Analyticity of the Dirichlet-to-Neumann Map for Maxwell's Equations in Passive Composite Media

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Joint work with Maxence Cassier & Graeme Milton (Univ. of Utah)

1. G. W. Milton, *The Theory of Composites*, Cambridge University Press, 2002.
2. G. W. Milton (editor), *Extending the Theory of Composites to Other Areas of Science*, to appear in 2016.
    3. G. W. Milton, *Canonical forms for many of the linear equations of physics and key identities*, Chap. 1 – Shows that many of the linear equations of physics, such as acoustics, elastodynamics, or electromagnetism, can be written in a unifying framework appropriate to the theory of composite materials.
    4. G. W. Milton, *A new perspective on boundary-value problems in mathematics, physics, and engineering*, Chap. 3. – Shows the problem of finding the DtN map in an inhomogeneous body can be reformulated as the problem of finding "effective tensor" associated with an abstract problem in the theory of composites.
    5. M. Cassier, A. Welters, and G. W. Milton, Analyticity of the Dirichlet-to-Neumann map for the time-harmonic Maxwell's equations, Chap. 4 (preprint available at http://arxiv.org/abs/1512.05838) – Studies in depth the analyticity properties of the DtN map for electromagnetism in composite media and the connection of this map to Herglotz functions.

Motivation

- A direct mathematical "isomorphism" between the theory of effective tensors of composites and the theory of the DtN map for bodies.
- As a result of this, many of the tools/machinery that have been developed in the theory of composite materials essentially carry over directly to DtN maps, e.g.,
  - variational principles for composites map over to variational principles for boundary-value problems;
  - theory of bounds on effective tensors carries over to an analogous theory of bounds on DtN maps;
  - analyticity properties of effective tensors as functions of the component moduli map over to analyticity properties of the DtN map as functions of the component tensors within the body.
- Advantage: Directly apply the theory of composites to bear on a much wider class of problems, e.g., inverse problems[6], which could indirectly have an impact on the theory of composites.

[6] G. W. Milton, *The inverse problem: Obtaining information about what's inside a body*, Chap. 5– Explores bounds on the DtN map and the associated inverse problem of what can be said about the distribution of materials inside a body from surface measurements, using connection between DtN maps and effective tensors in composites.

Herglotz functions

Herglotz functions: key role in analytic methods used to derive bounds/limitations in composites[1] & electromagnetism[7].

Definition (Herglotz function)

Let $m, n, d \in \mathbb{N}$. Define $\mathbb{C}^+ := \{\omega \in \mathbb{C} : \operatorname{Im} \omega > 0\}$, $M_n^+(\mathbb{C}) := \{\mathbf{M} \in M_n(\mathbb{C}) : \operatorname{Im} \mathbf{M} > 0\}$, and $\mathcal{T} = (\mathbb{C}^+)^d$ or $\mathcal{T} = (M_n^+(\mathbb{C}))^d$. A function $h : \mathcal{T} \to \mathbb{C}$ or $h : \mathcal{T} \to M_m(\mathbb{C})$ is a called a Herglotz function if h is an analytic function satisfying

$$\operatorname{Im} h(\mathbf{Z}) \geq 0, \ \forall \mathbf{Z} \in \mathcal{T},$$

where $\operatorname{Im} h(\mathbf{Z}) = (2i)^{-1}\left(h(\mathbf{Z}) - h(\mathbf{Z})^*\right)$ denotes the imaginary part.

Theorem (Integral representation)

A necessary and sufficient condition for $h : \mathbb{C}^+ \to \mathbb{C}$ to be a Herglotz function is that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$ and a positive regular Borel measure μ for which $\int_{\mathbb{R}} (1+\lambda^2)^{-1} d\mu(\lambda)$ is finite such that

$$h(\omega) = \beta + \alpha\omega + \int_{\mathbb{R}} \left(\frac{1}{\lambda - \omega} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda), \ \textit{for } \omega \in \mathbb{C}^+.$$

[1] See Chap. 27: Bounds using the analytic method.

[7] A. Welters, Y. Avniel, & S. G. Johnson, *Speed-of-light limitations in passive linear media*, Phys. Rev. A, vol. 90, no. 2, 2014.

Main Points

- For the time-harmonic Maxwell's equations in passive linear multicomponent media, the electromagnetic (EM) DtN map is an analytic function of frequency ω and, more generally, of the dielectric permittivities and magnetic permeabilities tensors of each phase.
    - Including tensors (as opposed to scalars) is necessary for bodies containing anisotropic materials, e.g., gyrotropic materials.
    - Essentially, the DtN map is a Herglotz function of these parameters.
- Many important impacts and consequences of theory of Herglotz functions.
    - Connection to the theory of holomorphic functions on tubular domains with nonnegative imaginary part and multivariate passive linear systems theory[8].
- Applications in electromagnetism, e.g., inverse problems, bounds/limitations on certain physical quantities, and studying EM phenomenon in composites with high-loss and lossless components.

