Szlenk indices of convex hulls

joint work with Gilles Lancien and Matías Raja

Tony Procházka

Université de Franche-Comté

Luminy, December 2015

The Coauthors

Figure: Gilles

Figure: Matías

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*, \\ \text{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K :\ \forall\ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*,$$
$$\mathrm{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \ \forall \ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \mathrm{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \ \forall \ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \mathrm{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,
- $s^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \ \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*,$$
$$\text{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,
- $s^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.

Then $Sz(X, \varepsilon) := \inf\{\alpha : \ s^\alpha_\varepsilon(B_{X^*}) = \emptyset\}$ with $\inf \emptyset = +\infty$.

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K :\ \forall\ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \operatorname{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,
- $s^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.

Then $Sz(X, \varepsilon) := \inf\{\alpha :\ s^\alpha_\varepsilon(B_{X^*}) = \emptyset\}$ with $\inf \emptyset = +\infty$.

Finally the Szlenk index of X is $Sz(X) = \sup_{\varepsilon>0} Sz(X, \varepsilon)$.

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \ \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*, \\ \text{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,
- $s^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.

Then $Sz(X,\varepsilon) := \inf\{\alpha : \ s^\alpha_\varepsilon(B_{X^*}) = \emptyset\}$ with $\inf \emptyset = +\infty$.

Finally the Szlenk index of X is $Sz(X) = \sup_{\varepsilon>0} Sz(X,\varepsilon)$.

Fact

$Sz(X) = 1$ *iff* $\dim X < \infty$

Definition

X ... a Banach space, $K \subset X^*$ ... w*-compact, $\varepsilon > 0$.
Define

$$s'_\varepsilon(K) := \{x^* \in K : \ \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*, \\ \text{diam}(K \cap U) \geq \varepsilon\}$$ ... Szlenk derivation of K

Define $s^\alpha_\varepsilon(K)$ for α ordinal as follows:

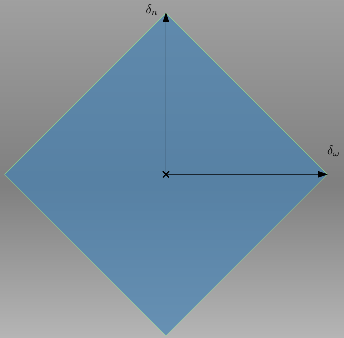
- $s^0_\varepsilon(K) := K$,
- $s^{\alpha+1}_\varepsilon(K) := s'_\varepsilon(s^\alpha_\varepsilon(K))$,
- $s^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.

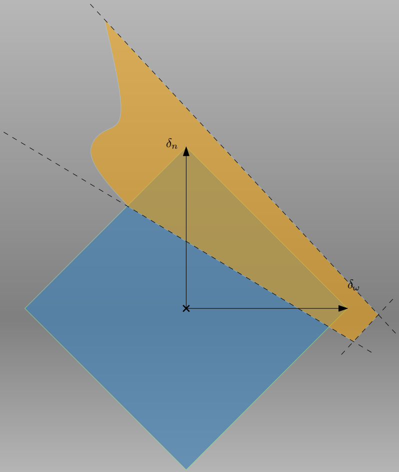
Then $Sz(X,\varepsilon) := \inf\{\alpha : \ s^\alpha_\varepsilon(B_{X^*}) = \emptyset\}$ with $\inf \emptyset = +\infty$.

Finally the Szlenk index of X is $Sz(X) = \sup_{\varepsilon>0} Sz(X,\varepsilon)$.

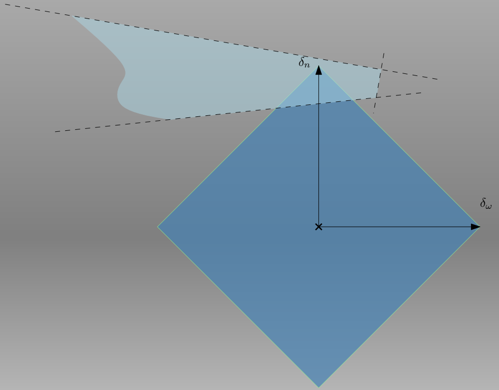
Fact

$Sz(X) = 1$ *iff* $\dim X < \infty$ *iff* $Sz(X) < \omega$.

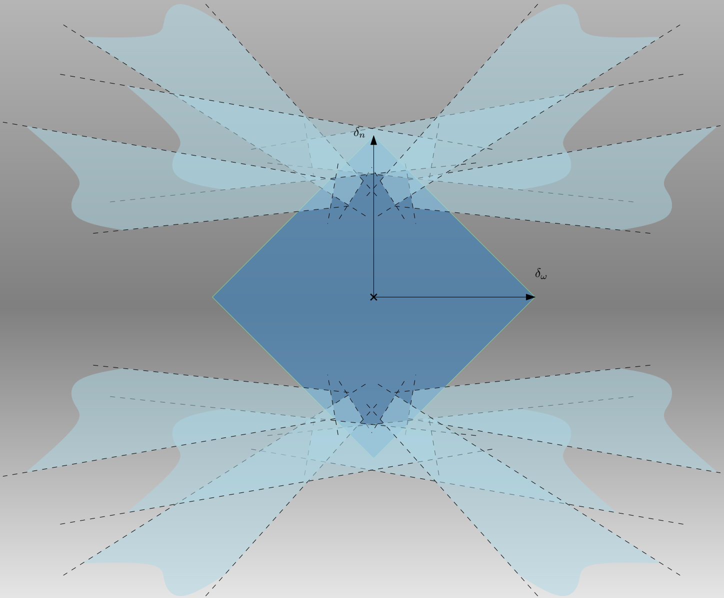

δ_n
δ_ω



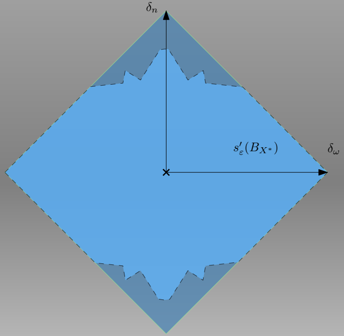
δ_n
δ_ω

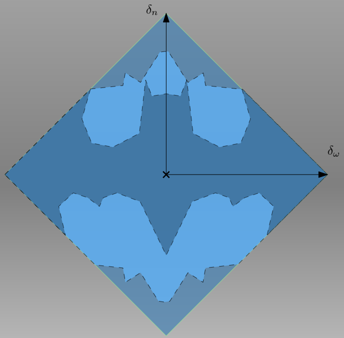


δ_n
δ_ω

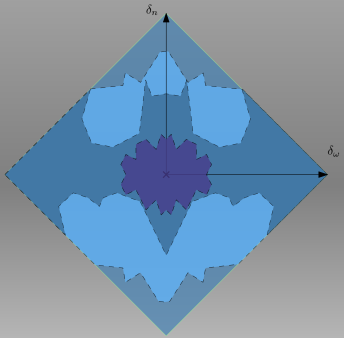


δ_n
δ_ω


δ_n
$s'_\varepsilon(B_{X^*})$
δ_ω



δ_n
δ_ω



δ_n
δ_ω

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.
- More generally, if $X \subseteq Y$, then $Sz(X) \leq Sz(Y)$.

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.
- More generally, if $X \subseteq Y$, then $Sz(X) \leq Sz(Y)$.
- $Sz(X) < +\infty$ iff X is Asplund.

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.
- More generally, if $X \subseteq Y$, then $Sz(X) \leq Sz(Y)$.
- $Sz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Sz(X) < +\infty$ iff $Sz(X) < \omega_1$.

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.
- More generally, if $X \subseteq Y$, then $Sz(X) \leq Sz(Y)$.
- $Sz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Sz(X) < +\infty$ iff $Sz(X) < \omega_1$.
- $\sup\{Sz(X) :\ X$ separable reflexive $\} = \omega_1$ (Szlenk '68)

Facts about the Szlenk index

- If $X \simeq Y$, then $Sz(X) = Sz(Y)$.
- More generally, if $X \subseteq Y$, then $Sz(X) \leq Sz(Y)$.
- $Sz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Sz(X) < +\infty$ iff $Sz(X) < \omega_1$.
- $\sup\{Sz(X) : \ X$ separable reflexive $\} = \omega_1$ (Szlenk '68)
- $\forall\ X$ Banach $\exists\ \alpha$ ordinal such that $Sz(X) = \omega^\alpha$ (Lancien '96)

UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

UKK* norms

Definition

The dual norm on X^* is *w^*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0 \,\exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK* then $Sz(X) = \omega$.

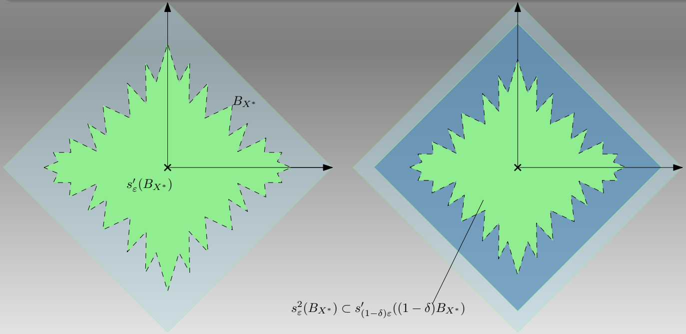
UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK then $Sz(X) = \omega$.*



UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK* then $Sz(X) = \omega$.

