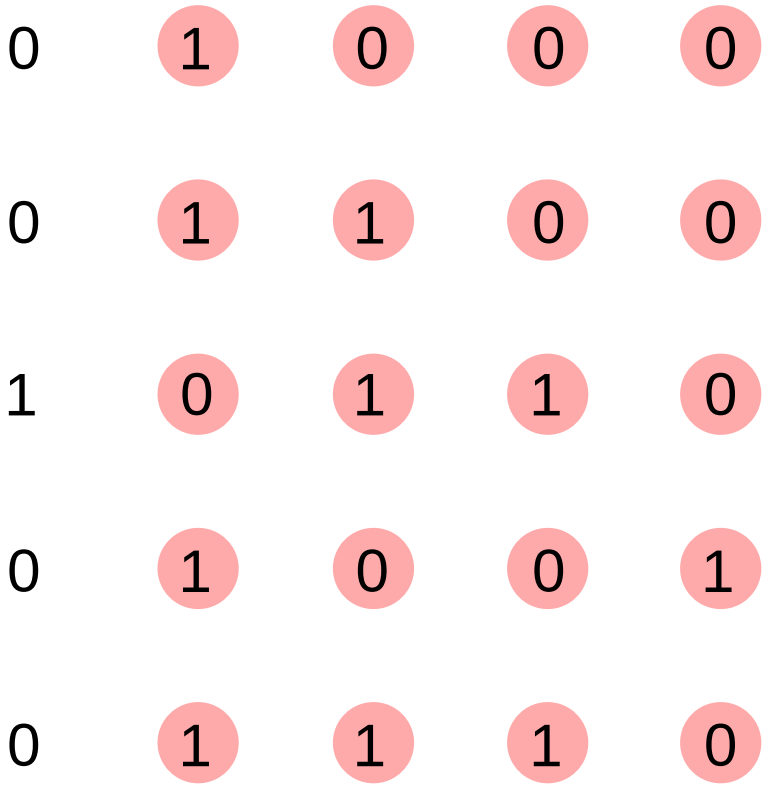
Gog

Gog



Gog

Gog



Gog

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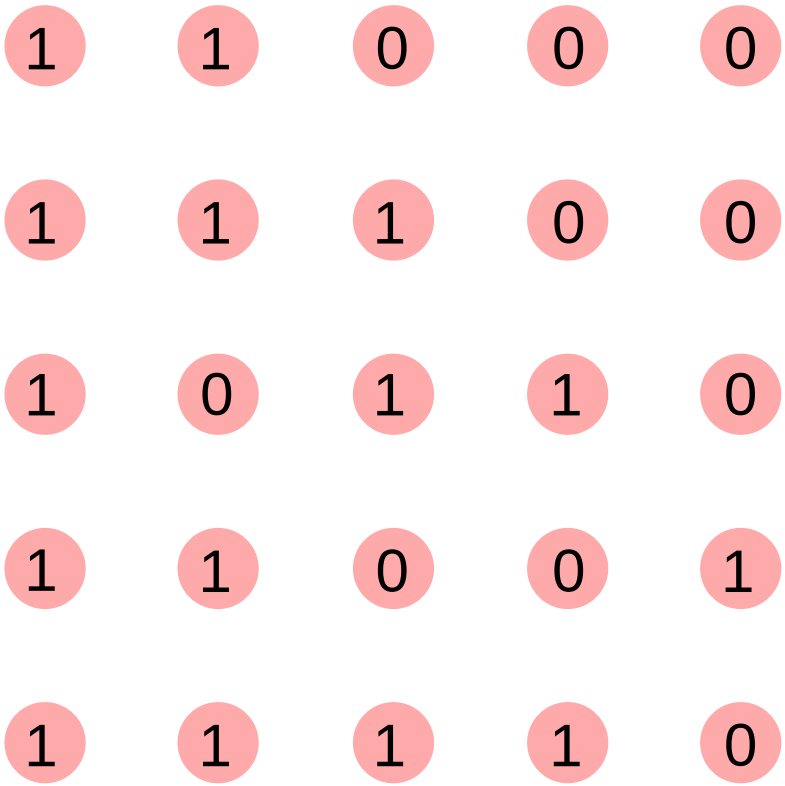


Gog

Gog

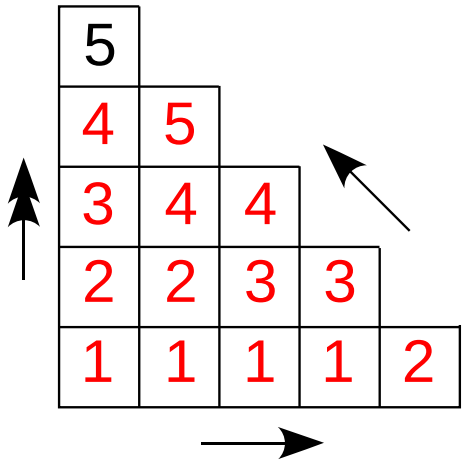
1	1	0	0	0
1	1	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

