CIRM, Luminy, 18 April 2016

Stability of Frustration-Free Ground States of Quantum Spin Systems

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Joint work with

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[1]Based on work supported by the U.S. National Science Foundation under grant # DMS-1515850.

Frustration-Free Quantum Spin Models

A quantum spin system is a collection of quantum systems labeled by x in a finite set Λ, each with a finite-dimensional Hilbert space of states $\mathcal{H}_x$. For concreteness, consider $\Lambda \subset \mathbb{Z}^\nu$.

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x.$$

The algebra of observables for the subsystem in $X \subset \Lambda$ is

$$\mathcal{A}_X = \bigotimes_{x\in X} \mathcal{B}(\mathcal{H}_x).$$

The Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$ is defined in terms of an interaction Φ: for any finite $X \subset \mathbb{Z}^\nu$, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, and

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X).$$

In this talk most interactions will be of finite range, i.e., there is $R \geq 0$, such that $\Phi(X) = 0$ if $\operatorname{diam} X > R$.

The model defined by a finite-range interaction Φ is Frustration-Free (FF) if for all finite $\Lambda \subset \mathbb{Z}^\nu$

$$\inf \operatorname{spec} H_\Lambda = \sum_{X \subset \Lambda} \inf \operatorname{spec} \Phi(X).$$

Equivalently, there is a ground state of H_Λ that is simultaneously a ground state of all $\Phi(X)$, for $X \subset \Lambda$.

Note that a frustration-free interaction may have infinite volume ground states in which some of the terms $\Phi(X)$ have expectation strictly greater than their minimal eigenvalue. In this situation we distinguish two types of ground states: frustration-free and non-frustration-free ground states.

Outline

- A few examples of frustration-free quantum spin models
- Gapped ground state phases
- Stability of the spectral gap
- Stability of gapped ground state phases
- Non-frustration-free ground states of the Toric Code Model
- Outlook

A few examples of frustration-free quantum spin models

1. The first quantum spin model we learn about is frustration-free (FF): the ferromagnetic spin-1/2 Heisenberg model (Heisenberg 1928).
For each $x \in \mathbb{Z}^\nu$, $\mathcal{H}_x = \mathbb{C}^2$ and

$$H_\Lambda = - \sum_{|x-y|=1} \mathbf{S}_x \cdot \mathbf{S}_y .$$

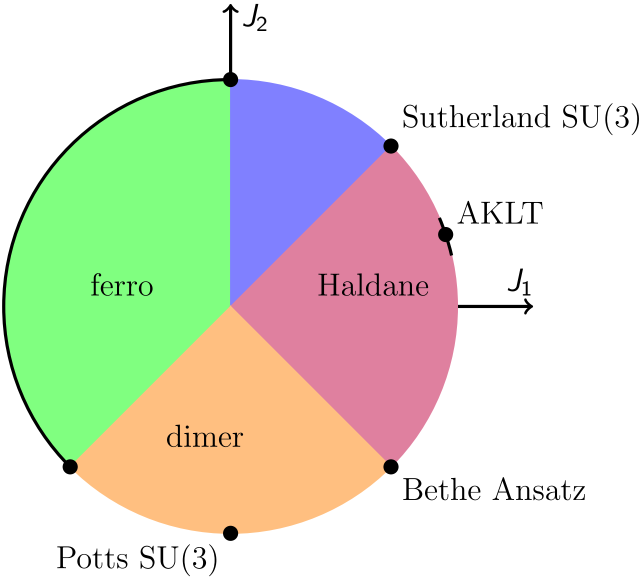
The ground states are easily found to be the states of maximal spin, which are common eigenvectors of all the terms $-\mathbf{S}_x \cdot \mathbf{S}_y$, with the minimal eigenvalue $-1/4$.
The ground state space is spanned by product states. The continuous symmetry of simultaneous rotations of the spins is broken; hence the there is no gap in the spectrum above the ground state in infinite volume.