- is the case for uniformly smooth spaces.

UKK^* norms

Definition

The dual norm on X^* is *w^*-uniformly Kadets-Klee (UKK^*)* if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK^* then $Sz(X) = \omega$.

- is the case for uniformly smooth spaces.
- is the case for c_0.

UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK* then $Sz(X) = \omega$.

- is the case for uniformly smooth spaces.
- is the case for c_0.(In fact X^* is UKK* iff X is AUS.)

UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK* then $Sz(X) = \omega$.

- is the case for uniformly smooth spaces.
- is the case for c_0.(In fact X^* is UKK* iff X is AUS.)

Theorem (Knaust-Odell-Schlumprecht '99, Godefroy-Kalton-Lancien '01)

If $Sz(X) \leq \omega$, then X admits a norm whose dual norm is UKK*.

UKK* norms

Definition

The dual norm on X^* is *w***-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK* then $Sz(X) = \omega$.

- is the case for uniformly smooth spaces.
- is the case for c_0.(In fact X^* is UKK* iff X is AUS.)

Theorem (Knaust-Odell-Schlumprecht '99, Godefroy-Kalton-Lancien '01)

If $Sz(X) \leq \omega$, then X admits a norm whose dual norm is UKK*.

- An important tool for showing that $Sz(X) \leq \omega$ is invariant under uniform homeomorphisms (G-K-L).

UKK* norms

Definition

The dual norm on X^* is *w*-uniformly Kadets-Klee (UKK*)* if $\forall\, \varepsilon > 0\; \exists\, \delta > 0$ such that $s'_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Fact

If the dual norm on X^* is UKK then $Sz(X) = \omega$.*

- is the case for uniformly smooth spaces.
- is the case for c_0.(In fact X^* is UKK* iff X is AUS.)

Theorem (Knaust-Odell-Schlumprecht '99, Godefroy-Kalton-Lancien '01)

If $Sz(X) \leq \omega$, then X admits a norm whose dual norm is UKK.*

- An important tool for showing that $Sz(X) \leq \omega$ is invariant under uniform homeomorphisms (G-K-L).
- It is still open whether $Sz(X) \leq \omega^\alpha$ is Lipschitz invariant.

Definition

Let $\alpha \in [0, \omega_1)$ be an ordinal. The dual norm on X^* is ω^α-*w***-uniformly Kadets-Klee* (ω^α-UKK*) if

$\forall\, \varepsilon > 0\; \exists\, \delta > 0$ such that $s_\varepsilon^{\omega^\alpha}(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Definition

Let $\alpha \in [0, \omega_1)$ be an ordinal. The dual norm on X^* is ω^α-*w*-uniformly Kadets-Klee* (ω^α-UKK*) if

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $s_\varepsilon^{\omega^\alpha}(B_{X^*}) \subset (1-\delta)B_{X^*}$.

Theorem (Lancien-P-Raja '15)

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm whose dual norm is ω^α-UKK*.

Definition

Let $\alpha \in [0, \omega_1)$ be an ordinal. The dual norm on X^* is ω^α-*w*-uniformly Kadets-Klee* (ω^α-UKK*) if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } s_\varepsilon^{\omega^\alpha}(B_{X^*}) \subset (1-\delta)B_{X^*}.$$

Theorem (Lancien-P-Raja '15)

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm whose dual norm is ω^α-UKK*.

Remark

There are Banach spaces with $Sz(X) = \omega^\alpha$, α limit (Causey '15) $\rightsquigarrow$ No result for such spaces.

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set.

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

- $K_\varepsilon^0 := K$,

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

- $K_\varepsilon^0 := K$,
- $K_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}^*}(s'_\varepsilon(K_\varepsilon^\alpha))$,

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

- $K_\varepsilon^0 := K$,
- $K_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}^*}(s'_\varepsilon(K_\varepsilon^\alpha))$,
- $K_\varepsilon^\alpha := \bigcap_{\beta<\alpha} K_\varepsilon^\beta$ if α is a limit ordinal.

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

- $K_\varepsilon^0 := K$,
- $K_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}^*}(s'_\varepsilon(K_\varepsilon^\alpha))$,
- $K_\varepsilon^\alpha := \bigcap_{\beta<\alpha} K_\varepsilon^\beta$ if α is a limit ordinal.

Then $Cz(K,\varepsilon) := \inf\{\alpha : \ K_\varepsilon^\alpha = \emptyset\}$ with $\inf \emptyset = +\infty$.

The convex Szlenk index

Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

- $K_\varepsilon^0 := K$,
- $K_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}^*}(s'_\varepsilon(K_\varepsilon^\alpha))$,
- $K_\varepsilon^\alpha := \bigcap_{\beta<\alpha} K_\varepsilon^\beta$ if α is a limit ordinal.

Then $Cz(K,\varepsilon) := \inf\{\alpha : \ K_\varepsilon^\alpha = \emptyset\}$ with $\inf\emptyset = +\infty$.
$Cz(K) := \sup_{\varepsilon>0} Cz(K,\varepsilon)$.

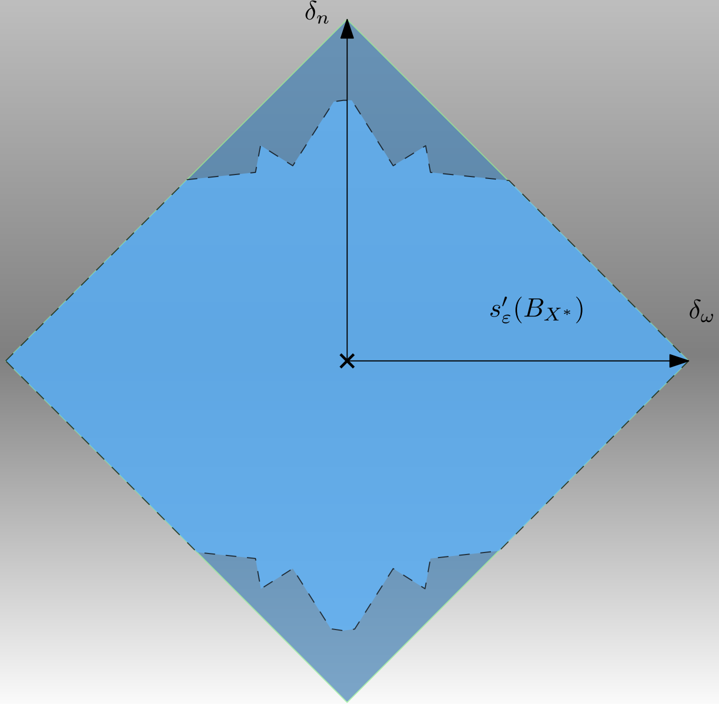
The convex Szlenk index

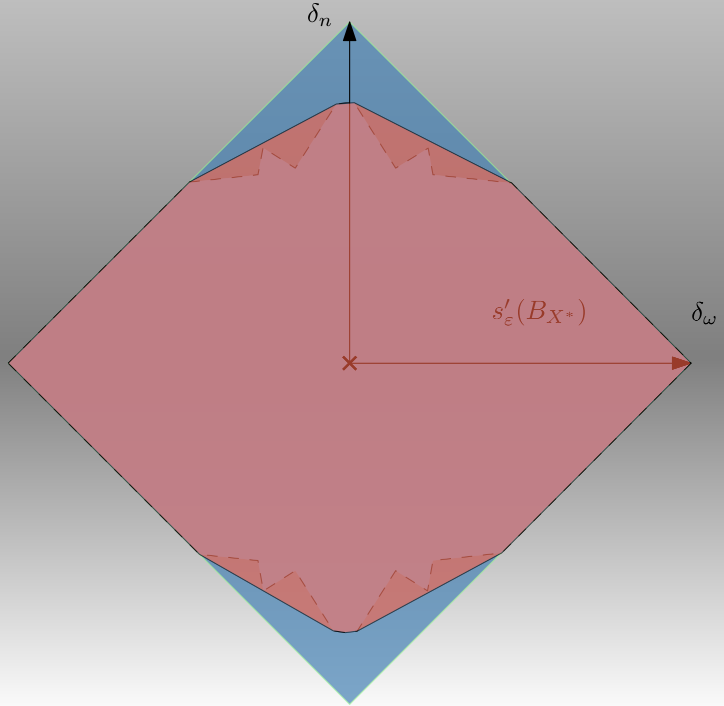
Definition

Let $K \subset X^*$ be a w*-compact convex set. Define K_ε^α for α ordinal as follows:

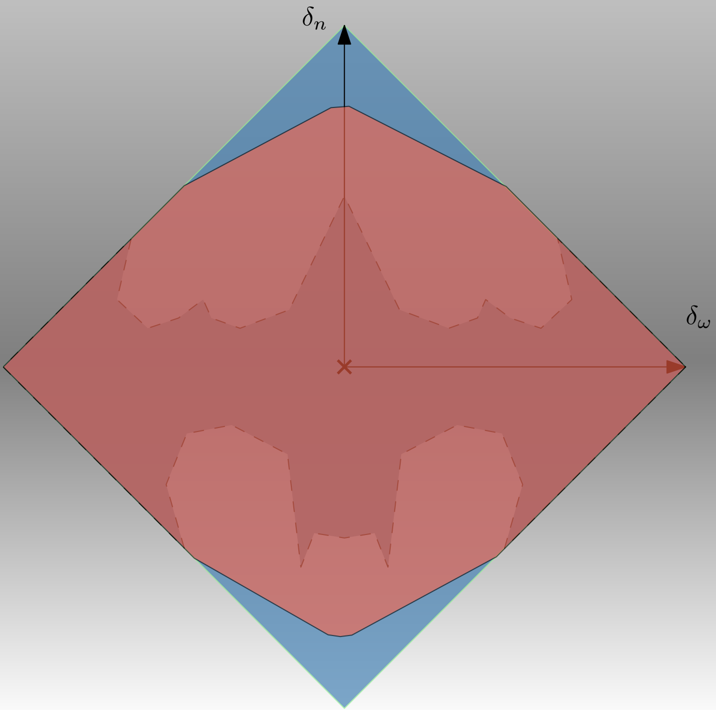
- $K_\varepsilon^0 := K$,
- $K_\varepsilon^{\alpha+1} := \overline{\text{conv}^*}(s'_\varepsilon(K_\varepsilon^\alpha))$,
- $K_\varepsilon^\alpha := \bigcap_{\beta<\alpha} K_\varepsilon^\beta$ if α is a limit ordinal.