5				
4	5			
3	4	4		
2	2	3	3	
1	1	1	1	2

Gog

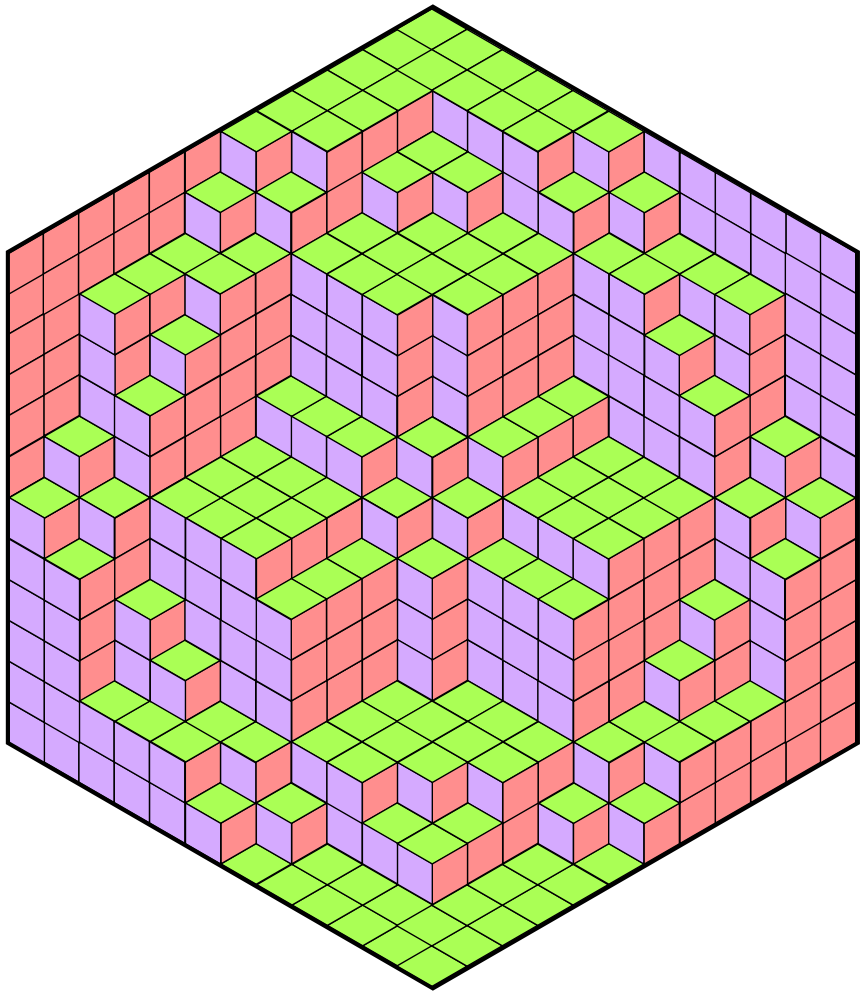
Gog

1	1	0	0	0
1	1	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0



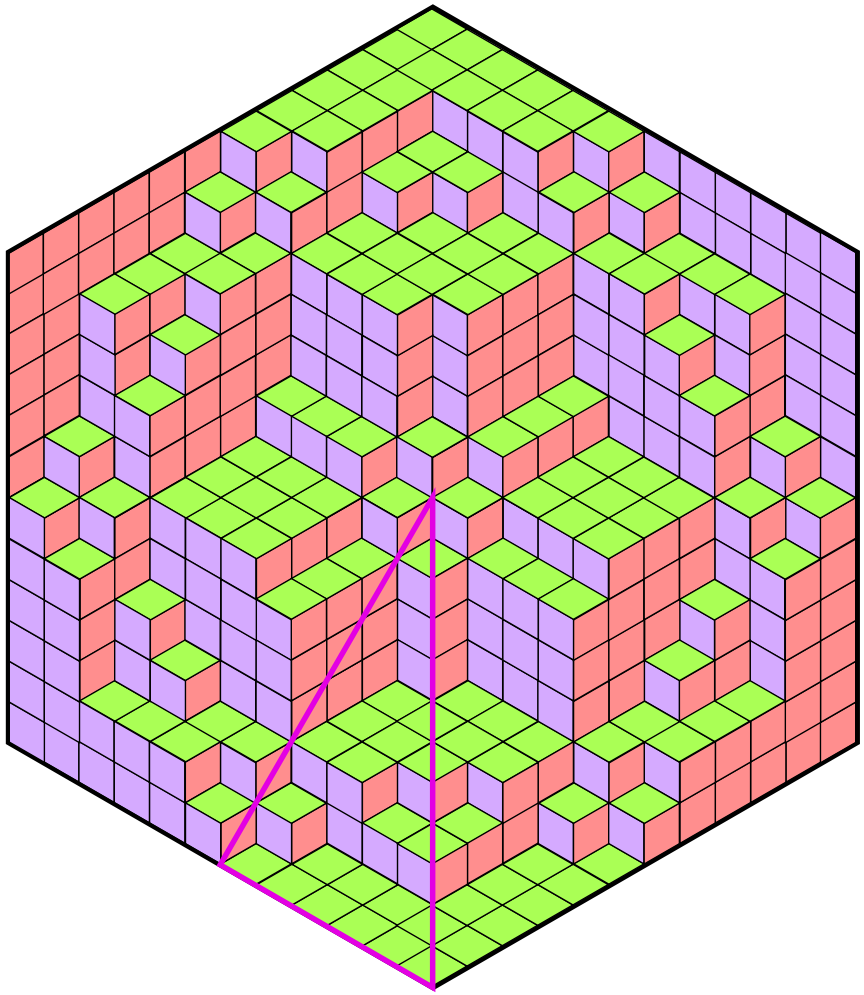
Gog

Magog



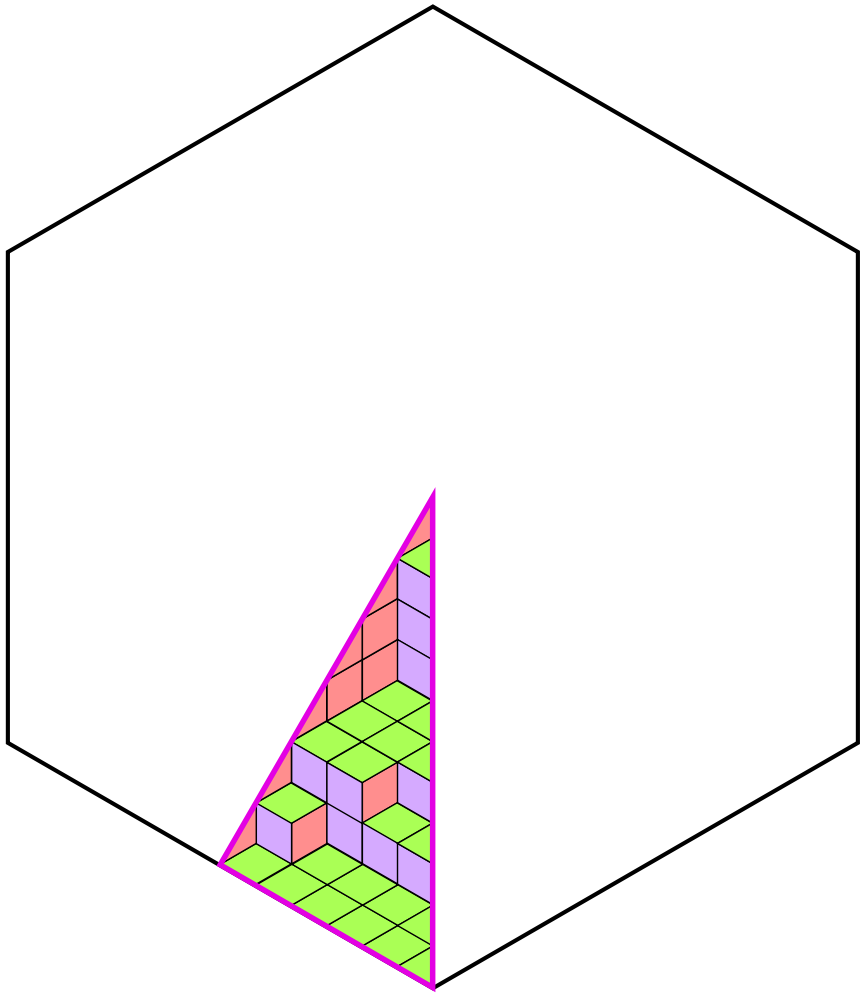
Totally symmetric self-complementary plane partitions

Magog



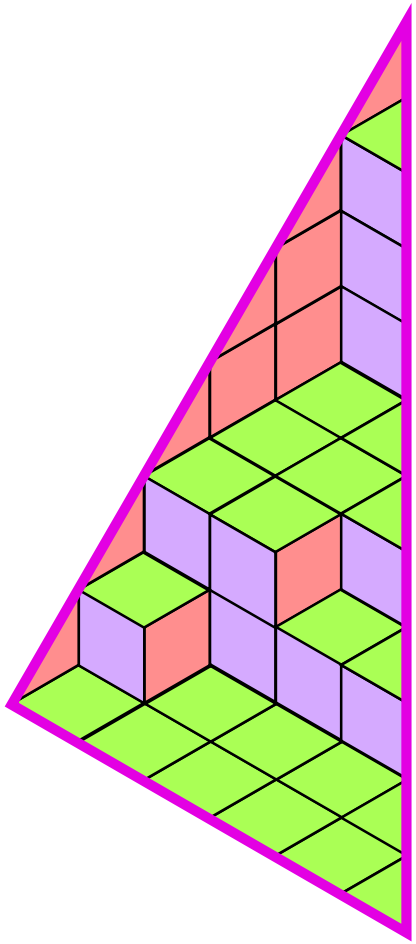
Totally symmetric self-complementary plane partitions

Magog



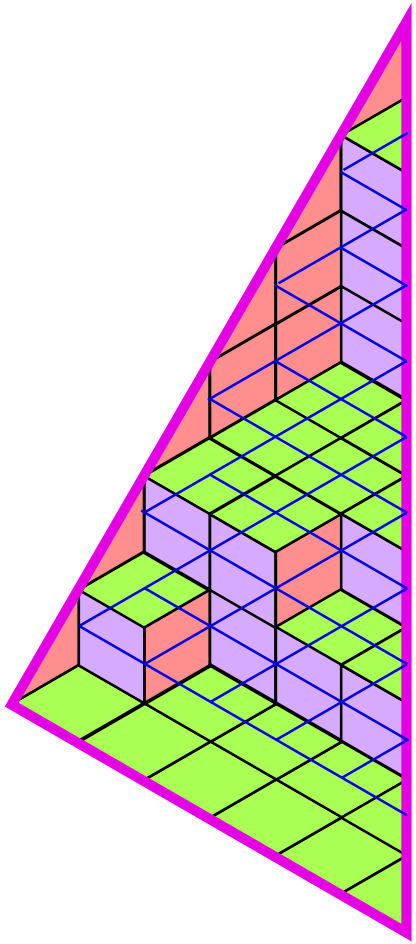
Totally symmetric self-complementary plane partitions