2. The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987-88).
$\Lambda \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$;

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L} P^{(2)}_{x,x+1}$$

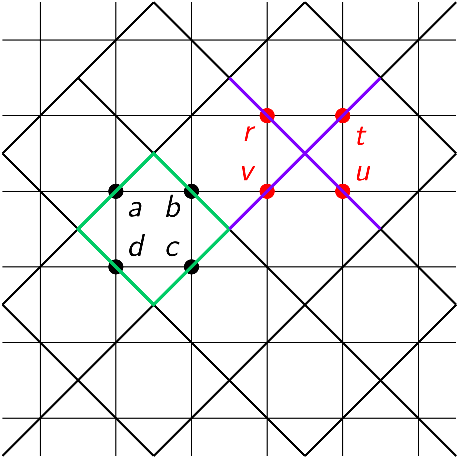
In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase). Ground state is frustration free (Valence Bond Solid state (VBS), aka Matrix Product State (MPS), aka Finitely Correlated State (FCS))., and has String Order (den Nijs-Rommelse 1989): support is span of

$$\cdots 0\underline{1}0010\underline{1}1000\underline{1}0000101\underline{1} \cdots$$



$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

3. Toric Code model (Kitaev, 2003). $\Lambda \subset \mathbb{Z}^2$, $\mathcal{H}_x = \mathbb{C}^2$.



$$H = -\sum_p h_p - \sum_s h_s$$

$$h_p = \sigma^3_a \sigma^3_b \sigma^3_c \sigma^3_d$$

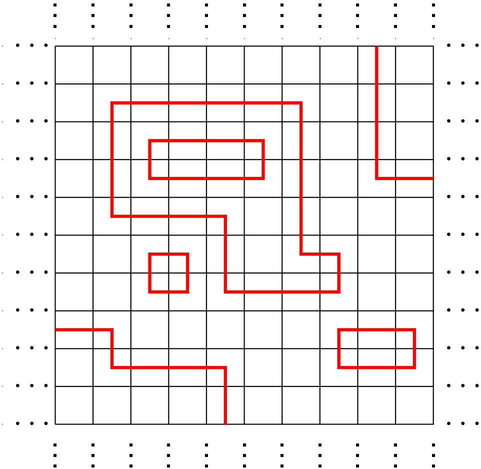
$$h_s = \sigma^1_r \sigma^1_t \sigma^1_u \sigma^1_v$$

On a surface of genus g, the model has 4^g frustration free ground states.

A ground state is a superposition (with equal coefficients) of all configurations of $\pm$'s satisfying the condition $h_p = 1$ for each plaquette p: each equivalence class of configurations related by the action of the star operators h_s gives rise to a ground state. The number of equivalence classes depends on the topology of the lattice:

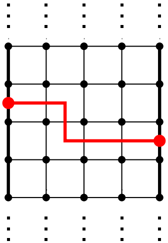
- On a finite square, with suitable boundary conditions, there is only one such equivalence class
- If the boundary has more than 1 connected component, multiple equivalence classes exist
- on a compact surface of genus g, the model has 4^g frustration free ground states.

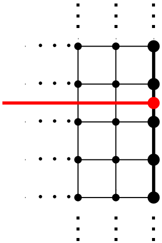
Picture: line-like defects (excitations) that bind to the boundary or form topologically nontrivial closed curves. In all cases there is a gap in the spectrum above the ground states. Example of "topological insulator".

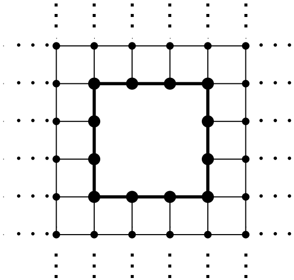


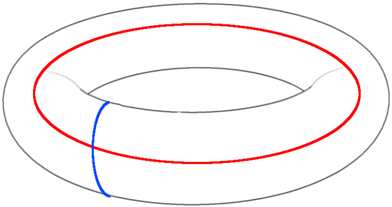
Dual edges in red denote a $-$ at intersection with the edge in black.

Some simple examples of non-trivial topologies:









etc.