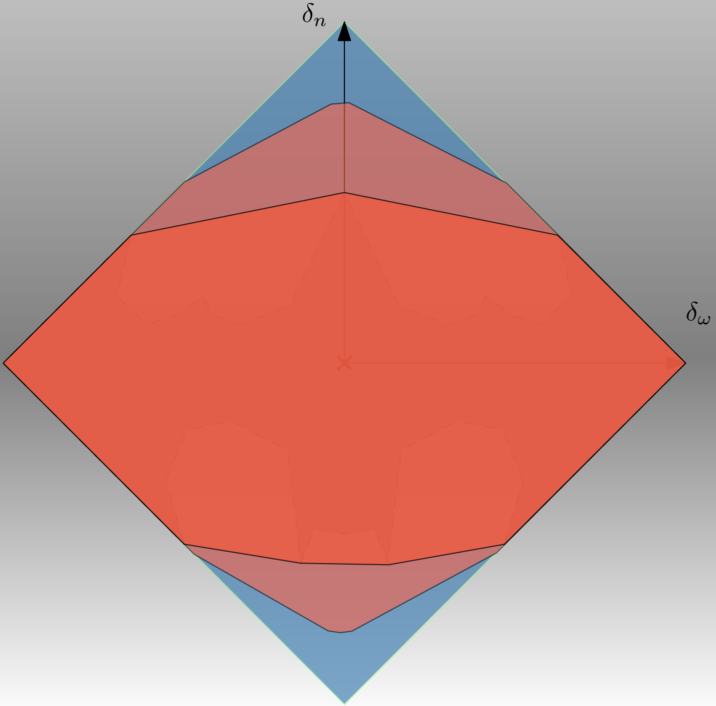
Then $Cz(K,\varepsilon) := \inf\{\alpha : \ K_\varepsilon^\alpha = \emptyset\}$ with $\inf\emptyset = +\infty$.
$Cz(K) := \sup_{\varepsilon>0} Cz(K,\varepsilon)$.
Finally the convex Szlenk index of X is $Cz(B_{X^*})$.


δ_n
$s'_\varepsilon(B_{X^*})$
δ_ω



δ_n
$s'_\varepsilon(B_{X^*})$
δ_ω


δ_n
δ_ω



δ_n
δ_ω

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.
- $\forall\ X$ Banach $\exists\ \alpha$ ordinal such that $Cz(X) = \omega^\alpha$ (Lancien '96)

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.
- $\forall\ X$ Banach $\exists\ \alpha$ ordinal such that $Cz(X) = \omega^{\alpha}$ (Lancien '96)
- $Sz(X) \leq Cz(X)$

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.
- $\forall\ X$ Banach $\exists\ \alpha$ ordinal such that $Cz(X) = \omega^\alpha$ (Lancien '96)
- $Sz(X) \leq Cz(X)$
- If $Sz(X) = \omega \Rightarrow X^*$ admits a dual UKK* norm $\Rightarrow$ $Cz(X) = \omega$.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.
- $\forall\ X$ Banach $\exists\ \alpha$ ordinal such that $Cz(X) = \omega^\alpha$ (Lancien '96)
- $Sz(X) \leq Cz(X)$
- If $Sz(X) = \omega \Rightarrow X^*$ admits a dual UKK* norm $\Rightarrow Cz(X) = \omega$.

Theorem (Hájek-Schlumprecht '13)

If $Sz(X) < \omega_1$ then $Cz(X) \leq \omega \cdot Sz(X)$. In particular, if $\omega^\omega \leq Sz(X) \leq \omega_1$, then $Cz(X) = Sz(X)$.

Facts about the convex Szlenk index

- If $X \simeq Y$, then $Cz(X) = Cz(Y)$.
- More generally, if $X \subseteq Y$, then $Cz(X) \leq Cz(Y)$.
- $Cz(X) < +\infty$ iff X is Asplund.
- If X is separable, $Cz(X) < +\infty$ iff $Cz(X) < \omega_1$.
- $\forall$ X Banach $\exists$ α ordinal such that $Cz(X) = \omega^\alpha$ (Lancien '96)
- $Sz(X) \leq Cz(X)$
- If $Sz(X) = \omega \Rightarrow X^*$ admits a dual UKK* norm $\Rightarrow$ $Cz(X) = \omega$.

Theorem (Hájek-Schlumprecht '13)

If $Sz(X) < \omega_1$ then $Cz(X) \leq \omega \cdot Sz(X)$. In particular, if $\omega^\omega \leq Sz(X) \leq \omega_1$, then $Cz(X) = Sz(X)$.

Theorem (L-P-R '15)

If $Sz(X) < \omega_1$ then $Cz(X) = Sz(X)$.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s_\varepsilon'(K)$.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s'_\varepsilon(K)$.
- if $x^* \in s'_\varepsilon(K)$ and $\varepsilon' < \varepsilon$, then $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \frac{\varepsilon'}{2}$

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s'_\varepsilon(K)$.
- if $x^* \in s'_\varepsilon(K)$ and $\varepsilon' < \varepsilon$, then $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \frac{\varepsilon'}{2}$

Definition (Family of trees $\mathcal{T}_\alpha$, $\alpha < \omega_1$)

- $\mathcal{T}_0 := \{\{\emptyset\}\}$

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s'_\varepsilon(K)$.
- if $x^* \in s'_\varepsilon(K)$ and $\varepsilon' < \varepsilon$, then $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \frac{\varepsilon'}{2}$

Definition (Family of trees $\mathcal{T}_\alpha$, $\alpha < \omega_1$)

- $\mathcal{T}_0 := \{\{\emptyset\}\}$
- $T \in \mathcal{T}_\alpha$ if there exists $(n_k) \subset \mathbb{N}$, $n_k \nearrow \infty$, such that

$$T = \{\emptyset\} \cup \bigcup_{k=0}^{\infty} (n_k)^\frown T_k,$$

where $T_k \in \mathcal{T}_{\alpha_k}$ for each $k \in \mathbb{N}$ and some α_k

Characterization of $s^\alpha_\varepsilon(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s'_\varepsilon(K)$.
- if $x^* \in s'_\varepsilon(K)$ and $\varepsilon' < \varepsilon$, then $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \frac{\varepsilon'}{2}$

Definition (Family of trees $\mathcal{T}_\alpha$, $\alpha < \omega_1$)

- $\mathcal{T}_0 := \{\{\emptyset\}\}$
- $T \in \mathcal{T}_\alpha$ if there exists $(n_k) \subset \mathbb{N}$, $n_k \nearrow \infty$, such that

$$T = \{\emptyset\} \cup \bigcup_{k=0}^{\infty} (n_k)^\frown T_k,$$

  where $T_k \in \mathcal{T}_{\alpha_k}$ for each $k \in \mathbb{N}$ and some α_k
  - if $\alpha = \beta + 1 \implies \alpha_k = \beta$ for all k

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).

- if $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \varepsilon$, then $x^* \in s'_\varepsilon(K)$.
- if $x^* \in s'_\varepsilon(K)$ and $\varepsilon' < \varepsilon$, then $\exists\,(x_n^*) \subset K$ such that $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^* - x^*\| \geq \frac{\varepsilon'}{2}$

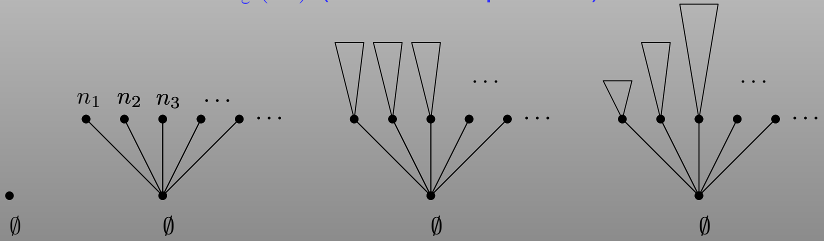
Definition (Family of trees $\mathcal{T}_\alpha$, $\alpha < \omega_1$)

- $\mathcal{T}_0 := \{\{\emptyset\}\}$
- $T \in \mathcal{T}_\alpha$ if there exists $(n_k) \subset \mathbb{N}$, $n_k \nearrow \infty$, such that

$$T = \{\emptyset\} \cup \bigcup_{k=0}^{\infty} (n_k)^\frown T_k,$$

where $T_k \in \mathcal{T}_{\alpha_k}$ for each $k \in \mathbb{N}$ and some α_k
  - if $\alpha = \beta + 1 \implies \alpha_k = \beta$ for all k
  - if α limit $\implies \alpha_k \nearrow \alpha$.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).