Magog



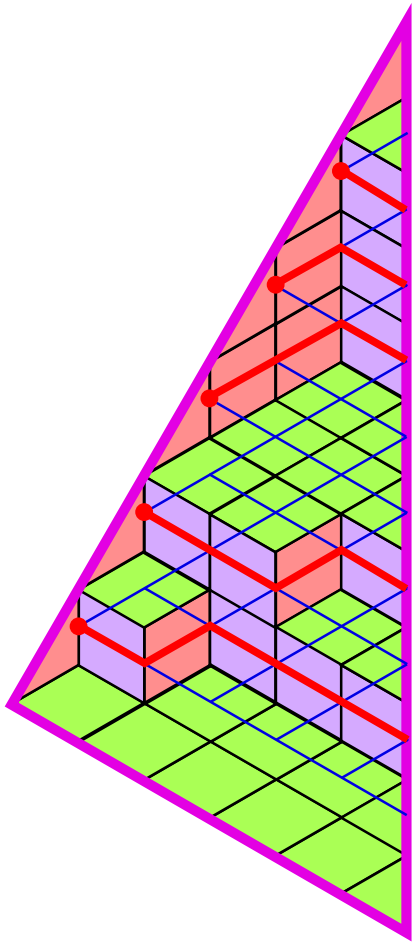
Non intersecting lattice paths

Magog



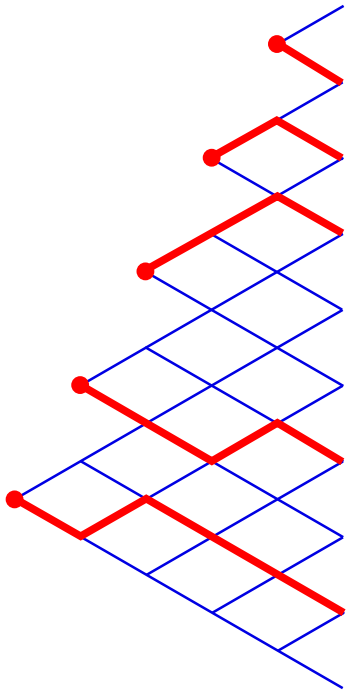
Non intersecting lattice paths

Magog



Non intersecting lattice paths

Magog



Non intersecting lattice paths

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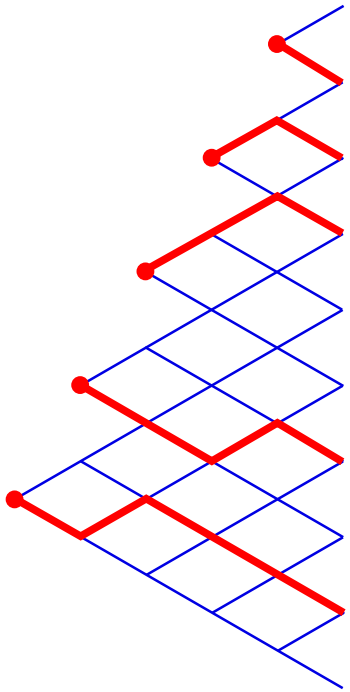
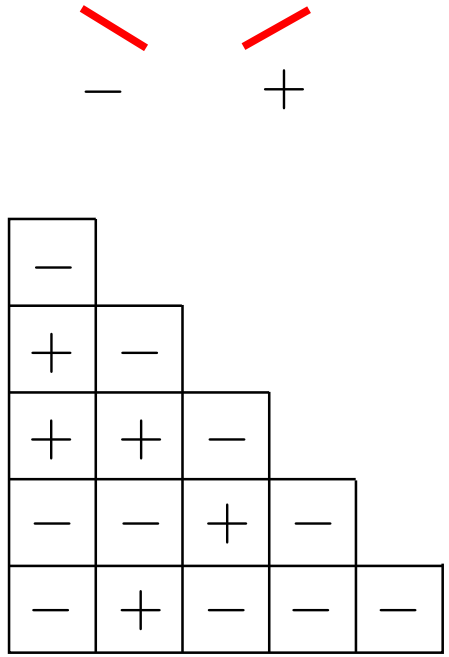
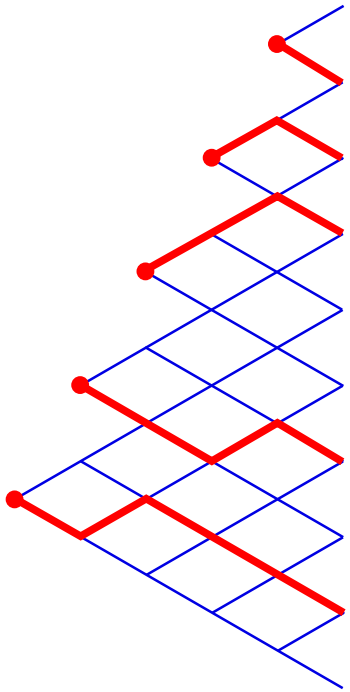
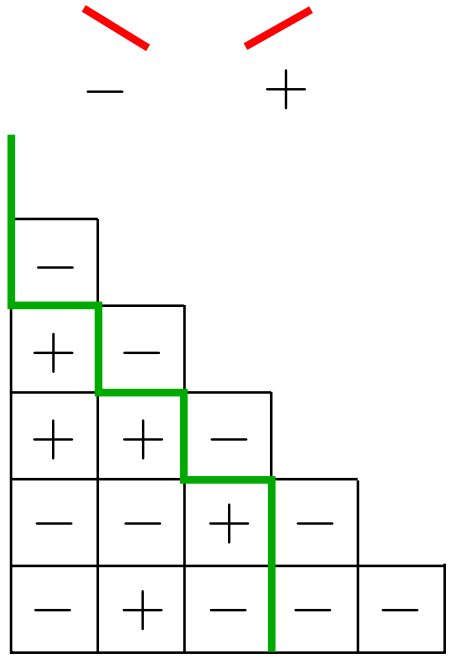