Gapped ground state phases

The main motivation for the current research on FF models stems from the surge of interest in gapped ground state phases, including topologically ordered phases.

The term gapped refers to the existence of a positive lower bound for the energy of excited states with respect to a ground state, uniformly in the size of the system. This implies a gap in the spectrum of the GNS Hamiltonian of the ground state of the infinite system.

The term phase refers to regions in a interaction space where the gap is positive (open). Phase transitions in interaction space can occur when the gap vanishes (closes).

Topological Order and Discrete Symmetry Breaking are usually accompanied by a non-vanishing spectral gap.

Proofs of a gap: Affleck-Kennedy-Lieb-Tasaki (1988), Fannes-N-Werner (1992), N (1996), Kitaev (2006), Bachmann-Hamza-N-Young (2014), Bravyi-Gosset (2015), Gosset-Mozgunov (2015), Bishop-N-Young (2016).

Proofs of stability:

'classical' results by Kennedy-Tasaki, Datta-Fernandez-Fröhlich, Borgs-Kotecky-Uetlschi, and others(1980-90s),

More recently: Yarotsky (2004), Bravyi-Hastings-Michalakis (2010), Michalakis-Zwolak (2013), Cirac-Michalakis-PerezGarcia-Schuch (2013), Szehr-Wolf (2015), N-Sims-Young (in prep)

Moreover it is believed that any type of gapped ground state can be well approximated by a ground state of a gapped FF model. Results for spin chains ($d = 1$): Fannes-N-Werner (1992), Hastings (2007), Schuch-Cirac-Verstraete (2008), Landau-Vazirani-Vidick (2013-15).

Structure of Ground State Spaces: Topological vs Landau Order

Consider a quantum spin Hamiltonian on a finite set Λ, $H_\Lambda \in \mathcal{A}_\Lambda$, defined in terms of a finite range interaction Φ:

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

In a 'gapped phase' (and with suitable boundary conditions), we often expect the spectrum of H_Λ to have the following structure:

$$\mathrm{spec}(H_\Lambda) \subset [E_\Lambda(0), E_\Lambda(0) + \delta_\Lambda] \cup [E_\Lambda(0) + \delta_\Lambda + \gamma_\Lambda, \infty)$$

for some $\delta_\lambda \geq 0$ and $\gamma_\Lambda > 0$. The simplest situation is when $\delta_\Lambda \to 0$ as $\Lambda \to \mathbb{Z}^\nu$, and $\gamma_\Lambda \geq \gamma > 0$, for all Λ.
Let $\mathcal{G}_\Lambda$ denote the spectral subspace associated with the spectrum in $[E_\Lambda(0), E_\Lambda(0) + \delta_\Lambda]$.

For concreteness, suppose Φ has a local symmetry described by a finite group G: for every $x \in \Lambda$, there is a unitary representation $u_x(g)$, $g \in G$, acting on $\mathcal{H}_x$, such that

$$[\Phi(X), U_X(g)] = 0, U_X(g) = \bigotimes_{x \in X} u_x(g), \text{ for all } g \in G.$$

If this symmetry is fully broken in the (infinite-volume) ground states, we expect a decomposition of $\mathcal{G}_\Lambda$ labeled by $g \in G$:

$$\mathcal{G}_\Lambda = \bigoplus_{g \in G} \mathcal{G}_\Lambda^g.$$

Example: the $\mathbb{Z}_2$-symmetry of the Ising model. In general, direct sum is not necessarily orthogonal.

Let P_Λ denote the $\perp$ projection onto $\mathcal{G}_\Lambda$ and P_Λ^g the $\perp$ projection onto $\mathcal{G}_\Lambda^g$. For a suitable sequence of finite volumes Λ_n we can obtain the symmetry broken ground states in the thermodynamic limit:

$$\omega^g(A) = \lim_{n\to\infty} \frac{\mathrm{Tr} P_\Lambda^g A}{\mathrm{Tr} P_\Lambda^g},$$

for any local observable A. Symmetry breaking means that there is a local order parameter that distinguishes the states:

$$\omega^g(m) = m^g.$$

If there is translation invariance it follows that any two states giving different values to m must become orthogonal in the infinite volume limit.