[8] N. K. Bose, *Multidimensional Systems Theory & Applications*, 2nd ed., Springer, 2003; V. S. Vladimirov, *Methods of the Theory of Generalized Functions*, Taylor & Francis, 2002.

Overview

I. Time-harmonic Maxwell's equations, Poynting's Theorem, and the connection to the EM DtN map
  - Integrating a conservation law in divergence form over a body, connects the EM fields in the body to the EM fields on the boundary via the EM "Dirichlet-to-Neumann" (DtN) map which depends on the materials in the body.

II. EM DtN map for layered media
  - Reduction of Maxwell's equations to ODEs.
  - Explicit expression for the EM DtN map in terms of the 4×4 transfer matrix.
  - Analytic properties of the DtN map in the context of matrix perturbation theory.

III. EM DtN map for bounded media
  - large class of different 3D geometries includes domains with nonsmooth boundary, e.g., polyhedra
  - variational reformulation of time-harmonic Maxwell's equations.
  - well-posedness and analyticity of the DtN map based on variational methods.

Time-harmonic Maxwell's equations for passive linear media

- The time-harmonic Maxwell's equations (in Gaussian units without sources) at a fixed frequency ω:

$$\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B},\ \nabla \times \mathbf{H} = -i\frac{\omega}{c}\mathbf{D},\ \mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},$$

  where c denotes the speed of light in a vacuum with the electric field $\mathbf{E}(\mathbf{x})$, electric displacement field $\mathbf{D}(\mathbf{x})$, magnetic field $\mathbf{H}(\mathbf{x})$, magnetic induction $\mathbf{B}(\mathbf{x})$.
- For passive linear (local) media filling a region Ω, the electric permittivity $\varepsilon(\omega, \mathbf{x})$ and magnetic permability $\mu(\omega, \mathbf{x})$ have the key properties:

$$\text{for fixed } \mathbf{x} \in \Omega \text{ and } \operatorname{Im}\omega > 0:$$

$$\varepsilon(\omega, \mathbf{x}),\ \mu(\omega, \mathbf{x}) \text{ are analytic functions of } \omega \text{ with}$$

$$\operatorname{Im}[\omega\varepsilon(\omega, \mathbf{x})] \geq 0,\ \operatorname{Im}[\omega\mu(\omega, \mathbf{x})] \geq 0.$$

Poynting's Theorem: divergence form

- From Maxwell's equations ($\nabla \times \mathbf{H} = -i\frac{\omega}{c}\mathbf{B}$) we have

$$0 = c\mathbf{E}\cdot\left(\nabla \times \overline{\mathbf{H}}\right) + \mathbf{E}\cdot\overline{i\omega\mathbf{D}}.$$

- Using the vector identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

with $\mathbf{a} = \mathbf{E}$ and $\mathbf{b} = \overline{\mathbf{H}}$ we get

$$0 = c\left[\overline{\mathbf{H}} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot \left(\mathbf{E} \times \overline{\mathbf{H}}\right)\right] + \mathbf{E}\cdot\overline{i\omega\mathbf{D}}.$$

- From Maxwell's equations ($\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}$) and the constitutive relations ($\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) we arrive at a divergence form of Poynting's Theorem:

$$0 = -c\nabla \cdot \left(\mathbf{E} \times \overline{\mathbf{H}}\right) + \overline{\mathbf{H}} \cdot (i\omega\mu\mathbf{H}) + \mathbf{E}\cdot\left(\overline{i\omega\varepsilon\mathbf{E}}\right).$$

Poynting's Theorem: integral form

- Integrating this form over the body Ω and applying the divergence theorem, we arrive at an integral form of Poynting's Theorem:

$$\int_{\Omega} \overline{\mathbf{H}} \cdot (i\omega\boldsymbol{\mu}\mathbf{H}) + \mathbf{E} \cdot \left(\overline{i\omega\boldsymbol{\varepsilon}\mathbf{E}}\right) d\mathbf{x} = \int_{\Omega} -c\nabla \cdot \left(\mathbf{E} \times \overline{\mathbf{H}}\right) d\mathbf{x}$$

$$= -\int_{\partial\Omega} \mathbf{n} \cdot \left(c\mathbf{E} \times \overline{\mathbf{H}}\right) dS,$$

  where $\mathbf{n}$ is the outward unit $(\mathbf{n} \cdot \mathbf{n} = 1)$ normal vector on the boundary $\partial\Omega$ of Ω.

- Also, we have

$$-\int_{\partial\Omega} \mathbf{n} \cdot \left(c\mathbf{E} \times \overline{\mathbf{H}}\right) dS = -c \int_{\partial\Omega} \mathbf{E} \cdot \left(\overline{\mathbf{H}} \times \mathbf{n}\right) dS$$

$$= c \int_{\partial\Omega} (\mathbf{E} \times \mathbf{n}) \cdot \left(\mathbf{n} \times \overline{\mathbf{H}} \times \mathbf{n}\right) dS,$$

  which follows from the vector identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

  with $\mathbf{a} = \mathbf{E}$, $\mathbf{b} = \mathbf{n}$, $\mathbf{c} = \mathbf{n}$, and $\mathbf{d} = \overline{\mathbf{H}} \times \mathbf{n}$.