Magog

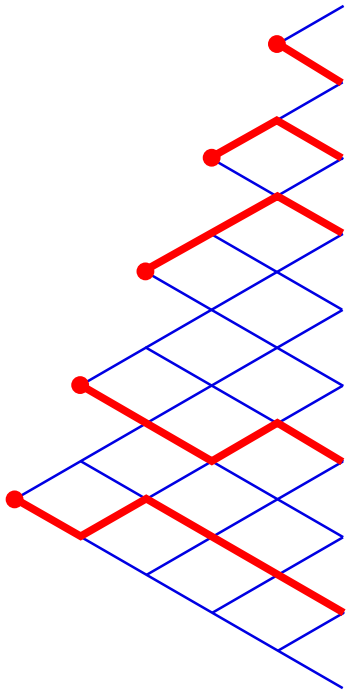
Magog

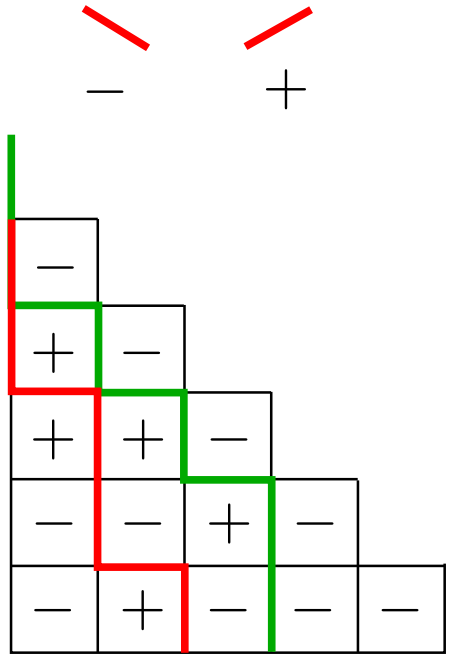




Magog

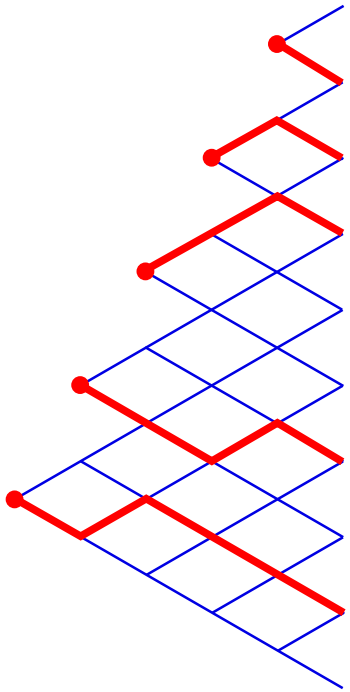
Magog

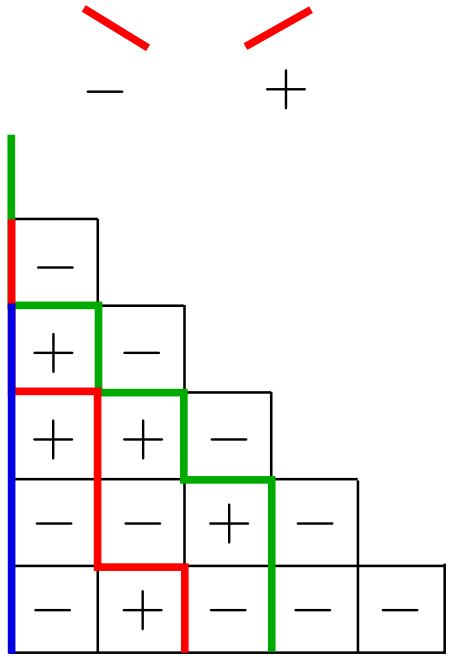




Magog

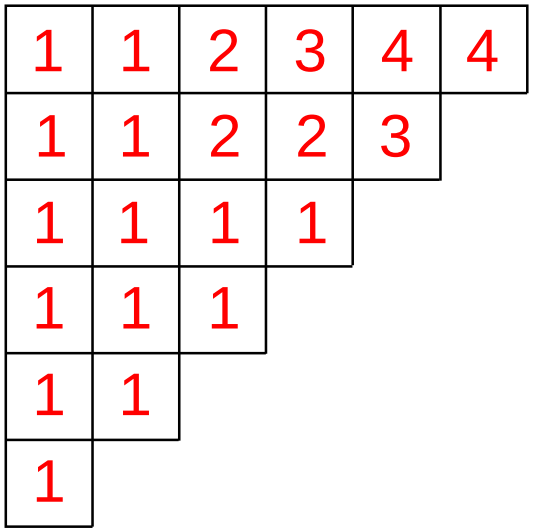
Magog

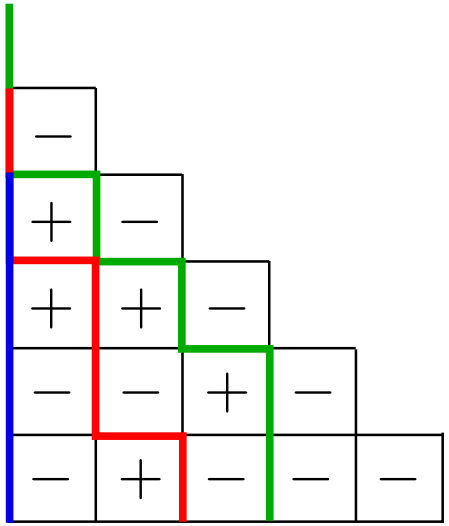




Magog

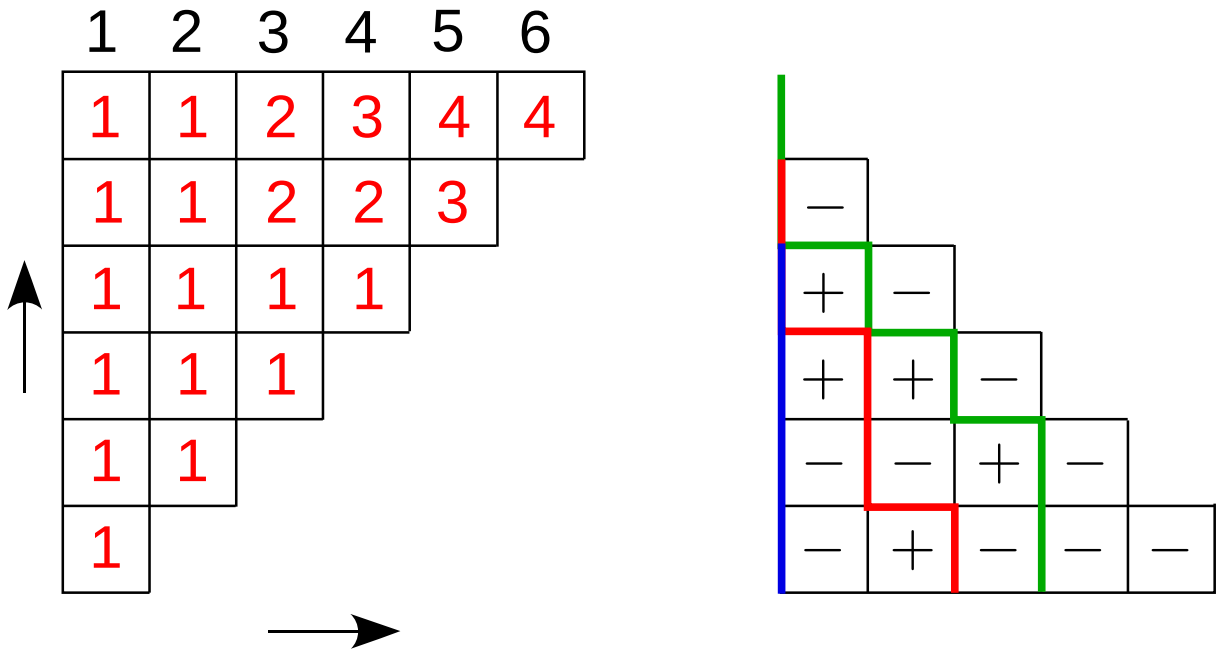
Magog





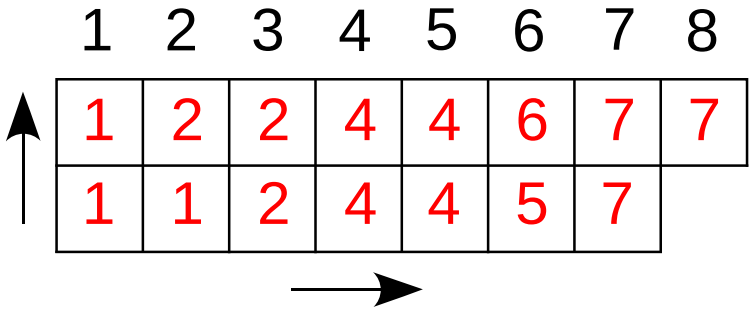
Magog

Magog

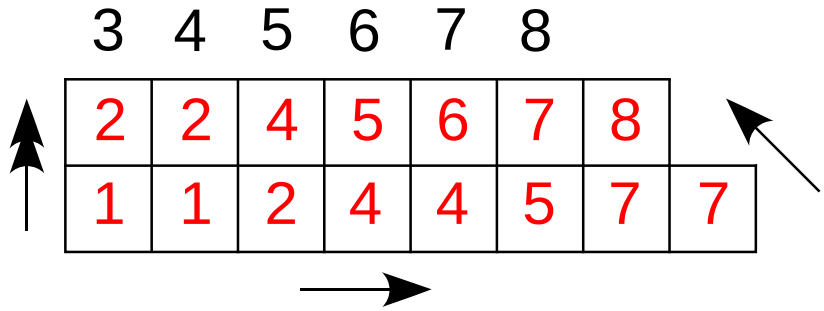


Magog

Trapezoids

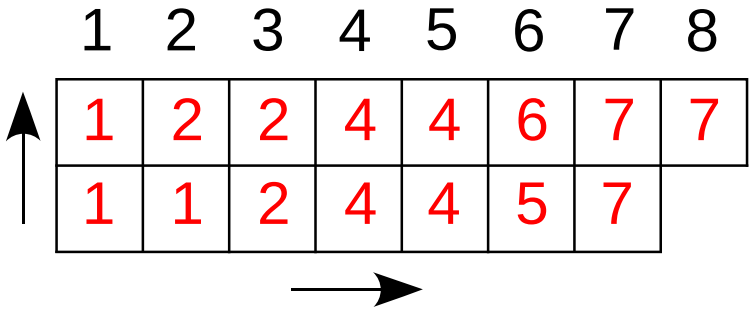


$(8, 2)$ Magog trapezoid



$(8, 2)$ Gog trapezoid

Trapezoids



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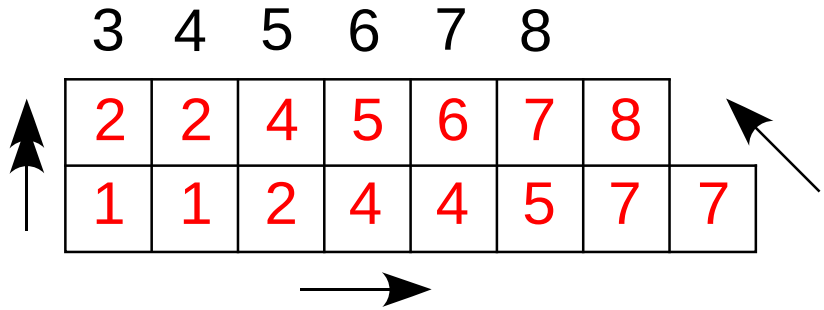
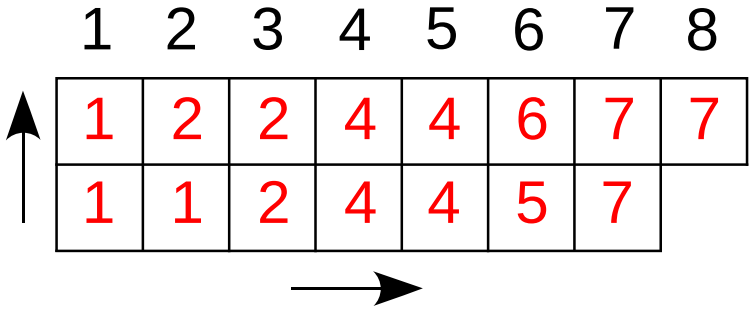


$(8,2)$ Gog trapezoid

Theorem (Zeilberger '96)

(n,k) Magog trapezoids and (n,k) Gog trapezoids are equinumerous.