Definition (Family of trees $\mathcal{T}_\alpha$, $\alpha < \omega_1$)

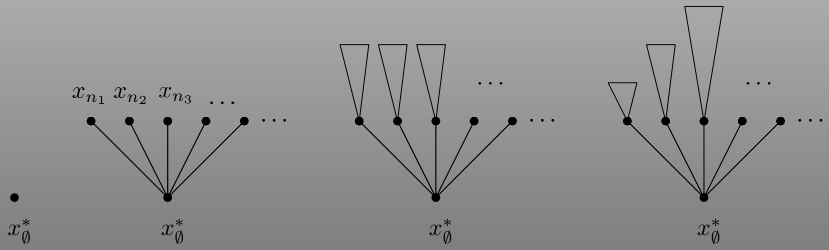
- $\mathcal{T}_0 := \{\{\emptyset\}\}$
- $T \in \mathcal{T}_\alpha$ if there exists $(n_k) \subset \mathbb{N}$, $n_k \nearrow \infty$, such that

$$T = \{\emptyset\} \cup \bigcup_{k=0}^{\infty} (n_k)^\frown T_k,$$

where $T_k \in \mathcal{T}_{\alpha_k}$ for each $k \in \mathbb{N}$ and some α_k

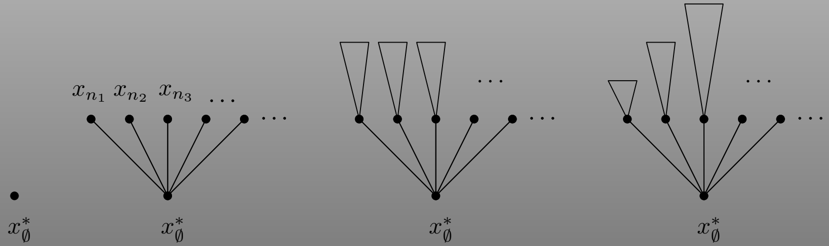
  - if $\alpha = \beta + 1 \Longrightarrow \alpha_k = \beta$ for all k
  - if α limit $\Longrightarrow \alpha_k \nearrow \alpha$.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).



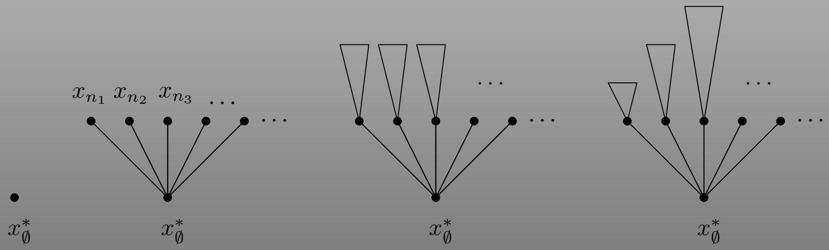
- if $T \in \mathcal{T}_\alpha$ and $\exists\, (x_s^*)_{s\in T} \subset K$ such that $x^*_{s^\frown n} \xrightarrow{w^*} x_s^*$ and $\left\|x^*_{s^\frown n} - x^*_s\right\| \geq \varepsilon$, then $x^*_\emptyset \in s^\alpha_\varepsilon(K)$.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).



- if $T \in \mathcal{T}_\alpha$ and $\exists\,(x^*_s)_{s\in T} \subset K$ such that $x^*_{s^\frown n} \xrightarrow{w^*} x^*_s$ and $\left\|x^*_{s^\frown n} - x^*_s\right\| \geq \varepsilon$, then $x^*_\emptyset \in s_\varepsilon^\alpha(K)$. Say that $(x^*_s)_{s\in T}$ is w*-continuous and ε-separated.

Characterization of $s_\varepsilon^\alpha(K)$ (when X separable).



- if $T \in \mathcal{T}_\alpha$ and $\exists\, (x_s^*)_{s\in T} \subset K$ such that $x^*_{s^\frown n} \xrightarrow{w^*} x_s^*$ and $\left\|x^*_{s^\frown n} - x_s^*\right\| \geq \varepsilon$, then $x^*_\emptyset \in s_\varepsilon^\alpha(K)$. Say that $(x_s^*)_{s\in T}$ is w*-continuous and ε-separated.
- if $x^* \in s_\varepsilon^\alpha(K)$ and $\varepsilon' < \varepsilon$, then $\exists\, T \in \mathcal{T}_\alpha$ and $\exists$ w*-continuous $\frac{\varepsilon'}{2}$-separated $(x_s^*)_{s\in T} \subset K$ such that $x^*_\emptyset = x^*$.

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK*.

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.
    - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\text{conv}}^*\left(s_{2^{-k}}^{\omega^\alpha}(A_n^k)\right).$$

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.
  - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\text{conv}}^*\left(s_{2^{-k}}^{\omega^\alpha}(A_n^k)\right).$$

  - $Cz(X) = Sz(X) \leq \omega^{\alpha+1} \Rightarrow \exists\, n \in \mathbb{N} : A_n^k = \emptyset$. Denote $N_k := \min\{n \in \mathbb{N} : A_n^k = \emptyset\} - 1$.

Proof of the renorming theorem

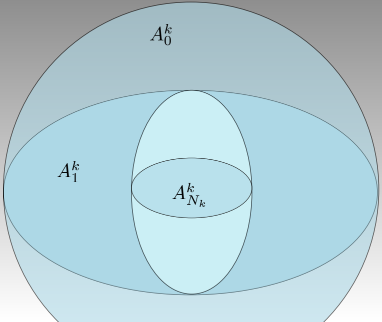
Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "⇐" Homogeneity argument as before in the UKK* case.
- "⇒" Assume $Sz(X) \leq \omega^{\alpha+1}$.
  - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\text{conv}}^*\left(s_{2^{-k}}^{\omega^\alpha}(A_n^k)\right).$$

  - $Cz(X) = Sz(X) \leq \omega^{\alpha+1} \Rightarrow \exists\, n \in \mathbb{N}\,:\, A_n^k = \emptyset$. Denote $N_k := \min\{n \in \mathbb{N} : A_n^k = \emptyset\} - 1$.



Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.
  - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\text{conv}}^* \left(s_{2^{-k}}^{\omega^\alpha}(A_n^k) \right).$$

  - $Cz(X) = Sz(X) \leq \omega^{\alpha+1} \Rightarrow \exists\, n \in \mathbb{N} \, : \, A_n^k = \emptyset$. Denote $N_k := \min\{n \in \mathbb{N} : A_n^k = \emptyset\} - 1$.
  - Define

$$f(x^*) = \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \text{d}(x^*, A_n^k).$$

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.
  - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\mathrm{conv}}^* \big(s_{2^{-k}}^{\omega^\alpha}(A_n^k) \big).$$

  - $Cz(X) = Sz(X) \leq \omega^{\alpha+1} \Rightarrow \exists\, n \in \mathbb{N} \,:\, A_n^k = \emptyset$. Denote $N_k := \min\{n \in \mathbb{N} : A_n^k = \emptyset\} - 1$.
  - Define

$$f(x^*) = \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x^*, A_n^k).$$

  - Finally, $|\cdot|$ is the Minkowski functional of $\{x : f(x) \leq 1\}$.

Proof of the renorming theorem

Theorem

Let X be a separable Banach space. Then $Sz(X) \leq \omega^{\alpha+1}$ if and only if X admits an equivalent norm $|\cdot|$ whose dual norm is ω^{α}-UKK.*

- "$\Leftarrow$" Homogeneity argument as before in the UKK* case.
- "$\Rightarrow$" Assume $Sz(X) \leq \omega^{\alpha+1}$.
  - Fix $k \in \mathbb{N}$. Inductively for $n \in \mathbb{N}$:

$$A_0^k := B_{X^*}, \quad A_{n+1}^k := \overline{\text{conv}}^*\left(s_{2^{-k}}^{\omega^\alpha}(A_n^k)\right).$$

  - $Cz(X) = Sz(X) \leq \omega^{\alpha+1} \Rightarrow \exists\, n \in \mathbb{N} \ : \ A_n^k = \emptyset$. Denote $N_k := \min\{n \in \mathbb{N} : A_n^k = \emptyset\} - 1$.
  - Define

$$f(x^*) = \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x^*, A_n^k).$$

  - Finally, $|\cdot|$ is the Minkowski functional of $\{x : f(x) \leq 1\}$.
  - $|\cdot|$ is dual and satisfies $\|\cdot\| \leq |\cdot| \leq 2\,\|\cdot\|$.

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.
- Let , let $\varepsilon < \varepsilon'$, $\varepsilon \sim \varepsilon'$.

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.
- Let , let $\varepsilon < \varepsilon'$, $\varepsilon \sim \varepsilon'$.
- then $\exists\, T \in \mathcal{T}_\alpha$ and $\exists$ w*-continuous $\frac{\varepsilon}{2}$-separated $(x^*_s)_{s \in T} \subset K$ such that $x^*_\emptyset = x^*$.