Poynting's Theorem: EM DtN map form

Definition (EM DtN map)

The electromagnetic Dirichlet-to-Neumann map
$\Lambda = \Lambda(\omega) = \Lambda(\omega\varepsilon(\omega,\cdot), \omega\mu(\omega,\cdot))$ is defined on EM fields $\mathbf{E}$, $\mathbf{H}$ by

$$\Lambda\left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right) = i\mathbf{n}\times\left(\mathbf{H}\times\mathbf{n}\right)|_{\partial\Omega}.$$

- Multiplying by $-i$ we arrive at the DtN form of Poynting's Theorem:

$$(\omega\mu\mathbf{H},\mathbf{H}) - (\mathbf{E},\omega\varepsilon\mathbf{E}) = \int_\Omega \overline{\mathbf{H}}\cdot(\omega\mu\mathbf{H}) - \mathbf{E}\cdot\left(\overline{\omega\varepsilon\mathbf{E}}\right)dx$$
$$= i\int_{\partial\Omega}\mathbf{n}\cdot\left(c\mathbf{E}\times\overline{\mathbf{H}}\right)dS = c\int_{\partial\Omega}\left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right)\cdot\overline{\left[\Lambda\left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right)\right]}dS$$
$$= \left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}, \Lambda\left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right)\right).$$

- Connection to EM energy loss:

$$0 \le (\mathrm{Im}(\omega\mu)\mathbf{H},\mathbf{H}) + (\mathbf{E},\mathrm{Im}(\omega\varepsilon)\mathbf{E}) = \int_{\partial\Omega}\mathbf{n}\cdot\mathrm{Re}\left(c\mathbf{E}\times\overline{\mathbf{H}}\right)dS$$
$$= \mathrm{Im}\left(\Lambda\left(\mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right), \mathbf{E}\times\mathbf{n}|_{\partial\Omega}\right).$$

Main problems

Problem (Well-posedness)

Is the EM DtN map Λ *well-defined? Is it a continuous linear operator?*

Problem (Dependence on parameters)

Is the map $\omega \mapsto \Lambda\left(\omega\boldsymbol{\varepsilon}\left(\omega,\cdot\right),\omega\boldsymbol{\mu}\left(\omega,\cdot\right)\right)$ *an analytic function? Is it a Herglotz function? For composite media, what is the analytic dependency of the map* Λ *on* $\omega\boldsymbol{\varepsilon}\left(\omega,\cdot\right),\omega\boldsymbol{\mu}\left(\omega,\cdot\right)$*?*

Definition (Analyticity)

Let E and F be two Banach spaces and U and open set of E. A function $h : U \to F$ is said to be analytic if it is differentiable on U.

- $L\left(E,F\right)$ – the Banach space of all continuous linear operators from E into F (with E,F Banach spaces) equipped with the operator norm; $\langle\cdot,\cdot\rangle$ – the duality product of F and its dual F^*; $\mathbb{C}^n$ – the Hilbert space with inner product $\left(\mathbf{u},\mathbf{v}\right) := \mathbf{u}\cdot\overline{\mathbf{v}} := \mathbf{v}^*\mathbf{u}$, where $\mathbf{v}^* = \overline{\mathbf{v}}^T$; recall, $M_n\left(\mathbb{C}\right)$, $L\left(\mathbb{C}^n,\mathbb{C}^n\right)$, and $\mathbb{C}^{n^2}$ as Banach spaces are isomorphic.
- In particular, the sets $\left(\mathbb{C}^+\right)^n$, and $M_n^+\left(\mathbb{C}\right)$ are open.

Key theorems on analyticity

Theorem (weak analyticity implies strong analyticity[9])

Let U be an open subset of $\mathbb{C}$ and $h : U \to L(E, F)$, where E and F are Banach spaces. If the function

$$h_{\phi,\psi}(\omega) := \langle h(\omega)\phi, \psi\rangle, \text{ for } \omega \in U$$

is analytic on U for all $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ in dense subsets of E and F^*, respectively, then h is analytic.

Theorem (Hartog's Theorem[10])

If $h : U \to E$ is a function on an open set $U \subseteq \mathbb{C}^n$ in to a Banach space E then h is a multivariate analytic function (i.e., jointly analytic) if and only if it is an analytic function of each variable separately.

Theorem (analytic and invertible implies analytic inverse[9])

If $h : U \to L(E, F)$ is an analytic function on an open set $U \subseteq \mathbb{C}^n$, where E and F are Banach spaces, and $h(\mathbf{Z})$ is invertible for all $\mathbf{Z} \in U$ then the function $\mathbf{Z} \mapsto h(\mathbf{Z})^{-1}$ is analytic from U into $L(F, E)$.

[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1995.

[10] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, 1986.