Trapezoids



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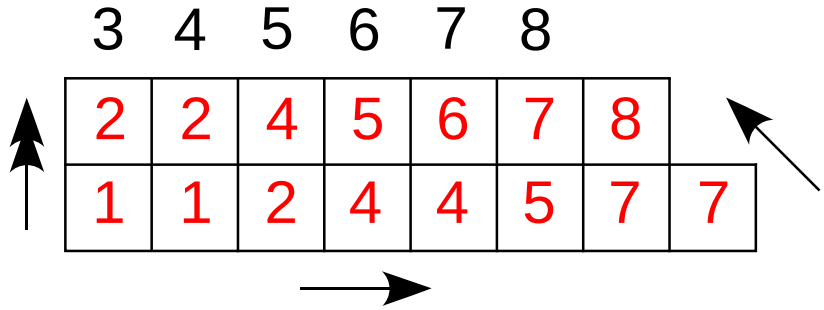

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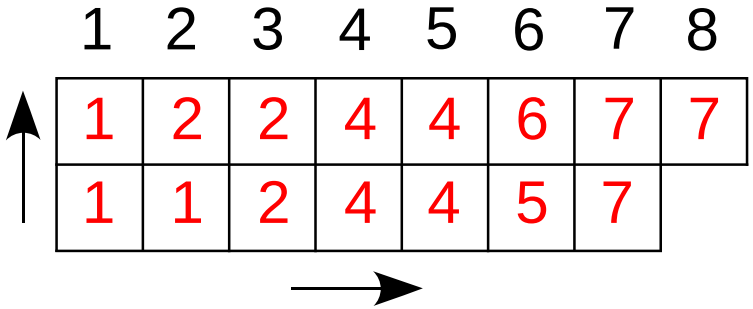
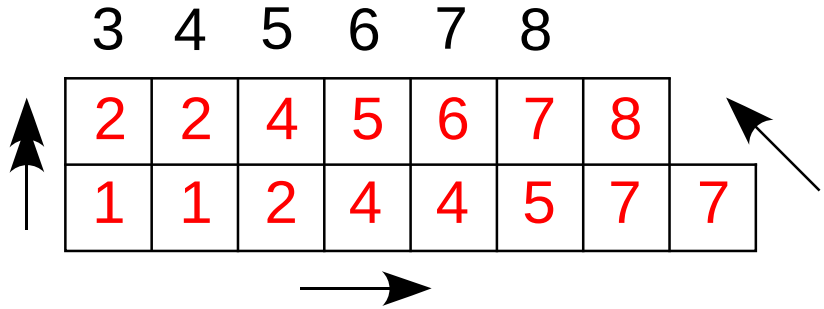
- $k = 1$
- Bijection by Krattenthaller *matching refined statistics*

Trapezoids



$(8,2)$ Magog trapezoid



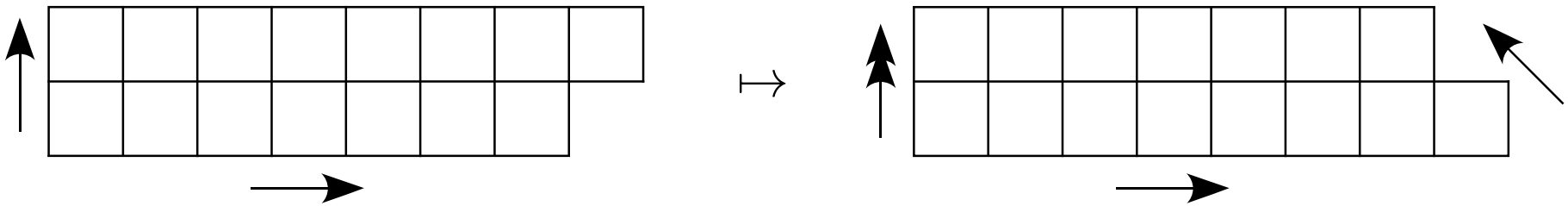
$(8,2)$ Gog trapezoid

Theorem (Zeilberger '96)

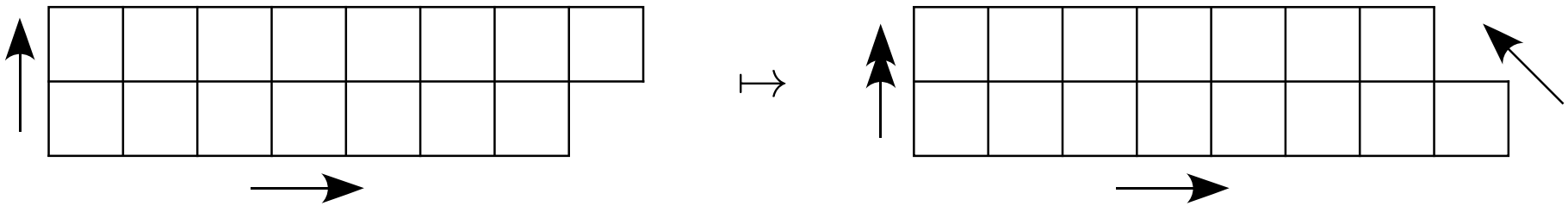
(n,k) Magog trapezoids and (n,k) Gog trapezoids are equinumerous.

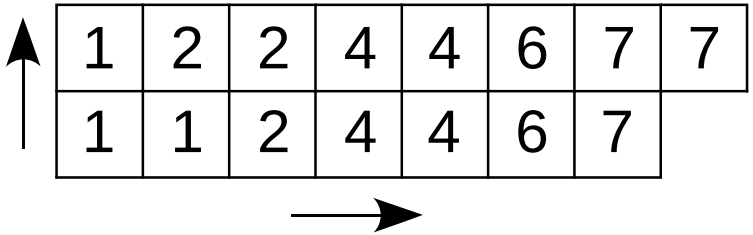
- $k = 1$
  - Bijection by Krattenthaller *matching refined statistics*
- $k = 2$
  - Bijection by Biane & Cheballah '12
  - This talk