Stronger even, we expect that for any local observable $A \in \mathcal{A}_X$, and unit vectors $\psi^i_{\Lambda_n} \in \mathcal{G}^{g_i}_{\Lambda_n}$, $i = 1, 2$, we have

$$\lim_n \langle \psi^1_{\Lambda_n}, A\psi^2_{\Lambda_n} \rangle = 0, \text{ if } g_1 \neq g_2.$$

What about the different $\psi_{\Lambda_n} \in \mathcal{G}^g_{\Lambda_n}$, with the same g? The Local Topological Quantum Order (LTQO) property first introduced by Bravyi, Hastings, and Michalakis, generalized to the situation with a symmetry G, asserts the following (simplifying slightly): for $X \subset \Gamma$ and $n \geq 0$, define

$$X(n) = \{y | d(X, y) \leq n\}.$$

Then for all $A \in \mathcal{A}_X$ s.t., $[A, U_X(g)] = 0$, for all $g \in G$,

$$\| P_{X(n)} A P_{X(n)} - \omega_\Lambda(A) P_{X(n)} \| \leq \Omega(n) \|A\|$$

as long as $d(X(n), \Lambda^c) \gg n$, and where $\Omega(n) \to 0$ as $n \to \infty$, faster than $1/n^p$ for some sufficiently large $p > 0$, and

$$\omega_\Lambda(A) = \text{Tr}(P_\Lambda A)/\text{Tr}(P_\Lambda).$$

If spontaneous symmetry breaking occurs, then for all $A \in \mathcal{A}_X$

$$\|P^g_{X(n)} A P^h_{X(n)} - \delta_{g,h}\omega^g_\Lambda(A) P^g_{X(n)}\| \leq \Omega(n)\|A\|$$

as long as $d(X(n), \Lambda^c) \gg n$, and where $\Omega(n) \to 0$ as $n \to \infty$, faster than $1/n^p$ for some sufficiently large $p > 0$, and

$$\omega^g_\Lambda(A) = \mathrm{Tr}(P^g_\Lambda A)/\mathrm{Tr}(P^g_\Lambda).$$

Stability under uniformly small perturbations

The Michalakis-Zwolak (née Pytel) stability result (CMP, 2013) applies to models with frustration-free finite-range interactions on periodic boxes in $\mathbb{Z}^\nu$. We (N-Sims-Young) recently obtained a generalization which includes situations with discrete symmetry breaking and more general lattices and boundary conditions.

Let $B_x(R)$ denote the ball of radius R centered at x in a discrete metric space (Γ, d), and Λ is a finite subset of Γ. Then,

$$H_\Lambda(0) = \sum_{\substack{x \in \Lambda \\ B_x(R) \subset \Lambda}} Q_x,$$

where each term $Q_x \in \mathcal{A}_{B_x(R)}$, satisfies $0 \leq Q_x \leq M\mathbb{1}$, and $[Q_x, U_{B_x(R)}(g)] = 0$, for all $g \in G$.

We consider perturbations of the following form:

$$H_\Lambda(\lambda) = H_\Lambda(0) + \lambda \sum_{X \subset \Lambda} \Phi(X).$$

and we will assume that there exists $a > 0$ such that

$$\|\Phi\|_a = \sup_{x,y \in \Gamma} e^{ad(x,y)} \sum_{\substack{X \subset \Gamma \\ x,y \in X}} \|\Phi(X)\| < \infty.$$

(Michalakis-Zwolak claim perturbations with power law decay can be treated too, but we have not been able to verify that.)