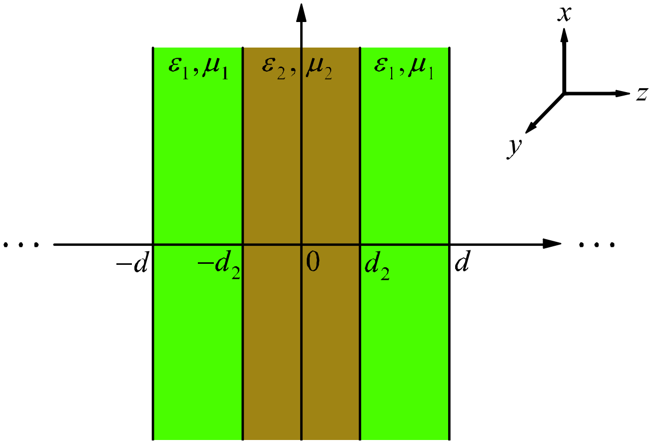

Figure: A plane-parallel layered medium consisting of two phases (ε_1, μ_1 in Ω_1 and ε_2, μ_2 in Ω_2) of passive linear materials with layers normal to z-axis occupying a region $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, where $-d \leq z \leq d$.

Layered media: formulation of the EM problem

- Consider passive linear two-component layered media (layers normal to z-axis) occupying a region $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ (Ω_1, Ω_2 measurable sets) with $z \in [-d, d]$, $\omega \in \mathbb{C}^+$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\omega, z) = \chi_{\Omega_1}(z)\,\boldsymbol{\varepsilon}_1(\omega) + \chi_{\Omega_2}(z)\,\boldsymbol{\varepsilon}_2(\omega),$$
$$\boldsymbol{\mu} = \boldsymbol{\mu}(\omega, z) = \chi_{\Omega_1}(z)\,\boldsymbol{\mu}_1(\omega) + \chi_{\Omega_2}(z)\,\boldsymbol{\mu}_2(\omega);$$

χ_{Ω_j} – indicator function of Ω_j (x,y independent);

Herglotz functions – $\omega\boldsymbol{\varepsilon}_j(\omega),\ \omega\boldsymbol{\mu}_j(\omega) : \mathbb{C}^+ \to M_3^+(\mathbb{C})$.

- Consider the solutions of the time-harmonic Maxwell's equations at a fixed frequency $\omega \in \mathbb{C}^+$ in the separable form

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{E}(z) \\ \mathbf{H}(z) \end{bmatrix} e^{i(k_1 x + k_2 y)},$$

where $x, y \in \mathbb{R}$, $z \in [-d, d]$, $\boldsymbol{\kappa} := (k_1, k_2) \in \mathbb{R}^2$, $\mathbf{E}(z)$, $\mathbf{H}(z) \in \mathbb{C}^3$.

- Boundary conditions require $\boldsymbol{\psi} \in (AC([-d, d]))^4$, where

$$\boldsymbol{\psi}(z) = \begin{bmatrix} E_1(z) & E_2(z) & H_1(z) & H_2(z) \end{bmatrix}^T$$

is the vector of tangential electric and magnetic field components.

Layered media: Reduction to ODEs

- EM solutions of this type correspond to the solutions of the linear ODEs:

$$\frac{d\boldsymbol{\psi}}{dz} = i\mathbf{J}\mathbf{A}(z)\,\boldsymbol{\psi}(z),\ \boldsymbol{\psi} \in (AC([-d,d]))^4,$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\rho} \\ \boldsymbol{\rho}^* & \mathbf{0} \end{bmatrix},\ \boldsymbol{\rho} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\mathbf{A} : z \mapsto \mathbf{A}(z),\ \mathbf{A}(\cdot) \in M_4\left(L^1([-d,d])\right).$$

- The matrix-valued function $\mathbf{A} : \mathbb{C}^+ \to M_4\left(L^1([-d,d])\right)$ is analytic, where

$$\begin{aligned} \mathbf{A} &: \omega \mapsto \mathbf{A}(\cdot, \mathbf{Z}(\omega)), \\ \mathbf{Z}(\omega) &= (\omega\boldsymbol{\varepsilon}_1(\omega), \omega\boldsymbol{\varepsilon}_2(\omega), \omega\boldsymbol{\mu}_1(\omega), \omega\boldsymbol{\mu}_2(\omega)). \end{aligned}$$

- More generally, $\mathbf{A} : \left(M_3^+(\mathbb{C})\right)^4 \to M_4\left(L^1([-d,d])\right)$ is analytic, where

$$\mathbf{A} : \mathbf{Z} \mapsto \mathbf{A}(\cdot, \mathbf{Z}).$$

Layered media: transfer matrix

Theorem (Analyticity of the transfer matrix)