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.
- Let , let $\varepsilon < \varepsilon'$, $\varepsilon \sim \varepsilon'$.
- then $\exists\, T \in \mathcal{T}_\alpha$ and $\exists$ w*-continuous $\frac{\varepsilon}{2}$-separated $(x_s^*)_{s\in T} \subset K$ such that $x_\emptyset^* = x^*$.
- $$f(x^*) = \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x^*, A_n^k)$$
  $$\leq \varliminf_{s\in T} \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \varliminf_{s\in T} \mathrm{d}(x^*, A_n^k)$$
  $$\leq \liminf_{s\in T} \left(\|x_s^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x_s^*, A_n^k) \right) \leq 1$$

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.
- Let , let $\varepsilon < \varepsilon'$, $\varepsilon \sim \varepsilon'$.
- then $\exists\, T \in \mathcal{T}_\alpha$ and $\exists$ w*-continuous $\frac{\varepsilon}{2}$-separated $(x_s^*)_{s\in T} \subset K$ such that $x_\emptyset^* = x^*$.
- $$\begin{aligned} f(x^*) &= \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x^*, A_n^k) \\ &\leq \underline{\lim}_{s\in T} \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \underline{\lim}_{s\in T} \mathrm{d}(x^*, A_n^k) \\ &\leq \liminf_{s\in T} \left(\|x_s^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x_s^*, A_n^k) \right) \leq 1 \end{aligned}$$
- Let p such that $\frac{\varepsilon}{8} \leq 2^{-p} < \frac{\varepsilon}{4}$. We will show that for some $l \in \{1, \ldots, N_p\}$ there is a jump $\mathrm{d}(x^*, A_l^p) + \gamma < \liminf_{s\in T} \mathrm{d}(x_s^*, A_l^p)$ of size $\gamma \sim \varepsilon$.

Proof of the renorming theorem

- Let $\varepsilon' > 0$, it is enough to find $\delta > 0$ such that $\forall\, x^* \in s'_{\varepsilon'}(B_{|\cdot|})$ (w.r.t. the original norm) $f(x^*) \leq 1 - \delta$.
- Let , let $\varepsilon < \varepsilon'$, $\varepsilon \sim \varepsilon'$.
- then $\exists\, T \in \mathcal{T}_\alpha$ and $\exists$ w*-continuous $\frac{\varepsilon}{2}$-separated $(x_s^*)_{s\in T} \subset K$ such that $x_\emptyset^* = x^*$.
- $$f(x^*) + \frac{\gamma}{2^p N_p} = \frac{\gamma}{2^p N_p} + \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x^*, A_n^k)$$
$$\leq \varliminf_{s\in T} \|x^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \varliminf_{s\in T} \mathrm{d}(x^*, A_n^k)$$
$$\leq \liminf_{s\in T} \left(\|x_s^*\| + \sum_{k=1}^{\infty} \frac{1}{2^k N_k} \sum_{n=1}^{N_k} \mathrm{d}(x_s^*, A_n^k) \right) \leq 1$$
- Let p such that $\frac{\varepsilon}{8} \leq 2^{-p} < \frac{\varepsilon}{4}$. We will show that for some $l \in \{1, \ldots, N_p\}$ there is a jump $\mathrm{d}(x^*, A_l^p) + \gamma < \liminf_{s\in T} \mathrm{d}(x_s^*, A_l^p)$ of size $\gamma \sim \varepsilon$.

The jump

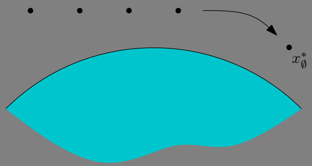
Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b - 2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

The jump

Lemma

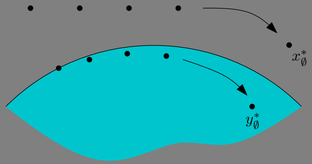
Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.



The jump

Lemma

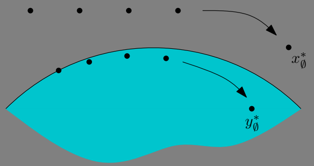
Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s\in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s\in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\ S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s\in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.



The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s\in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s\in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s\in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

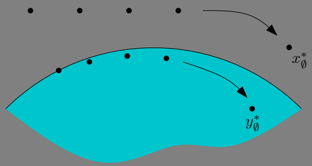


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

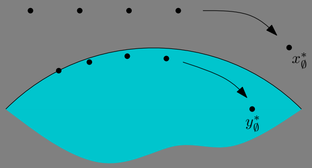


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact*, $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b - 2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

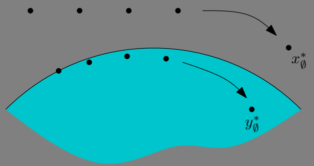


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b - 2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

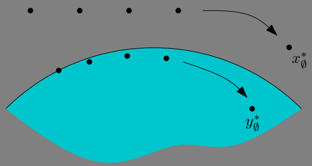


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p)$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\ S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

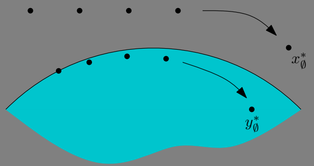


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\ l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p)\ \Rightarrow \mathrm{d}(x^*, A_3^p) < 2\gamma$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact*, $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\ S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

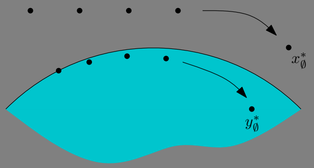


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\ l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p)\ \Rightarrow \mathrm{d}(x^*, A_3^p) < 2\gamma$
- $\ldots$

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b - 2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

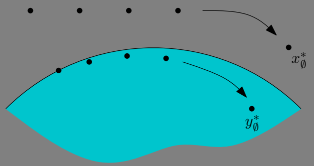


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p) \ \Rightarrow \mathrm{d}(x^*, A_3^p) < 2\gamma$
- $\cdots\ \mathrm{d}(x^*, A_{N_p+1}^p) < N_p \gamma$ contradiction as $A_{N_p+1}^k = \emptyset$.

The jump

Lemma

Let $0 \leq \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s\in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s\in T} \mathrm{d}(x_s^*, A) \leq a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b-2a)$*-separated* $(y_s^*)_{s\in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \leq a$.

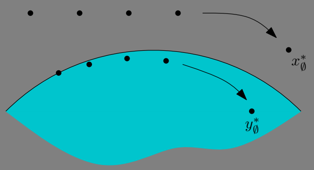


- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \leq N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p)\ \Rightarrow \mathrm{d}(x^*, A_3^p) < 2\gamma$
- $\cdots\ \mathrm{d}(x^*, A_{N_p+1}^p) < N_p\gamma$ contradiction as $A_{N_p+1}^k = \emptyset$.
- enough that $\frac{\varepsilon}{2} - 2N_p\gamma \geq \frac{\varepsilon}{4} > 2^{-p}$.

The jump

Lemma

Let $0 \le \alpha < \omega_1$, $0 < 2a < b$ *and* $T \in \mathcal{T}_\alpha$. *Let* $A \subset X^*$ *w*-compact,* $(x_s^*)_{s \in T} \subset X^*$ *be* b*-separated, w*-cº such that* $\sup_{s \in T} \mathrm{d}(x_s^*, A) \le a$. *Then* $\exists\, S \in \mathcal{T}_\alpha$, $S \subset T$ *and w*-cº,* $(b - 2a)$*-separated* $(y_s^*)_{s \in S} \subset A$ *such that* $\|x_\emptyset^* - y_\emptyset^*\| \le a$.



- Suppose that $\gamma + d(x^*, A_l^p) > \liminf_s \mathrm{d}(x_s^*, A_l^p)\ \forall\, l \le N_p$
- $x^* \in A_1^p \Rightarrow \gamma > \liminf_s \mathrm{d}(x_s^*, A_1^p)$
- Lemma $\Longrightarrow \mathrm{d}(x^*, A_2^p) < \gamma$
- $2\gamma > \gamma + \mathrm{d}(x^*, A_2^p) > \liminf_s \mathrm{d}(x_s^*, A_2^p) \ \Rightarrow \mathrm{d}(x^*, A_3^p) < 2\gamma$
- $\cdots\ \mathrm{d}(x^*, A_{N_p+1}^p) < N_p\gamma$ contradiction as $A_{N_p+1}^k = \emptyset$.
- enough that $\frac{\varepsilon}{2} - 2N_p\gamma \ge \frac{\varepsilon}{4} > 2^{-p}$. $\square$

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Definition

$$[\sigma]'_{\varepsilon}(K) := \{x^* \in K : \ \forall \ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \sigma(\overline{K \cap U}^*) \geq \varepsilon\}$$

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Definition

$$[\sigma]'_\varepsilon(K) := \{x^* \in K : \ \forall \ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \sigma(\overline{K \cap U}^*) \geq \varepsilon\}$$

Define $[\sigma]^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $[\sigma]^0_\varepsilon(K) := K$,

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Definition

$$[\sigma]'_\varepsilon(K) := \{x^* \in K : \ \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*, \\ \sigma(\overline{K \cap U}^*) \geq \varepsilon\}$$

Define $[\sigma]^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $[\sigma]^0_\varepsilon(K) := K$,
- $[\sigma]^{\alpha+1}_\varepsilon(K) := [\sigma]'_\varepsilon([\sigma]^\alpha_\varepsilon(K))$,

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \le \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Definition

$$[\sigma]'_{\varepsilon}(K) := \{x^* \in K : \ \forall \ \mathrm{w}^* - \text{neighborhood } U \text{ of } x^*, \\ \sigma(\overline{K \cap U}^*) \ge \varepsilon\}$$

Define $[\sigma]^{\alpha}_{\varepsilon}(K)$ for α ordinal as follows:

- $[\sigma]^{0}_{\varepsilon}(K) := K$,
- $[\sigma]^{\alpha+1}_{\varepsilon}(K) := [\sigma]'_{\varepsilon}([\sigma]^{\alpha}_{\varepsilon}(K))$,
- $[\sigma]^{\alpha}_{\varepsilon}(K) := \bigcap_{\beta<\alpha} [\sigma]^{\beta}_{\varepsilon}(K)$ if α is a limit ordinal.