The assumptions on the unperturbed model are:
- It is Frustration Free: $\ker H_\Lambda(0) \neq \{0\}$; Let $P_\Lambda(\lambda)$ denote the orthogonal projection onto $\ker H_\Lambda(\lambda)$. Assume convergence

$$\omega(A) = \lim_n \frac{1}{\dim \ker H_{\Lambda_n}} \mathrm{Tr} P_{\Lambda_n}(0) A, \quad A \in \mathcal{A}_{\mathrm{loc}},$$

for a suitable sequence $\Lambda_n \nearrow \Gamma$.
- Local Gap: there is $\gamma > 0$ such that the gap above the ground state of $H_{B_x(r)} \geq \gamma$ for all x and r;
- Local Topological Quantum Order (LTQO): there is a scale L^*, $L^* \to \infty$ as $L \to \infty$ such that, for all $r \leq L^*$, and all $A \in \mathcal{A}_{B_x(r)}$, s.t. $[A, U_{B_x(r)}(g)] = 0, g \in G$,

$$\|P_{B_x(r+\ell)} A P_{B_x(r+\ell)} - \omega(A) P_{B_x(r+\ell)}\| \leq \Omega(\ell) \|A\|$$

with $\Omega(\ell)$ decaying as a sufficiently large inverse power.

Stability of the Spectral Gap

Let $E_\Lambda(\lambda) = \inf \mathrm{spec}(H_\Lambda(\lambda))$. The gap of $H_\Lambda(\lambda)$ is defined taking into account that the perturbation may produce a splitting up to an amount δ_Λ of the zero eigenvalue of $H_\Lambda(0)$, which is in general degenerate:

$$\gamma_\delta(H_\Lambda(\lambda)) = \sup\{\eta > 0 \mid (\delta, \delta+\eta) \cap \mathrm{spec}(H_\Lambda(\lambda) - E_\Lambda(\lambda)\mathbb{1}) = \emptyset\}$$

Theorem (Bravyi-Hastings, Michalakis-Zwolak, N-Sims-Young)

Let $H_\Lambda(0)$ be a finite-range G-symmetric Hamiltonian satisfying the assumptions of above and Φ an exponentially decaying G-symmetric perturbation. Then, for any $0 < \gamma_0 < \gamma(H_\Lambda(0))$ there is an $\lambda_0 > 0$ such that for sufficiently large Λ,

$$\gamma_{\delta_\Lambda}(H_\Lambda(\lambda)) \geq \gamma_0, \; \text{if } |\lambda| \leq \lambda_0,$$

where $\delta_\Lambda \leq C\,\mathrm{diam}(L)^{-q}$, for some $q > 0$.

Stability of the Ground State Phases

Next, consider the situation where in the unperturbed model we have spontaneous breaking of the symmetry G in the frustration-free ground states. Concretely, we will assume the following:

The unperturbed model is defined on finite volume Λ as before:

$$H_\Lambda(0) = \sum_{\substack{x \in \Lambda \\ B_x(R) \subset \Lambda}} Q_x,$$

where each term $Q_x \in \mathcal{A}_{B_x(R)}$, satisfies $0 \leq Q_x \leq M\mathbb{1}$, and $[Q_x, U_{B_x(R)}(g)] = 0$, for all $g \in G$.

We now assume that there are N pure infinite-volume frustration-free ground states, $\omega^1, \ldots, \omega^N$, and the symmetries, $g \in G$, act transitively as permutations on this set.

For sufficiently large m, there are N non-zero orthogonal projections $P^1_{b_x(m)}, \ldots, P^N_{b_x(m)}$, onto subspaces of $\ker H_{b_x(m)}$ such that the following properties hold:

1. $\mathrm{ran} \sum_{i=1}^N P^i_{b_x(m)} = \ker H_{b_x(m)}$ and
$$\left\| P_{b_x(m)} - \sum_{i=1}^N P^i_{b_x(m)} \right\| \leq \Omega(m); \qquad (1)$$
2. There is a one-to-one correspondence between the projections $P^i_{b_x(m)}$ and the pure infinite-volume ground states ω^i as follows:
$$\omega^i(A) = \lim_{m\to\infty} \frac{\mathrm{Tr} P^i_{b_x(m)} A}{\mathrm{Tr} P^i_{b_x(m)}}, \qquad (2)$$
3. For any $A \in \mathcal{A}_{b_x(k)}$ we have
$$\left\| P^i_{b_x(k+\ell)} A P^j_{b_x(k+\ell)} - \delta_{ij} \omega^i(A) P^i_{b_x(k+\ell)} \right\| \leq \|A\| \Omega(\ell). \qquad (3)$$

We say that the model satisfies Local Topological Quantum Order with N G-broken phases.