For each $z_0 \in [-d, d]$, *the IVP*

$$\frac{d\boldsymbol{\psi}}{dz} = i\mathbf{J}\mathbf{A}(z)\,\boldsymbol{\psi}(z)\,,\ \boldsymbol{\psi}(z_0) = \boldsymbol{\psi}_0,$$

has a unique solution $\boldsymbol{\psi}$ *in* $(AC([-d,d]))^4$ *for each* $\boldsymbol{\psi}_0 \in \mathbb{C}^4$. *The solution is given in terms of the transfer matrix* $\mathbf{T}(z_0, z) \in M_4(\mathbb{C})$ *by*

$$\boldsymbol{\psi}(z) = \mathbf{T}(z_0, z)\,\boldsymbol{\psi}_0, \text{ for all } z \in [-d, d]\,.$$

Furthermore, for the function $\mathbf{T}(z_0, \cdot) : z \mapsto T(z_0, z)$ *we have* $\mathbf{T}(z_0, \cdot) \in M_4(AC([-d,d]))$ *and, for each* $z \in [-d, d]$, *the function* $\mathbf{T}(z_0, z, \cdot) : \mathbb{C}^+ \to M_4(\mathbb{C})$ *defined by*

$$\mathbf{T}(z_0, z, \cdot) : \omega \mapsto T(z_0, z, \mathbf{Z}(\omega))\,,$$
$$\mathbf{Z}(\omega) = (\omega\boldsymbol{\varepsilon}_1(\omega)\,, \omega\boldsymbol{\varepsilon}_2(\omega)\,, \omega\boldsymbol{\mu}_1(\omega)\,, \omega\boldsymbol{\mu}_2(\omega))$$

is analytic. More generally, $\mathbf{T}(z_0, z, \cdot) : \left(M_3^+(\mathbb{C})\right)^4 \to M_4(\mathbb{C})$ *is analytic, where*

$$\mathbf{T}(z_0, z, \cdot) : \mathbf{Z} \mapsto T(z_0, z, \mathbf{Z})\,.$$

Layered media: EM DtN Map

- For the region $\Omega(z_0, z_1)$ lying between the planes $z = z_0$ and $z = z_1$ (e.g., $\Omega = \Omega(-d, d)$), the outward pointing unit normal vector $\mathbf{n}$ to $\partial\Omega(z_0, z_1)$ is $\mathbf{n} = \mathbf{e}_3 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ on the plane $z = z_1$ and $\mathbf{n} = -\mathbf{e}_3$ on plane $z = z_0$. With respect to the standard orthonormal basis vectors of $\mathbb{C}^3$, we have

$$\mathbf{e}_3\times = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad -\mathbf{e}_3 \times \mathbf{e}_3\times = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- For layered media and EM fields $\mathbf{E}$, $\mathbf{H}$ in the separable form, we can define the EM DtN map $\Lambda = \Lambda\left(z_0, z_1\right) \in M_6\left(\mathbb{C}\right)$ to be the block operator matrix

$$\Lambda \begin{bmatrix} \mathbf{E} \times \mathbf{n}\big|_{z=z_1} \\ \mathbf{E} \times \mathbf{n}\big|_{z=z_0} \end{bmatrix} = \begin{bmatrix} i\mathbf{n} \times \mathbf{H} \times \mathbf{n}\big|_{z=z_1} \\ i\mathbf{n} \times \mathbf{H} \times \mathbf{n}\big|_{z=z_0} \end{bmatrix},$$

where $\mathbf{E} \times \mathbf{n} = -\mathbf{n} \times \mathbf{E}$ and $\mathbf{n} \times \mathbf{H} \times \mathbf{n} = -\mathbf{n} \times \mathbf{n} \times \mathbf{H}$.

Layered media: EM DtN Map

- Define the projection $\mathbf{P}_t$ and, with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$, define the matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(z_0, z_1) \in M_4(\mathbb{C})$ in terms of the transfer matrix $\mathbf{T} = \mathbf{T}(z_0, z_1)$ as

$$\mathbf{P}_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{21} \end{bmatrix}, \ \boldsymbol{\Gamma} := \begin{bmatrix} \mathbf{T}_{22}\mathbf{T}_{12}^{-1} & \mathbf{T}_{21} - \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11} \\ \mathbf{T}_{12}^{-1} & -\mathbf{T}_{12}^{-1}\mathbf{T}_{11} \end{bmatrix},$$

Theorem (DtN map is well-defined and analytic)

For each $\omega \in \mathbb{C}^+$, the matrix $\mathbf{T}_{12}(z_0, z_1, \mathbf{Z}(\omega))$, [where $\mathbf{Z}(\omega) = (\omega\varepsilon_1(\omega), \omega\varepsilon_2(\omega), \omega\boldsymbol{\mu}_1(\omega), \omega\boldsymbol{\mu}_2(\omega))$] is invertible and the EM DtN map $\Lambda = \Lambda(z_0, z_1, \mathbf{Z}(\omega))$ belongs to $M_6^+(\mathbb{C})$ and is given by

$$\Lambda(z_0, z_1, \mathbf{Z}(\omega)) = i \begin{bmatrix} \mathbf{P}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_t \end{bmatrix} \boldsymbol{\Gamma}(z_0, z_1, \mathbf{Z}(\omega)) \begin{bmatrix} \mathbf{P}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_t \end{bmatrix}^T \begin{bmatrix} \mathbf{e}_3 \times & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}_3 \times \end{bmatrix}.$$