Proof of $Sz(K) = Cz(K)$ when $Sz(K) \leq \omega^{n+1}$

First we show that $Sz(K) = KSz(K)$ and $Cz(K) = KCz(K)$, second $KSz(K) = KCz(K)$.

Definition (Kuratowski measure of non-compactness)

$\sigma(K) = \inf\{\varepsilon > 0 : K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}$

Definition

$$[\sigma]'_\varepsilon(K) := \{x^* \in K : \ \forall \text{ w}^* - \text{neighborhood } U \text{ of } x^*,$$
$$\sigma(\overline{K \cap U}^*) \geq \varepsilon\}$$

Define $[\sigma]^\alpha_\varepsilon(K)$ for α ordinal as follows:

- $[\sigma]^0_\varepsilon(K) := K$,
- $[\sigma]^{\alpha+1}_\varepsilon(K) := [\sigma]'_\varepsilon([\sigma]^\alpha_\varepsilon(K))$,
- $[\sigma]^\alpha_\varepsilon(K) := \bigcap_{\beta<\alpha} [\sigma]^\beta_\varepsilon(K)$ if α is a limit ordinal.

Then $KSz(K, \varepsilon) := \inf\{\alpha : \ [\sigma]^\alpha_\varepsilon(K) = \emptyset\}$ with $\inf \emptyset = +\infty$.

- $\mathrm{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$

- $\operatorname{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K)\ \exists\ (x^*_n) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x^*_n\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.

- $\mathrm{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K) \; \exists \; (x^*_n) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x^*_n\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.

- $\operatorname{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K) \; \exists \, (x_n^*) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x_n^*\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

- $\operatorname{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K)\ \exists\ (x_n^*) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x_n^*\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle_\varepsilon^0(K) := K$,

- $\operatorname{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K) \; \exists \; (x^*_n) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x^*_n\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle^0_\varepsilon(K) := K$,
- $\langle\sigma\rangle^{\alpha+1}_\varepsilon := \overline{\text{conv}^*}([\sigma]'_\varepsilon(\langle\sigma\rangle^\alpha_\varepsilon))$,

- $\mathrm{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K) \; \exists \; (x^*_n) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x^*_n\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle^0_\varepsilon(K) := K$,
- $\langle\sigma\rangle^{\alpha+1}_\varepsilon := \overline{\mathrm{conv}^*}([\sigma]'_\varepsilon(\langle\sigma\rangle^\alpha_\varepsilon))$,
- $\langle\sigma\rangle^\alpha_\varepsilon := \bigcap_{\beta<\alpha} \langle\sigma\rangle^\beta_\varepsilon$ if α is a limit ordinal.

- $\mathrm{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K)\ \exists\ (x_n^*) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x_n^*\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle_\varepsilon^0(K) := K$,
- $\langle\sigma\rangle_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}}^*([\sigma]'_\varepsilon(\langle\sigma\rangle_\varepsilon^\alpha))$,
- $\langle\sigma\rangle_\varepsilon^\alpha := \bigcap_{\beta<\alpha} \langle\sigma\rangle_\varepsilon^\beta$ if α is a limit ordinal.

Then $KCz(K,\varepsilon) := \inf\{\alpha : \ \langle\sigma\rangle_\varepsilon^\alpha = \emptyset\}$ with $\inf\emptyset = +\infty$.

- $\mathrm{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K)\ \exists\ (x_n^*) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x_n^*\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle_\varepsilon^0(K) := K$,
- $\langle\sigma\rangle_\varepsilon^{\alpha+1} := \overline{\mathrm{conv}^*}([\sigma]'_\varepsilon(\langle\sigma\rangle_\varepsilon^\alpha))$,
- $\langle\sigma\rangle_\varepsilon^\alpha := \bigcap_{\beta<\alpha} \langle\sigma\rangle_\varepsilon^\beta$ if α is a limit ordinal.

Then $KCz(K,\varepsilon) := \inf\{\alpha :\ \langle\sigma\rangle_\varepsilon^\alpha = \emptyset\}$ with $\inf\emptyset = +\infty$.
$KCz(K) := \sup_{\varepsilon>0} KCz(K,\varepsilon)$.

- $\operatorname{diam}(K) \geq \sigma(K) \Rightarrow [\sigma]'_\varepsilon(K) \subset s'_\varepsilon(K)$
- $x^* \in s'_\varepsilon(K)\ \exists\ (x^*_n) \subset K$, $x^* \xrightarrow{w^*} x^*$ and $\liminf \|x^* - x^*_n\| \geq \frac{\varepsilon}{2} \Rightarrow$ no w*-neighborhood of x^* can be covered by finitely many balls of radius $< \frac{\varepsilon}{4}$.
- $\Rightarrow s_\varepsilon(K) \subset [\sigma]'_{\varepsilon/5}(K)$.
- $\Rightarrow KSz(K) = Sz(K)$.

Definition

$K \subset X^*$ be a w*-compact convex set.

- $\langle\sigma\rangle^0_\varepsilon(K) := K$,
- $\langle\sigma\rangle^{\alpha+1}_\varepsilon := \overline{\operatorname{conv}^*}([\sigma]'_\varepsilon(\langle\sigma\rangle^\alpha_\varepsilon))$,
- $\langle\sigma\rangle^\alpha_\varepsilon := \bigcap_{\beta<\alpha} \langle\sigma\rangle^\beta_\varepsilon$ if α is a limit ordinal.

Then $KCz(K,\varepsilon) := \inf\{\alpha :\ \langle\sigma\rangle^\alpha_\varepsilon = \emptyset\}$ with $\inf\emptyset = +\infty$.
$KCz(K) := \sup_{\varepsilon>0} KCz(K,\varepsilon)$.

- $\overline{\operatorname{conv}^*}[\sigma]'_\varepsilon(K) \subset \overline{\operatorname{conv}^*} s'_\varepsilon(K) \subset \overline{\operatorname{conv}^*}[\sigma]'_{\varepsilon/5}(K) \Rightarrow$ $KCz(K) = Cz(K)$.

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

Definition

A map η from the weak*-compacts of X^* into $[0,\infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

(iv) $\eta(\lambda A) = |\lambda| \, \eta(A)$

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

(iv) $\eta(\lambda A) = |\lambda| \, \eta(A)$

(v) $\eta(A + B) \leq \eta(A) + \eta(B)$

Definition

A map η from the weak*-compacts of X^* into $[0, \infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

(iv) $\eta(\lambda A) = |\lambda|\, \eta(A)$

(v) $\eta(A + B) \leq \eta(A) + \eta(B)$

Fact

- *σ is a nc-measure.*

Definition

A map η from the weak*-compacts of X^* into $[0,\infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

(iv) $\eta(\lambda A) = |\lambda|\, \eta(A)$

(v) $\eta(A+B) \leq \eta(A) + \eta(B)$

Fact

- σ *is a nc-measure.*
- *If* η *is a nc-measure, then*

$$\eta^\omega(A) = \inf\{\varepsilon > 0 : [\eta]^\omega_\varepsilon(A) = \emptyset\}$$

is a nc-measure of non-compactness.

Definition

A map η from the weak*-compacts of X^* into $[0,\infty)$ is a *sublinear measure of non-compactness* (nc-measure) if

(o) $\eta(\{x^*\}) = 0$ for any $x^* \in X^*$.

(i) $A \subset B \Rightarrow \eta(A) \leq \eta(B)$.

(ii) $\eta(\bigcup_{i=1}^n A_i) = \max_i \eta(A_i)$.