From Magog to Gog

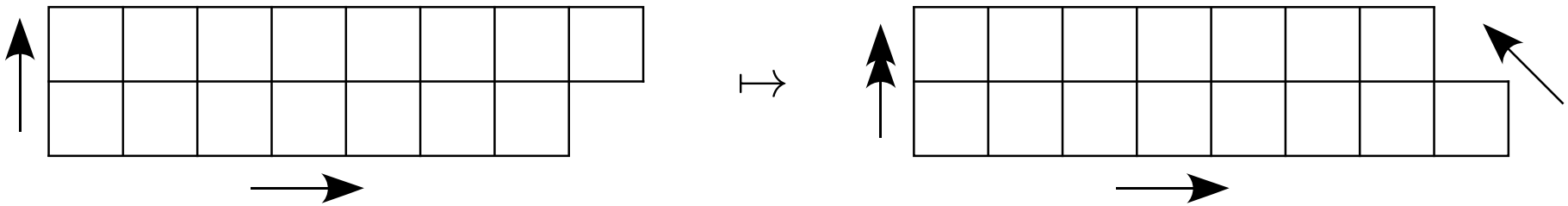


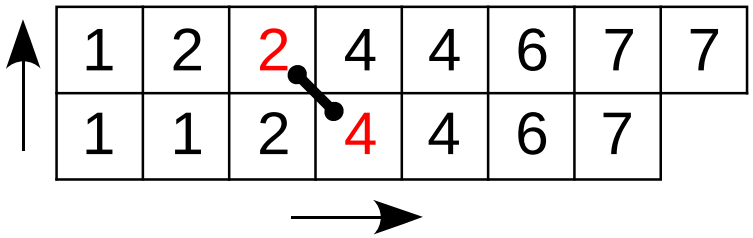
From Magog to Gog

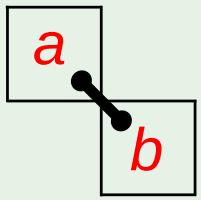




From Magog to Gog

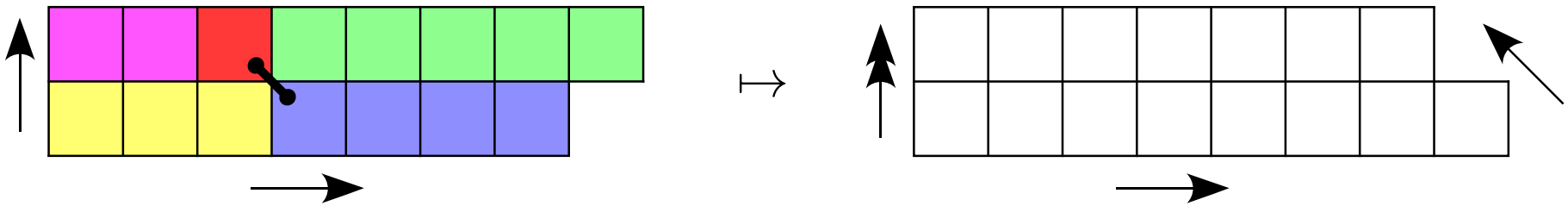


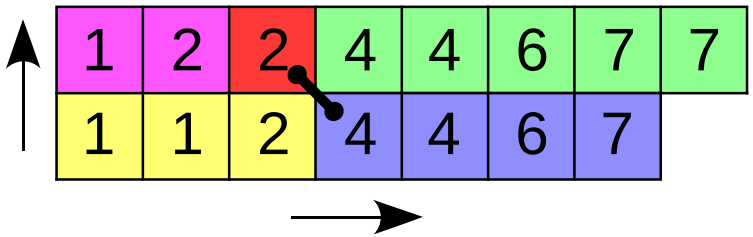


leftmost 

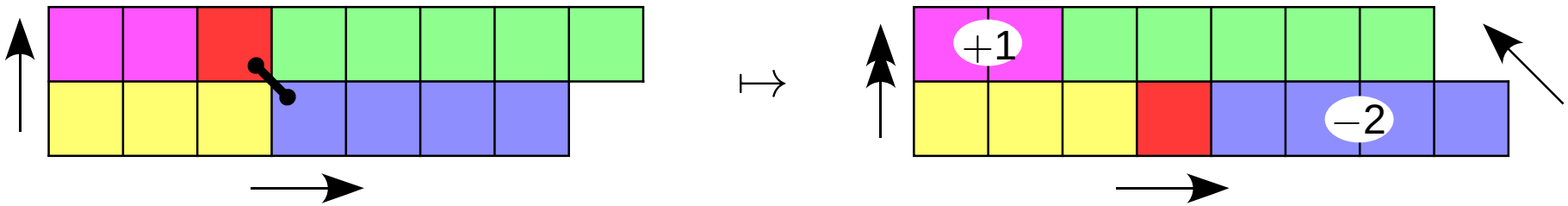
such that $b > a + 1$

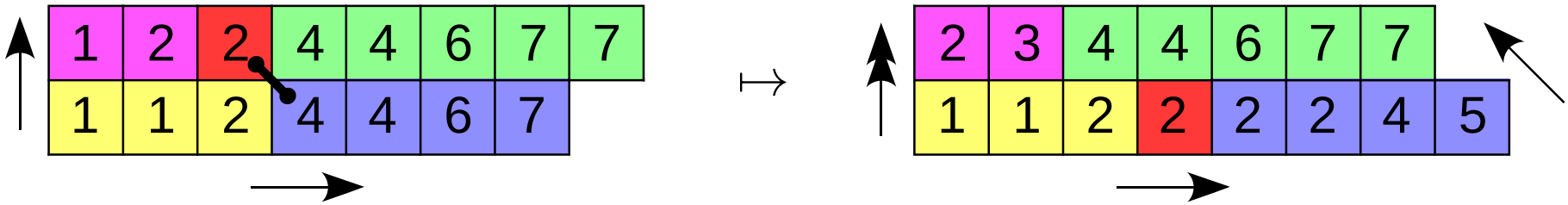
From Magog to Gog



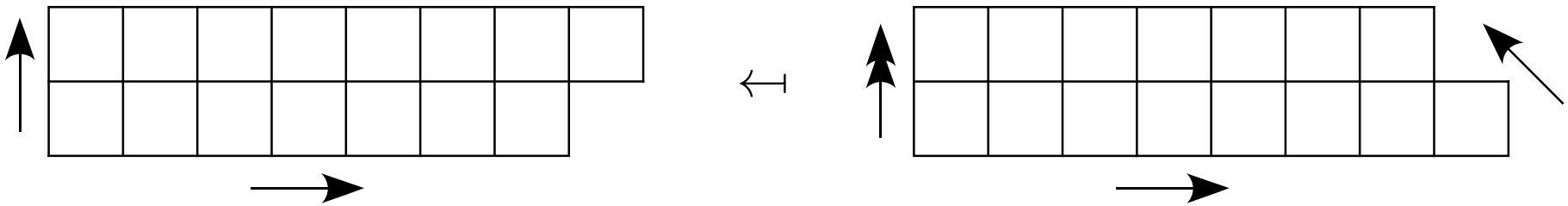


From Magog to Gog

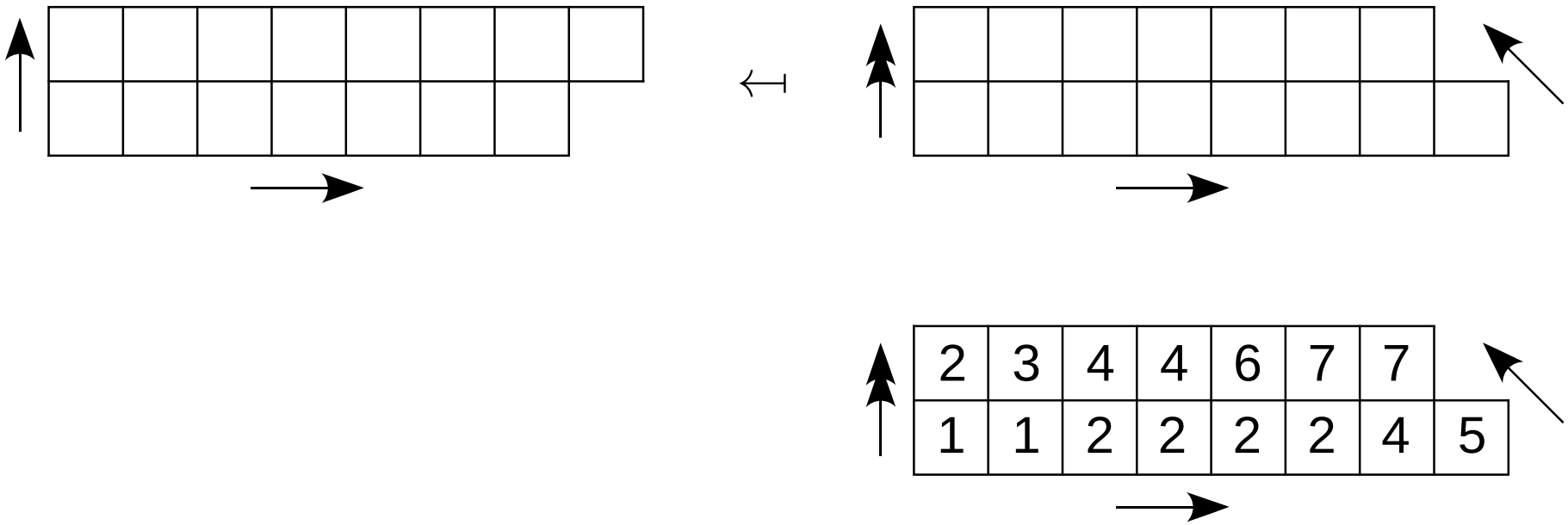