Furthermore, the function $\Lambda : \mathbb{C}^+ \to M_6^+(\mathbb{C})$ defined by $\omega \mapsto \Lambda(z_0, z_1, \mathbf{Z}(\omega))$ is a Herglotz function. More generally, the function $\Lambda : (M_3^+(\mathbb{C}))^4 \to M_6^+(\mathbb{C})$ defined by $\mathbf{Z} \mapsto \Lambda(z_0, z_1, \mathbf{Z})$ is a Herglotz function.

Bounded media: composite geometry

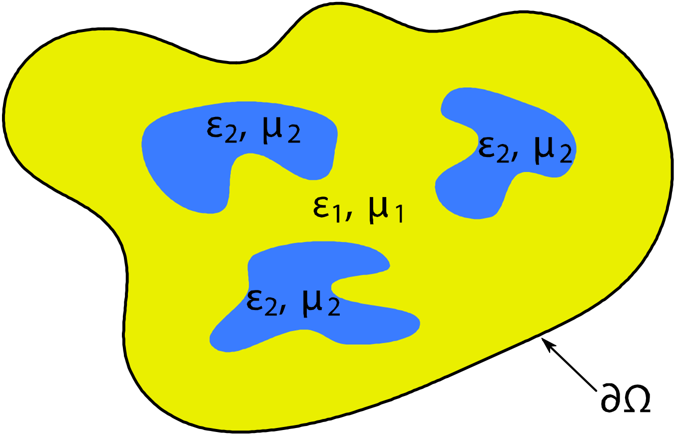

Figure: An electromagnetic medium composed of two isotropic homogeneous materials which fills an open connected bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$. The two phases ($\varepsilon_1$, μ_1 in Ω_1 and ε_2, μ_2 in Ω_2) of passive linear materials occupy the region $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$.

Bounded media: formulation of the EM problem

- Consider passive linear two-component bounded media occupying an open connected bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ (Ω_1, Ω_2 measurable sets) with $\mathbf{x} \in \Omega$, $\omega \in \mathbb{C}^+$:

$$\varepsilon = \varepsilon(\omega, \mathbf{x}) = \chi_{\Omega_1}(\mathbf{x})\,\varepsilon_1(\omega) + \chi_{\Omega_2}(\mathbf{x})\,\varepsilon_2(\omega),$$
$$\mu = \mu(\omega, \mathbf{x}) = \chi_{\Omega_1}(\mathbf{x})\,\mu_1(\omega) + \chi_{\Omega_2}(\mathbf{x})\,\mu_2(\omega);$$

χ_{Ω_j} – indicator function of Ω_j;

Herglotz functions – $\omega\varepsilon_j(\omega)$, $\omega\mu_j(\omega) : \mathbb{C}^+ \to \mathbb{C}^+$.

- The time-harmonic Maxwell's equations (in Gaussian) units for the electromagnetic fields $\mathbf{E}$, $\mathbf{H}$ in Ω:

$$(\mathcal{P}) \begin{cases} \nabla \times \mathbf{E} - ic^{-1}\omega\mu\mathbf{H} = \mathbf{0} & \text{in } \Omega, \\ \nabla \times \mathbf{H} + ic^{-1}\omega\varepsilon\mathbf{E} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} = \mathbf{f} & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$ of Ω and $\mathbf{f}$ is given boundary data.[11]

[11] For classical functional spaces associated with the solutions of $(\mathcal{P})$ see, for instance: A. Kirsch and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwell's Equations*, Springer-Verlag, 2015.

- We seek solutions $(\mathbf{E},\mathbf{H}) \in H(\mathrm{curl},\Omega)^2$ of the problem $(\mathcal{P})$ for data $\mathbf{f} \in H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)$, where $L^2(\Omega)$ is the Hilbert space with inner product

$$(\mathbf{u},\mathbf{v})_{L^2(\Omega)} = \int_\Omega \mathbf{u}(x)\cdot\overline{\mathbf{v}(x)}\,d\mathbf{x}, \text{ for } \mathbf{u},\mathbf{v} \in L^2(\Omega),$$

$$H(\mathrm{curl},\Omega) = \left\{\mathbf{u} \in L^2(\Omega) : \nabla\times\mathbf{u} \in L^2(\Omega)\right\},$$

$$H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega) = \left\{(\mathbf{u}\times\mathbf{n})_{\partial\Omega} : \mathbf{u} \in H(\mathrm{curl},\Omega)\right\}.$$

- To define the EM DtN map, we introduce the spaces

$$\begin{aligned} H_0(\mathrm{curl},\Omega) &= \{\mathbf{u} \in H(\mathrm{curl},\Omega) : \mathbf{u}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}, \\ H^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega) &= \{\mathbf{n}\times(\mathbf{u}\times\mathbf{n})_{\partial\Omega} : \mathbf{u} \in H(\mathrm{curl},\Omega)\}. \end{aligned}$$