(iv) $\eta(\lambda A) = |\lambda|\, \eta(A)$

(v) $\eta(A+B) \leq \eta(A) + \eta(B)$

Fact

- *σ is a nc-measure.*
- *If η is a nc-measure, then*

$$\eta^\omega(A) = \inf\{\varepsilon > 0 : [\eta]_\varepsilon^\omega(A) = \emptyset\}$$

  is a nc-measure of non-compactness.
- *Notation: $\sigma^{\omega^n} := (\sigma^{\omega^{n-1}})^\omega$*

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_{\varepsilon}^{\omega^{n+1}}(K) = \emptyset$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_\varepsilon^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_\varepsilon^\omega(A) = [\sigma^\omega]'_\varepsilon(A)$ so $[\sigma^{\omega^n}]^\omega(K) = \emptyset$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_\varepsilon^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_\varepsilon^\omega(A) = [\sigma^\omega]'_\varepsilon(A)$ so $[\sigma^{\omega^n}]^\omega(K) = \emptyset$.
- We will (maybe) prove that $\exists\, \kappa_n \geq 1$: $[\sigma^{\omega^n}]_\varepsilon^\omega(K) = \emptyset \Rightarrow \langle\sigma^{\omega^n}\rangle_{\kappa_n\varepsilon}^\omega(K) = \emptyset$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_\varepsilon^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_\varepsilon^\omega(A) = [\sigma^\omega]'_\varepsilon(A)$ so $[\sigma^{\omega^n}]^\omega(K) = \emptyset$.
- We will (maybe) prove that $\exists\ \kappa_n \geq 1$: $[\sigma^{\omega^n}]_\varepsilon^\omega(K) = \emptyset \Rightarrow \langle\sigma^{\omega^n}\rangle_{\kappa_n\varepsilon}^\omega(K) = \emptyset$.
- Unfortunately $\langle\sigma\rangle_\varepsilon^\omega(A) \neq \langle\sigma\rangle'_\varepsilon(A)$ but

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_{\varepsilon}^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_{\varepsilon}^{\omega}(A) = [\sigma^{\omega}]'_{\varepsilon}(A)$ so $[\sigma^{\omega^n}]^{\omega}(K) = \emptyset$.
- We will (maybe) prove that $\exists\, \kappa_n \geq 1$: $[\sigma^{\omega^n}]_{\varepsilon}^{\omega}(K) = \emptyset \Rightarrow \langle \sigma^{\omega^n} \rangle_{\kappa_n \varepsilon}^{\omega}(K) = \emptyset$.
- Unfortunately $\langle \sigma \rangle_{\varepsilon}^{\omega}(A) \neq \langle \sigma \rangle'_{\varepsilon}(A)$ but

$$\forall\, m\; \exists\, \kappa_m \geq 1\; \forall\, \alpha :\; \langle \sigma^{\omega^m} \rangle_{\kappa_m \varepsilon}^{\omega \cdot \alpha}(A) \subset \langle \sigma^{\omega^{m+1}} \rangle_{\varepsilon}^{\alpha}(A)$$

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_\varepsilon^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_\varepsilon^\omega(A) = [\sigma^\omega]'_\varepsilon(A)$ so $[\sigma^{\omega^n}]^\omega(K) = \emptyset$.
- We will (maybe) prove that $\exists\, \kappa_n \geq 1$: $[\sigma^{\omega^n}]_\varepsilon^\omega(K) = \emptyset \Rightarrow \langle\sigma^{\omega^n}\rangle_{\kappa_n\varepsilon}^\omega(K) = \emptyset$.
- Unfortunately $\langle\sigma\rangle_\varepsilon^\omega(A) \neq \langle\sigma\rangle'_\varepsilon(A)$ but

$$\forall\, m\, \exists\, \kappa_m \geq 1\, \forall\, \alpha :\ \langle\sigma^{\omega^m}\rangle_{\kappa_m\varepsilon}^{\omega\cdot\alpha}(A) \subset \langle\sigma^{\omega^{m+1}}\rangle_\varepsilon^\alpha(A)$$

- Hence $\langle\sigma\rangle_{\kappa_0\cdots\kappa_n\varepsilon}^{\omega^{n+1}}(K) = \emptyset$.

Upper level of the proof of $KSz(K) = KCz(K)$

- Enough to prove $KSz(K) \leq KCz(K)$.
- Let $KSz(K) \leq \omega^{n+1}$. Need to prove $KCz(K) \leq \omega^{n+1}$.
- Let $\varepsilon > 0$. Then $[\sigma]_{\varepsilon}^{\omega^{n+1}}(K) = \emptyset$.
- We have almost $[\sigma]_{\varepsilon}^{\omega}(A) = [\sigma^{\omega}]'_{\varepsilon}(A)$ so $[\sigma^{\omega^n}]^{\omega}(K) = \emptyset$.
- We will (maybe) prove that $\exists\, \kappa_n \geq 1$: $[\sigma^{\omega^n}]_{\varepsilon}^{\omega}(K) = \emptyset \Rightarrow \langle \sigma^{\omega^n} \rangle_{\kappa_n \varepsilon}^{\omega}(K) = \emptyset$.
- Unfortunately $\langle \sigma \rangle_{\varepsilon}^{\omega}(A) \neq \langle \sigma \rangle'_{\varepsilon}(A)$ but

$$\forall\, m\, \exists\, \kappa_m \geq 1\, \forall\, \alpha :\ \langle \sigma^{\omega^m} \rangle_{\kappa_m \varepsilon}^{\omega \cdot \alpha}(A) \subset \langle \sigma^{\omega^{m+1}} \rangle_{\varepsilon}^{\alpha}(A)$$

- Hence $\langle \sigma \rangle_{\kappa_0 \cdots \kappa_n \varepsilon}^{\omega^{n+1}}(K) = \emptyset$.
- $\Longrightarrow KCz(K) \leq \omega^{n+1}$. $\square$

Proof of the “maybe” implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\text{conv}^*}(A)) \leq \kappa\eta(A)$.

Proof of the “maybe” implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\mathrm{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^\omega_\varepsilon(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^\omega_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

Proof of the "maybe" implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\mathrm{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.

Proof of the “maybe” implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\mathrm{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.
- Assume that σ^{ω^n} is κ-convexifiable. Let A w*-compact such that $\sigma^{\omega^{n+1}}(A) < \varepsilon$.

Proof of the “maybe” implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\mathrm{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.
- Assume that σ^{ω^n} is κ-convexifiable. Let A w*-compact such that $\sigma^{\omega^{n+1}}(A) < \varepsilon$.
- Then $[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset$ by the definition.

Proof of the "maybe" implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\mathrm{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.
- Assume that σ^{ω^n} is κ-convexifiable. Let A w*-compact such that $\sigma^{\omega^{n+1}}(A) < \varepsilon$.
- Then $[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset$ by the definition.
- Lemma $\Rightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$

Proof of the “maybe” implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\text{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.
- Assume that σ^{ω^n} is κ-convexifiable. Let A w*-compact such that $\sigma^{\omega^{n+1}}(A) < \varepsilon$.
- Then $[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset$ by the definition.
- Lemma $\Rightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset$
- $\Longrightarrow [\sigma^{\omega^n}]^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset \Rightarrow \sigma^{\omega^{n+1}}(\overline{\text{conv}^*}(A)) \leq 5\kappa\varepsilon$.

Proof of the "maybe" implication

Definition

A nc-measure η is κ-convexifiable ($\kappa \geq 1$) if $\forall A$ w*-compact $\eta(\overline{\text{conv}^*}(A)) \leq \kappa\eta(A)$.

Lemma

If σ^{ω^n} is κ-convexifiable, then

$$[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset$$

- Exercise: σ is 1-convexifiable.
- Assume that σ^{ω^n} is κ-convexifiable. Let A w*-compact such that $\sigma^{\omega^{n+1}}(A) < \varepsilon$.
- Then $[\sigma^{\omega^n}]^{\omega}_{\varepsilon}(A) = \emptyset$ by the definition.
- Lemma $\Rightarrow \langle\sigma^{\omega^n}\rangle^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset$
- $\Longrightarrow [\sigma^{\omega^n}]^{\omega}_{5\kappa\varepsilon}(\overline{\text{conv}^*}(A)) = \emptyset \Rightarrow \sigma^{\omega^{n+1}}(\overline{\text{conv}^*}(A)) \leq 5\kappa\varepsilon$.
- $\Rightarrow \sigma^{\omega^{n+1}}$ is 6κ-convexifiable.

Proof of the lemma

Sublemma 1

Let X be a Banach space and η a κ-convexifiable nc-measure. Assume that A is a weak*-compact symmetric and radial such that $[\eta]'_\varepsilon(A) \subset \lambda A$ for some $\lambda \in (0,1)$ and $\varepsilon > 0$. Then

$$\forall\, \varepsilon' > \varepsilon \quad \langle \eta \rangle^{\omega}_{\kappa\varepsilon'}(\overline{\mathrm{conv}^*}(A)) = \emptyset.$$

Proof of the lemma

Sublemma 1

Let X be a Banach space and η a κ-convexifiable nc-measure. Assume that A is a weak*-compact symmetric and radial such that $[\eta]'_\varepsilon(A) \subset \lambda A$ for some $\lambda \in (0,1)$ and $\varepsilon > 0$. Then

$$\forall\, \varepsilon' > \varepsilon \quad \langle \eta \rangle^\omega_{\kappa\varepsilon'}(\overline{\mathrm{conv}^*}(A)) = \emptyset.$$

- Let $\varepsilon' > \varepsilon$. Let $\lambda < \zeta < \xi < 1$ (ζ arbitrary fixed, $\xi \to 1$).