It is natural to assume that

$$\omega_0 = \frac{1}{N} \sum_{i=1}^{N} \omega^i \tag{4}$$

is the unique G-invariant frustration-free ground state.
The perturbations are of the form

$$H_\Lambda(\lambda) = H_\Lambda(0) + \lambda \sum_{X \subset \Lambda} \Phi(X).$$

such that $[\Phi(X), U_X(g)] = 0$, for all $g \in G$, and $\|\Phi\|_a < \infty$ for some $a > 0$.
Let $\mathcal{S}_\lambda$ denote the set of all thermodynamic limits of ground states of $H_\Lambda(\lambda)$

Theorem (N-Sims-Young, in prep)

There exists $\lambda_0 > 0$ *such that if* $|\lambda| \leq \lambda_0$, *then the set* $\mathcal{S}_\lambda$ *is an N-dimensional simplex. Each of the extreme points (pure states) satisfies LTQO and has a non-vanishing spectral gap in the spectrum of its GNS Hamiltonian.*

The main tool in the proof is the spectral flow, which is contructed using Lieb-Robinson bounds for the dynamics and related transformations.

We also prove that the thermodynamic limit of the spectral flow yields quasi-local automorphisms α_λ such that

$$\mathcal{S}_\lambda = \{\omega \circ \alpha_\lambda \mid \omega \in \mathcal{S}_0\}.$$

Therefore, the entire phase structure is preserved under the perturbations.

Non-FF ground state of the Toric Code model

For infinite systems, ground states are those states ω on $\mathcal{A}_{\mathrm{loc}} = \bigcup_{X \subset \Gamma} \mathcal{A}_X$, such that

$$\lim_{\Lambda \uparrow \Gamma} \omega(A^*[H_\Lambda, A]) \geq 0, \quad \text{for all } A \in \mathcal{A}_{\mathrm{loc}}. \tag{5}$$

Alicki, Fannes, and Horodecki (2007) showed that the Toric Code model on $\mathbb{Z}^2$ has a unique FF ground state.
However, any ω obtained as a weak*-limit of ground states of a sequence of Hamiltonians $\tilde{H}_\Lambda$ with

$$\lim_{\Lambda \uparrow \Gamma} [\tilde{H}_\Lambda, A] = \lim_{\Lambda \uparrow \Gamma} [H_\Lambda, A], \quad \text{for all } A \in \mathcal{A}_{\mathrm{loc}},$$

satisfies (5).

Using this, it is not hard to show that the Toric Code model has 3 families of additional non-FF ground states, with topological charge, *e*, *m*, and $\epsilon = em$, respectively (half-infinite strings) . Together with the FF ground state these correspond to the four superselection sectors of the Toric Code model, as shown by Naaijkens (2011-13).
Since these are not FF ground states, the previous stability theorem cannot be directly applied. They are not expected to be stable as ground states, but there is a framework in which we can show stability of the 4 super-selection sectors and their structure as abelian anyons. (work in progress with Matthew Cha and Pieter Naaijkens).

Outlook

- Frustration free (FF) models turned out to be an essential tool to help us understand gapped ground state phases and their classification.
- Progress in estimating the spectral gap above the ground state has come from studying FF models.
  Well-understood in one dimension. Next step: higher dimensions (so far only special classes of examples in d dimensions (Bishop-N-Young, JSP 2016).
- Stability results of ground states is based on FF models. Next step: relax the FF condition.
- Progress in stability results of superselection sectors is based on FF models with commuting terms: next step: treat more physically realistic interactions.