- $H(\mathrm{curl},\Omega)$ and $H_0(\mathrm{curl},\Omega)$ are Hilbert spaces with inner product

$$(\mathbf{u},\mathbf{v})_{H(\mathrm{curl},\Omega)} = (\mathbf{u},\mathbf{v})_{L^2(\Omega)} + (\nabla\times\mathbf{u},\nabla\times\mathbf{v})_{L^2(\Omega)}.$$

- $H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)$ and $H^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)$ are Banach spaces and satisfy the duality relation $\left(H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)\right)^* = H^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)$ with respect to their duality product $\langle\cdot,\cdot\rangle$:

$$\langle \mathbf{n}\times(\mathbf{v}\times\mathbf{n})_{\partial\Omega}, (\mathbf{u}\times\mathbf{n})_{\partial\Omega}\rangle = \int_\Omega \mathbf{u}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{u})\,d\mathbf{x},$$

for all $\mathbf{u},\mathbf{v} \in H(\mathrm{curl},\Omega)$.

Bounded media: EM DtN Map

Theorem (DtN map is well-defined and analytic)

For each $\omega \in \mathbb{C}^+$*, the EM DtN map*

$$\Lambda\left(\mathbf{Z}\left(\omega\right)\right) : H^{-\frac{1}{2}}\left(\mathrm{div}, \partial\Omega\right) \to H^{-\frac{1}{2}}\left(\mathrm{curl}, \partial\Omega\right)$$

[where $\mathbf{Z}\left(\omega\right) = \left(\omega\varepsilon_1\left(\omega\right), \omega\varepsilon_2\left(\omega\right), \omega\mu_1\left(\omega\right), \omega\mu_2\left(\omega\right)\right)$*] defined by*

$$\Lambda\left(\mathbf{Z}\left(\omega\right)\right)\left(\mathbf{f}\right) = i\mathbf{n} \times \left(\mathbf{H} \times \mathbf{n}\right)_{\partial\Omega}, \ \mathbf{f} \in H^{-\frac{1}{2}}\left(\mathrm{div}, \partial\Omega\right)$$

where $\mathbf{H}$ *is the magnetic field in the solution of problem* $(\mathcal{P})$ *with boundary data* $\mathbf{f}$*, is well-defined and*

$$\Lambda\left(\mathbf{Z}\left(\omega\right)\right) \in L(H^{-\frac{1}{2}}\left(\mathrm{div}, \partial\Omega\right), H^{-\frac{1}{2}}\left(\mathrm{curl}, \partial\Omega\right)).$$

Furthermore, the function $\Lambda : \mathbb{C}^+ \to L(H^{-\frac{1}{2}}\left(\mathrm{div}, \partial\Omega\right), H^{-\frac{1}{2}}\left(\mathrm{curl}, \partial\Omega\right))$ *defined by* $\Lambda : \omega \mapsto \Lambda\left(\mathbf{Z}\left(\omega\right)\right)$ *is an analytic function. Moreover, the function* $h_{\mathbf{f}} : \mathbb{C}^+ \to \mathbb{C}^+$ *(*$\mathbf{f} \neq \mathbf{0}$*) defined by*

$$h_{\mathbf{f}} : \omega \mapsto \left\langle \Lambda\left(\mathbf{Z}\left(\omega\right)\right)\left(\mathbf{f}\right), \overline{\mathbf{f}} \right\rangle$$

is a Herglotz function.

- Similarly for $\Lambda : \left(\mathbb{C}^+\right)^4 \to L(H^{-\frac{1}{2}}\left(\mathrm{div}, \partial\Omega\right), H^{-\frac{1}{2}}\left(\mathrm{curl}, \partial\Omega\right))$ defined by $\Lambda : \mathbf{Z} \mapsto \Lambda\left(\mathbf{Z}\right)$.

EM DtN map representation

- This theorem essentially follows from the representation of the EM DtN map $\Lambda(\mathbf{Z}(\omega))$:

$$\Lambda(\mathbf{Z}(\omega))(\mathbf{f}) = i\gamma_T P T(\mathbf{Z}(\omega))(\mathbf{f}), \text{ for all } \mathbf{f} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega),$$

where $\gamma_T \in L\left(H(\mathrm{curl}, \Omega), H^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)\right)$,

$P \in L\left(H(\mathrm{curl}, \Omega)^2, H(\mathrm{curl}, \Omega)\right)$, and

$T(\mathbf{Z}(\omega)) \in L\left(H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega), H(\mathrm{curl}, \Omega)^2\right)$ are defined by

$$\begin{aligned} \gamma_T(\mathbf{H}) &= \mathbf{n} \times (\mathbf{H} \times \mathbf{n})_{\partial\Omega} \text{ for all } \mathbf{H} \in H(\mathrm{curl}, \Omega), \\ P(\mathbf{E}, \mathbf{H}) &= \mathbf{H} \text{ for all } (\mathbf{E}, \mathbf{H}) \in H(\mathrm{curl}, \Omega)^2, \\ T(\mathbf{Z}(\omega))(\mathbf{f}) &= (\mathbf{E}, \mathbf{H}) \text{ for each } \mathbf{f} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega), \end{aligned}$$

where $(\mathbf{E}, \mathbf{H}) \in H(\mathrm{curl}, \Omega)^2$ is the unique solution to problem $(\mathcal{P})$ with boundary data $\mathbf{f}$.