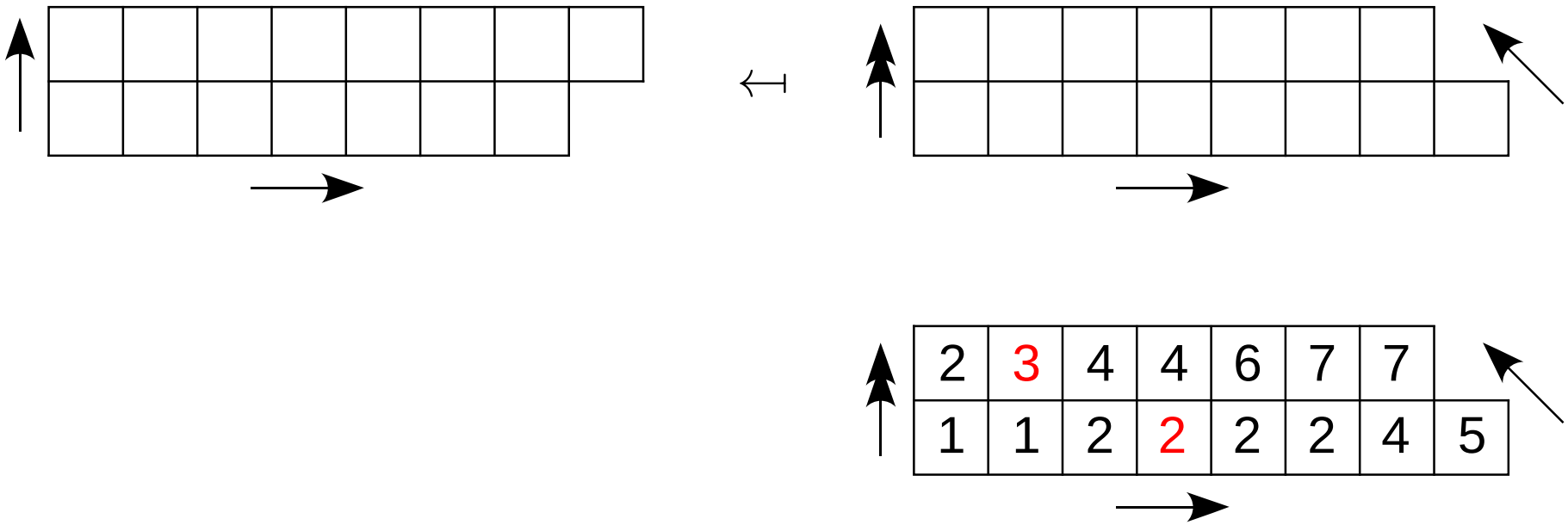
From Gog to Magog

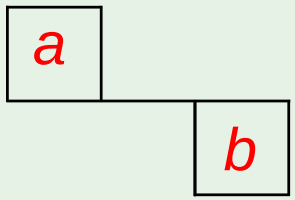


From Gog to Magog

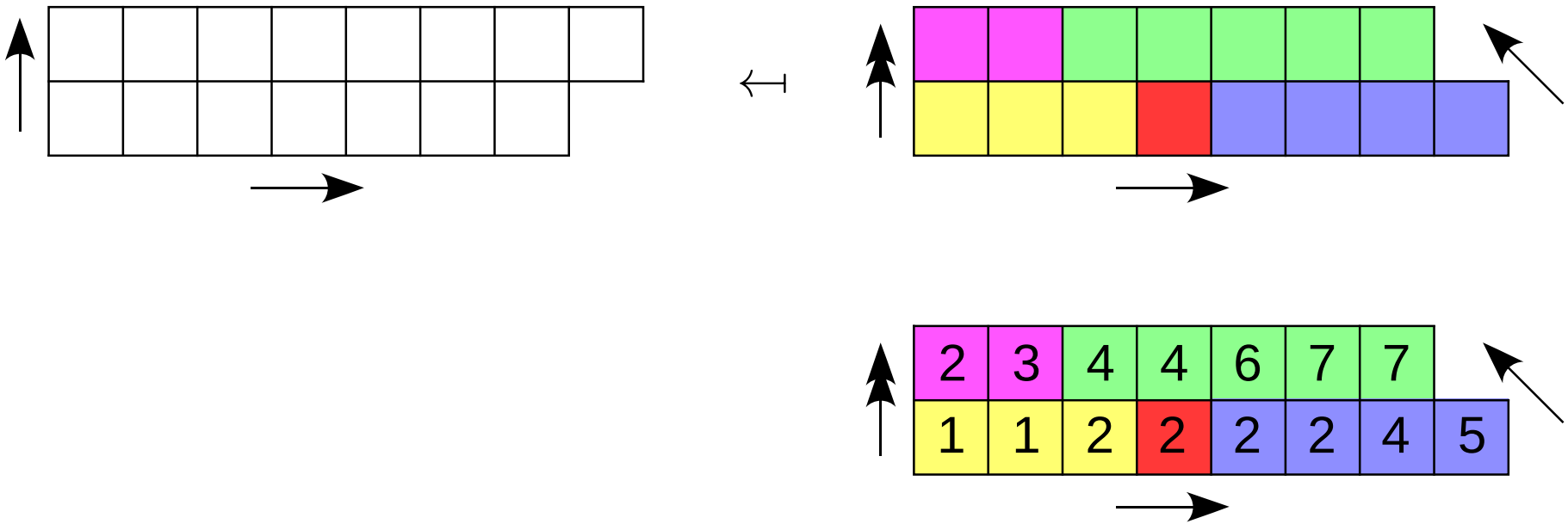


From Gog to Magog

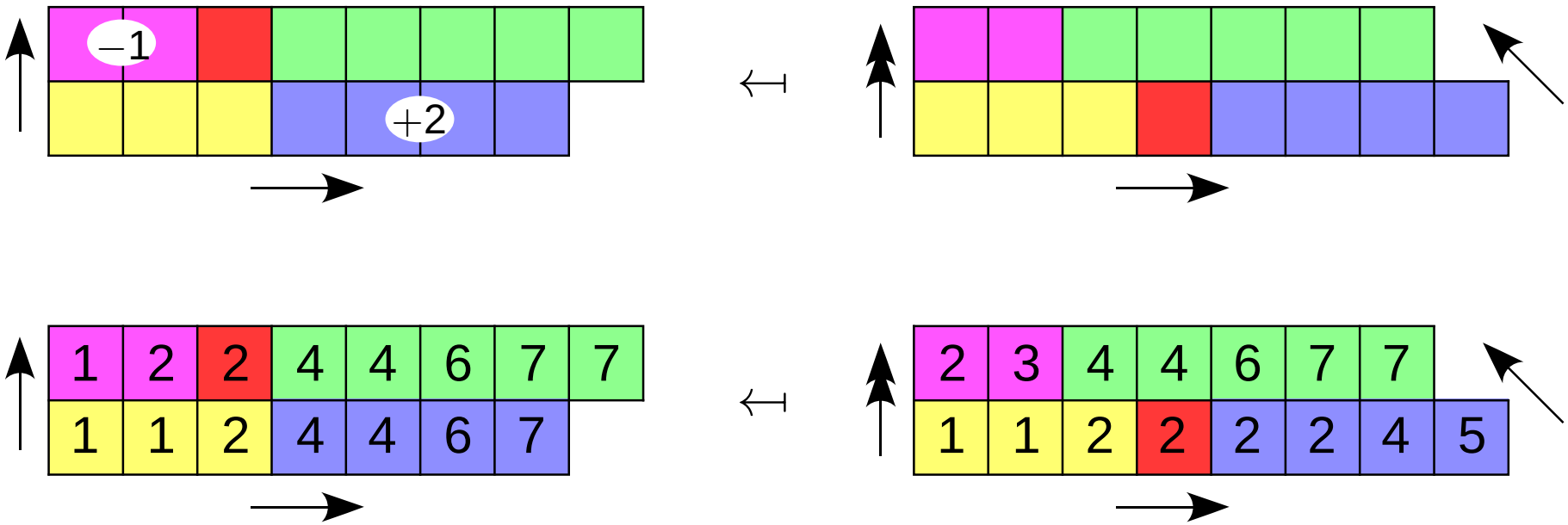


rightmost  *such that* $a \leq b + 1$

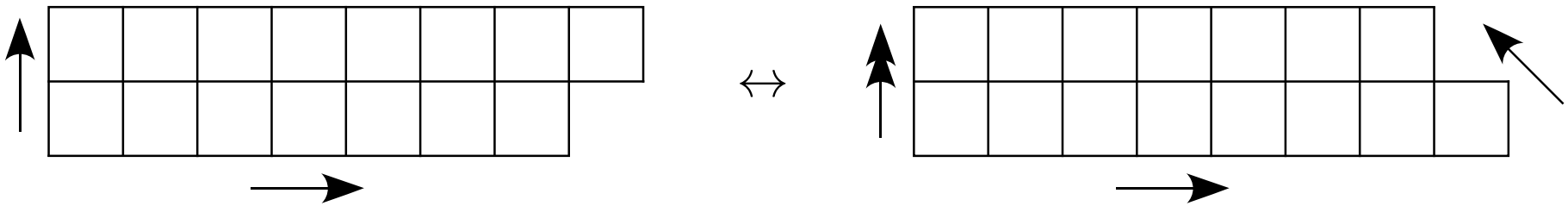
From Gog to Magog

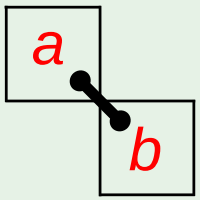


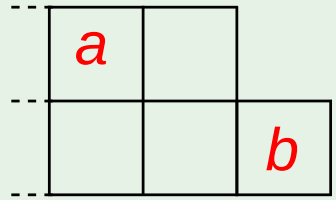
From Gog to Magog



Degenerate case

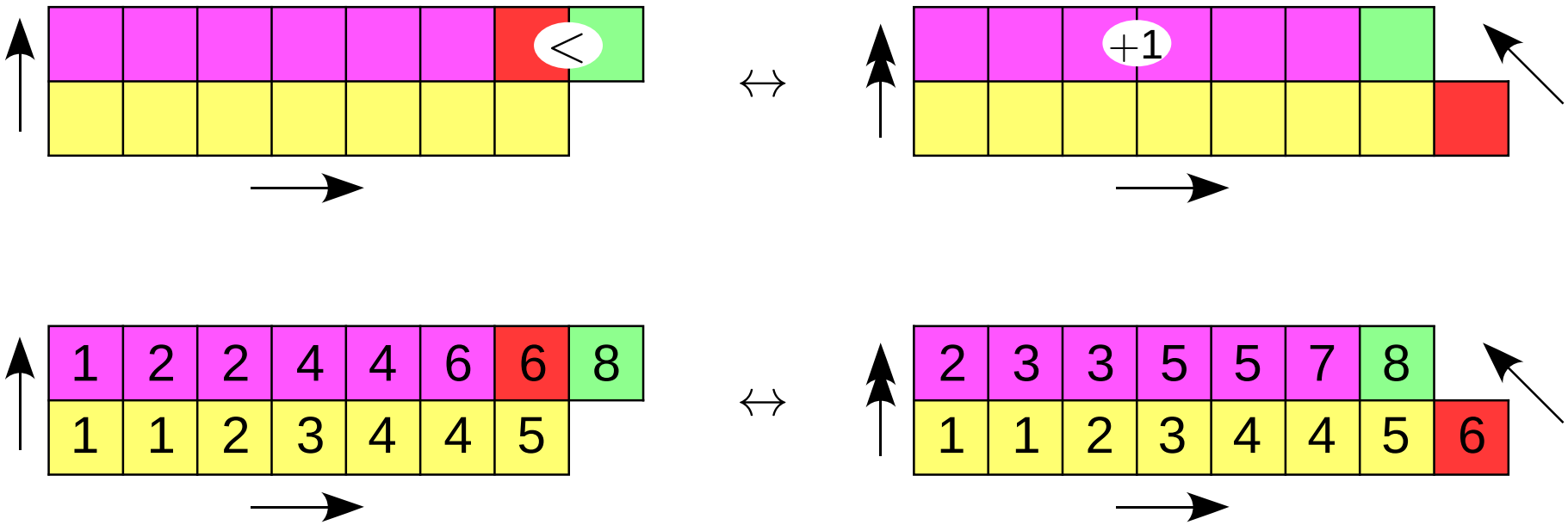