Proof of the lemma

Sublemma 1

Let X be a Banach space and η a κ-convexifiable nc-measure. Assume that A is a weak*-compact symmetric and radial such that $[\eta]'_\varepsilon(A) \subset \lambda A$ for some $\lambda \in (0,1)$ and $\varepsilon > 0$. Then

$$\forall\, \varepsilon' > \varepsilon \quad \langle\eta\rangle^{\omega}_{\kappa\varepsilon'}(\overline{\mathrm{conv}^*}(A)) = \emptyset.$$

- Let $\varepsilon' > \varepsilon$. Let $\lambda < \zeta < \xi < 1$ (ζ arbitrary fixed, $\xi \to 1$).
- We will show that $\langle\eta\rangle'_{\kappa\varepsilon'}\overline{\mathrm{conv}^*}(A) \subset \xi\overline{\mathrm{conv}^*}(A)$

Proof of the lemma

Sublemma 1

Let X be a Banach space and η a κ-convexifiable nc-measure. Assume that A is a weak*-compact symmetric and radial such that $[\eta]'_\varepsilon(A) \subset \lambda A$ for some $\lambda \in (0,1)$ and $\varepsilon > 0$. Then

$$\forall\, \varepsilon' > \varepsilon \quad \langle\eta\rangle^{\omega}_{\kappa\varepsilon'}(\overline{\mathrm{conv}^*}(A)) = \emptyset.$$

- Let $\varepsilon' > \varepsilon$. Let $\lambda < \zeta < \xi < 1$ (ζ arbitrary fixed, $\xi \to 1$).
- We will show that $\langle\eta\rangle'_{\kappa\varepsilon'}\overline{\mathrm{conv}^*}(A) \subset \xi\overline{\mathrm{conv}^*}(A)$
- It is enough to show that when S is a slice of $\overline{\mathrm{conv}^*}(A)$, $S \cap \xi\overline{\mathrm{conv}^*}(A) = \emptyset \Rightarrow \eta(S) < \kappa\varepsilon'$.

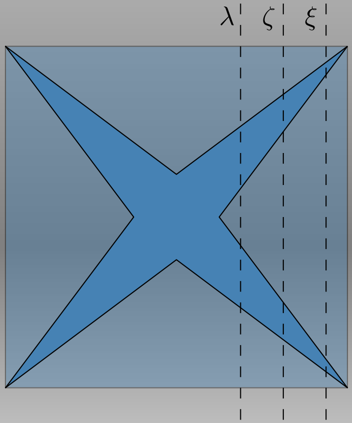
Proof of the lemma

Sublemma 1

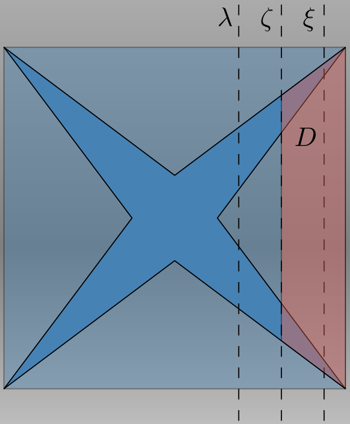
Let X be a Banach space and η a κ-convexifiable nc-measure. Assume that A is a weak*-compact symmetric and radial such that $[\eta]'_{\varepsilon}(A) \subset \lambda A$ for some $\lambda \in (0,1)$ and $\varepsilon > 0$. Then

$$\forall\, \varepsilon' > \varepsilon \quad \langle \eta \rangle^{\omega}_{\kappa\varepsilon'}(\overline{\mathrm{conv}^*}(A)) = \emptyset.$$

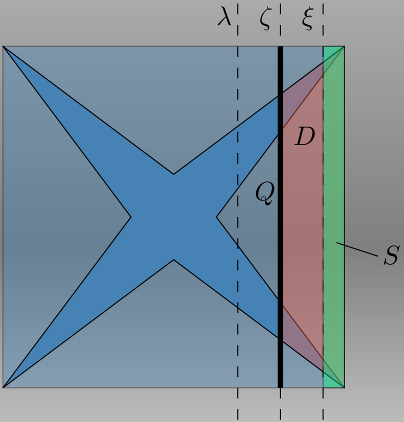
- Let $\varepsilon' > \varepsilon$. Let $\lambda < \zeta < \xi < 1$ (ζ arbitrary fixed, $\xi \to 1$).
- We will show that $\langle \eta \rangle'_{\kappa\varepsilon'} \overline{\mathrm{conv}^*}(A) \subset \xi \overline{\mathrm{conv}^*}(A)$
- It is enough to show that when S is a slice of $\overline{\mathrm{conv}^*}(A)$, $S \cap \xi \overline{\mathrm{conv}^*}(A) = \emptyset \Rightarrow \eta(S) < \kappa\varepsilon'$.
- So let $x \in X$, $\sup x(\overline{\mathrm{conv}^*}(A)) = 1$ and $S = \{x^* \in \overline{\mathrm{conv}^*}(A) : x^*(x) > \xi\}$.



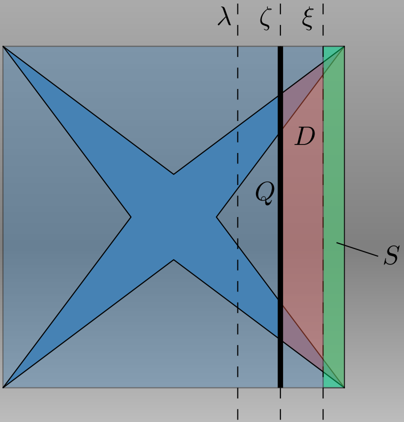
λ
ζ
ξ



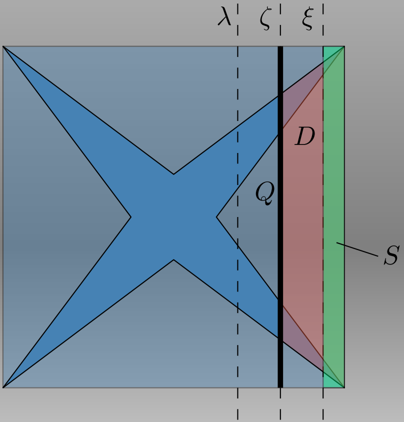
- $D = \overline{\text{conv}^*}\{x^* \in A : x^*(x) \geq \zeta\}$, $\eta(D) \leq \kappa\varepsilon$.



- $D = \overline{\text{conv}^*}\{x^* \in A : x^*(x) \geq \zeta\}$, $\eta(D) \leq \kappa\varepsilon$.
- $\forall\ x^* \in S\ \exists t > t_\xi = \frac{\xi - \zeta}{1 - \zeta}$ such that $x^* \in tD + (1-t)Q$.



- $D = \overline{\mathrm{conv}^*}\{x^* \in A : x^*(x) \geq \zeta\}$, $\eta(D) \leq \kappa\varepsilon$.
- $\forall\ x^* \in S\ \exists t > t_\xi = \frac{\xi-\zeta}{1-\zeta}$ such that $x^* \in tD + (1-t)Q$.
- $S \subset [t_\xi, 1]D + [0, 1-t_\xi]Q$



- $D = \overline{\mathrm{conv}^*}\{x^* \in A : x^*(x) \geq \zeta\}$, $\eta(D) \leq \kappa\varepsilon$.
- $\forall\ x^* \in S\ \exists t > t_\xi = \frac{\xi-\zeta}{1-\zeta}$ such that $x^* \in tD + (1-t)Q$.
- $S \subset [t_\xi, 1]D + [0, 1-t_\xi]Q$
- $\implies \eta(S) < \kappa\varepsilon'$. $\square$

Sublemma 2

Let $A \subset X^*$ be a weak*-compact such that $[\sigma^{\omega^n}]_\varepsilon^m(A) = \emptyset$. Then there is a symmetric radial weak*-compact set $B \supset A$ such that

$$[\sigma^{\omega^n}]'_{4\varepsilon}(B) \subset \Big(1 - \frac{1}{32(m+1)}\Big)B.$$

Sublemma 2

Let $A \subset X^*$ be a weak*-compact such that $[\sigma^{\omega^n}]^m_\varepsilon(A) = \emptyset$. Then there is a symmetric radial weak*-compact set $B \supset A$ such that

$$[\sigma^{\omega^n}]'_{4\varepsilon}(B) \subset \Big(1 - \frac{1}{32(m+1)}\Big)B.$$

- Sublemma 1 + Sublemma 2 $\Rightarrow$ Lemma

Sublemma 2

Let $A \subset X^*$ be a weak*-compact such that $[\sigma^{\omega^n}]_\varepsilon^m(A) = \emptyset$. Then there is a symmetric radial weak*-compact set $B \supset A$ such that

$$[\sigma^{\omega^n}]'_{4\varepsilon}(B) \subset \Big(1 - \frac{1}{32(m+1)}\Big)B.$$

- Sublemma 1 + Sublemma 2 $\Rightarrow$ Lemma
- i.e. (with $\kappa_n = 6^n$):

$$[\sigma^{\omega^n}]_\varepsilon^\omega(A) = \emptyset \implies \langle \sigma^{\omega^n} \rangle_{5\kappa_n\varepsilon}^\omega(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

Sublemma 2

Let $A \subset X^*$ be a weak*-compact such that $[\sigma^{\omega^n}]^m_\varepsilon(A) = \emptyset$. Then there is a symmetric radial weak*-compact set $B \supset A$ such that

$$[\sigma^{\omega^n}]'_{4\varepsilon}(B) \subset \Big(1 - \frac{1}{32(m+1)}\Big)B.$$

- Sublemma 1 + Sublemma 2 $\Rightarrow$ Lemma
- i.e. (with $\kappa_n = 6^n$):

$$[\sigma^{\omega^n}]^\omega_\varepsilon(A) = \emptyset \Longrightarrow \langle\sigma^{\omega^n}\rangle^\omega_{5\kappa_n\varepsilon}(\overline{\mathrm{conv}^*}(A)) = \emptyset$$

- This finishes the proof of $Sz(K) = Cz(K)$ when $Sz(K) < \omega^\omega$.

Thank you!