- The operator $T(\mathbf{Z}(\omega))(\mathbf{f}) = (\mathbf{E}, \mathbf{H})$ is given by

$$\mathbf{E} = R(\mathbf{f}) + A(\mathbf{Z}(\omega))^{-1} L(\mathbf{Z}(\omega)) R(\mathbf{f}), \quad \mathbf{H} = c(i\omega\mu)^{-1} \nabla \times \mathbf{E}.$$

EM DtN map representation

- The operator $R \in L\left(H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega), H(\mathrm{curl}, \Omega)\right)$ is the lifting operator, i.e.,

$$(R(\mathbf{f}) \times \mathbf{n})_{\partial\Omega} = \mathbf{f} \text{ for all } \mathbf{f} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega).$$

- The operators $A(\mathbf{Z}(\omega)) \in L\left(H_0(\mathrm{curl}, \Omega), H_0(\mathrm{curl}, \Omega)^*\right)$, $L(\mathbf{Z}(\omega)) \in L\left(H(\mathrm{curl}, \Omega), H_0(\mathrm{curl}, \Omega)^*\right)$ [where $H_0(\mathrm{curl}, \Omega)^*$ is the dual to $H_0(\mathrm{curl}, \Omega)$ with duality product $\langle \cdot, \cdot \rangle_{H_0}$] are defined by

$$\langle A(\mathbf{Z}(\omega))\phi, \psi\rangle_{H_0} = \int_\Omega -c^2(\omega\mu)^{-1}(\nabla \times \phi) \cdot (\nabla \times \psi) + (\omega\varepsilon)\phi \cdot \psi d\mathbf{x},$$

$$\langle L(\mathbf{Z}(\omega))\varphi, \psi\rangle_{H_0} = \int_\Omega c^2(\omega\mu)^{-1}(\nabla \times \varphi) \cdot (\nabla \times \psi) - (\omega\varepsilon)\varphi \cdot \psi d\mathbf{x},$$

  for all $\phi, \psi \in H_0(\mathrm{curl}, \Omega)$ and all $\varphi \in H(\mathrm{curl}, \Omega)$.
- Invertibility of $A(\mathbf{Z}(\omega))$ follows from Lax-Milgram Theorem by coercivity following from the hypothesis: for $\omega \in \mathbb{C}^+$, $\mathbf{Z}(\omega) = (\omega\varepsilon_1(\omega), \omega\varepsilon_2(\omega), \omega\mu_1(\omega), \omega\mu_2(\omega)) \in (\mathbb{C}^+)^4$.

Conclusion

- For bounded media, can generalize results to anisotropic media with $\omega \in \mathbb{C}^+$, $\mathbf{Z}(\omega) = (\omega\varepsilon_1(\omega), \omega\varepsilon_2(\omega), \omega\boldsymbol{\mu}_1(\omega), \omega\boldsymbol{\mu}_2(\omega)) \in \left(M_3^+(\mathbb{C})\right)^4$, $\omega \mapsto \omega\varepsilon_j(\omega)$ and $\omega \mapsto \omega\boldsymbol{\mu}_j(\omega)$ are Herglotz functions for $j = 1, 2$.
- For both layered and bounded media, can extend results to case of medium Ω composed with N anisotropic homogeneous phases with passive media in sense: the jth material $\omega \mapsto \omega\varepsilon_j(\omega)$ and $\omega \mapsto \omega\boldsymbol{\mu}_j(\omega)$ are Herglotz functions, for $j = 1, 2, \ldots, N$.
- Generalization for analyticity of EM DtN map: $\omega \mapsto \Lambda(\mathbf{Z}(\omega))$, $\mathbf{Z}(\omega) = (\omega\varepsilon_1(\omega), \ldots \omega\varepsilon_N(\omega), \omega\boldsymbol{\mu}_1(\omega), \ldots, \omega\boldsymbol{\mu}_N(\omega)) \in \left(M_3^+(\mathbb{C})\right)^{2N}$ to $\mathbf{Z} \mapsto \Lambda(\mathbf{Z})$, $\mathbf{Z} \in \left(M_3^+(\mathbb{C})\right)^{2N}$.
- Looking forward: Apply the theory of composites to bear on a much wider class of electromagnetic problems, e.g., inverse problems, bounds/limitations on certain physical quantities, and studying EM phenomenon in composites with high-loss and lossless components.

Auxiliary Slides