no 

with $b > a + 1$ $\leftrightarrow$ 

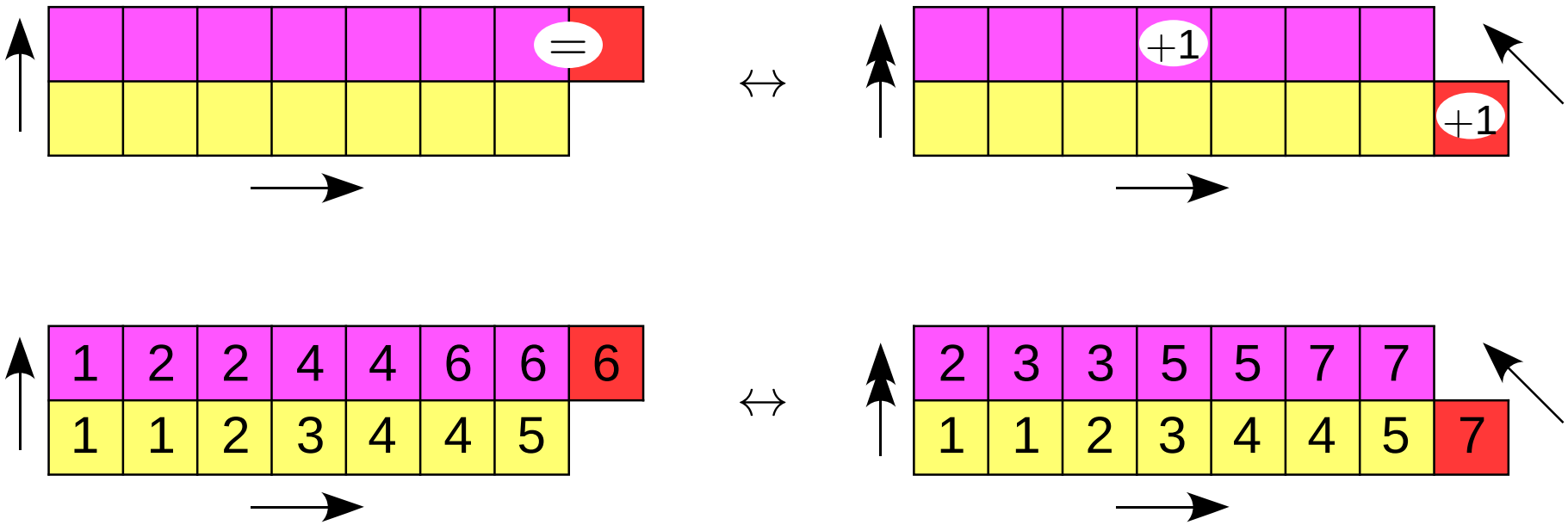
with $a \leq b + 1$

Degenerate case



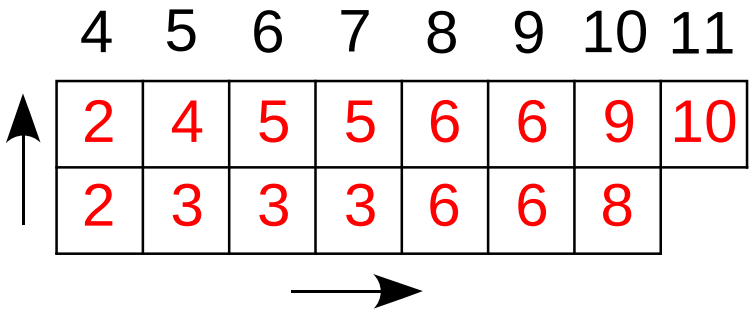
first possibility

Degenerate case

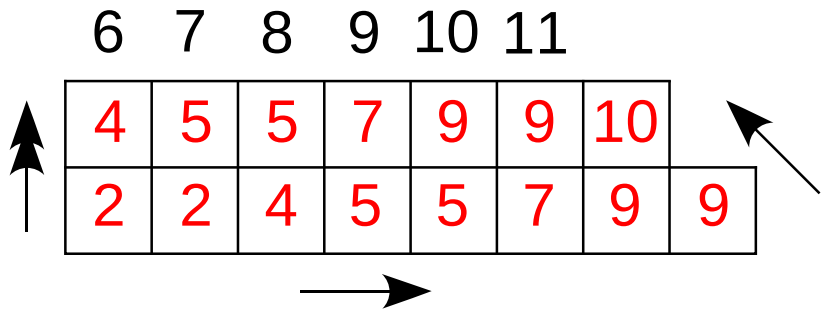


second possibility

Extension to $(\ell, n, 2)$ trapezoids

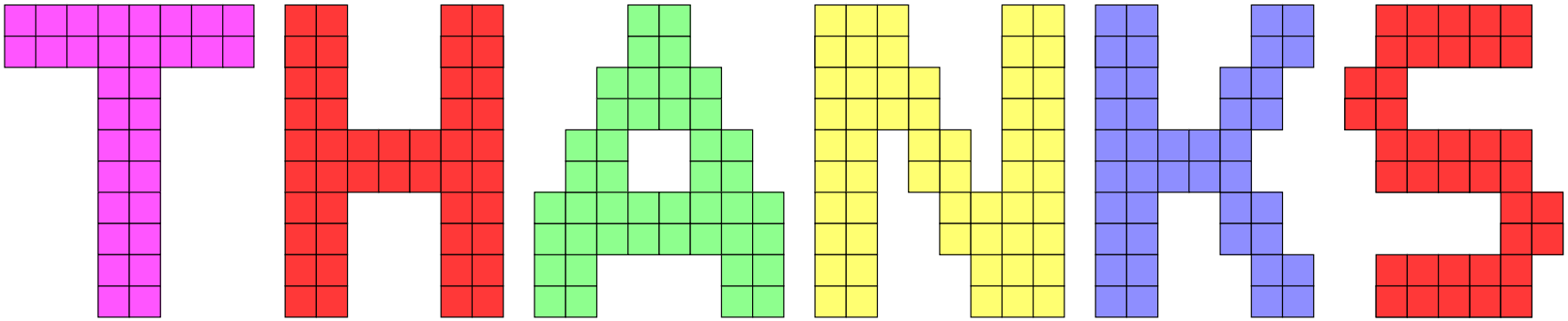


$(3, 8, 2)$ Magog trapezoid



$(3, 8, 2)$ Gog trapezoid

The previous bijection can be trivially extended to $(\ell, n, 2)$ *trapezoids.*


THANKS