Rigorous Analogies Between Quantum Systems and Certain Wave Equations

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2016.04.09@CIRM

Idea

Realizing Quantum Effects with Classical Waves

Making Quantum Analogies Rigorous

Develop and explore the *Schrödinger formalism* for certain *classical wave equations*

- Allows for adaptation of techniques from quantum mechanics to other wave equations
- Also differences, e. g. classical waves $\mathbb{R}$-valued

Advances in Understanding of Quantum Systems

- Spectral theory
- Scattering theory
- Semiclassical limits
- Perturbation theory
- Non-linear effects
- Periodic operators
- Random operators
- Topological insulators

$\rightsquigarrow$ Adapt and apply these techniques to other wave equations

Some Relevant Wave Equations

Classical electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} - \begin{pmatrix} j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla \cdot \varepsilon \mathbf{E} \\ \nabla \cdot \mu \mathbf{H} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

Transverse acoustic waves

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \rho_0 \\ -\rho_0^{-1} \nabla \gamma v_s^2 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

Magnons

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix} = \sigma_3 \, H(k) \begin{pmatrix} \beta(k) \\ \beta^\dagger(-k) \end{pmatrix}$$

Characteristics

1. First order in *time*
2. Product structure of operators
3. Waves take values in $\mathbb{R}^N$

Other examples

Plasmons, magnetoplasmons, van Alvén waves, etc.

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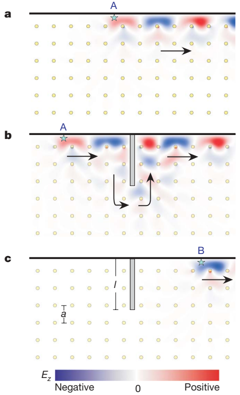
Experimental Realizations of *Quantum* Analogies

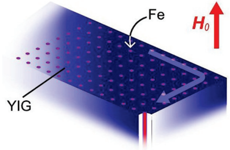
Topological Boundary States

Photonic



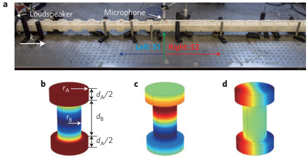
Joannopoulos, Soljačić et al (2009)

Magnonic



Shindou, Matsumoto et al (2013)

Acoustic



Xiao, Ma et al (2015)

Despite Experiments ...

... **first-principle derivations** are scarce, be it rigorous or non-rigorous!

$\rightsquigarrow$ Open field with lots of interesting problems!

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H, typical examples are

$$H = \tfrac{1}{2m}\bigl(-\mathrm{i}\nabla - A\bigr)^2 + V$$
$$H = m\,\beta + \bigl(-\mathrm{i}\nabla - A\bigr)\cdot\alpha + V$$

2. Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^N)$ where $\langle \phi, \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x\, \phi(x) \cdot \psi(x)$
3. Dynamics given by Schrödinger equation

$$\mathrm{i}\partial_t \psi(t) = H\psi(t), \qquad \psi(0) = \phi$$

Recap: Schrödinger Equation on $\mathbb{R}^d$

Fundamental Constituents

1. Hamilton/Schrödinger operator H
2. Hilbert space
3. Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}t\,H} \phi$
- $\left\|\psi(t)\right\|^2 = \|\phi\|^2$ (conservation of propability)

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W_L \, D \, W_R$ where
   - $W := W_R \, W_L^{-1} = W^* \, , 0 < c \, \mathrm{id} \leq W \leq C \, \mathrm{id}$
     (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. Complex (!) *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^N)$ where

$$\langle \phi, \psi \rangle_W = \langle \phi, W^{-1} \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot W^{-1} \psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i} \partial_t \psi(t) = M \psi(t) , \qquad \psi(0) = \phi$$

4. *Even particle-hole symmetry* K, i. e.
   K antiunitary, $K^2 = +\mathrm{id}$ and $K \, M \, K = -M$

Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* $M = W \, D$ where
   - $W = W^*$, $0 < c \, \mathrm{id} \leq W \leq C \, \mathrm{id}$ (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. Complex (!) **weighted** *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^N)$ where

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Schrödinger Formalism for Classical Waves

Fundamental Constituents

1. *"Hamilton" operator* with **product structure**
2. Complex (!) **weighted** *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^N)$
3. Dynamics given by *Schrödinger equation*
4. *Even particle-hole symmetry* K

Properties

- $M^{*_W} = M$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}t\,M} \phi$
- $\lVert \psi(t) \rVert_W^2 = \lVert \phi \rVert_W^2$ (conserved quantity, e. g. energy)
- $\mathrm{Re}\; \mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}$ where $\mathrm{Re} = \frac{1}{2}(\mathrm{id} + K)$
  (support of real solutions)

Quantum-Light Analogies

	Wave Equation	**Quantum Mechanics**
Generator dynamics	Maxwell-type operator $M = W\,D = M^*$	hamiltonian $H = -\Delta + V = H^*$
Necessary symmetry	+PH	none
Hilbert space	weighted L^2	L^2
Wave function	$\mathbb{R}$-valued	$\mathbb{C}$-valued
Conserved quantity $\lVert\Psi\rVert^2$	e. g. field energy	probability

Aim of this Section

Make a first-principles derivation of the Schrödinger formalism for electromagnetic waves, i. e. identify

1. "Hamilton" operator $M = W\,D$
2. Hilbert space
3. Schrödinger equation
4. Even particle-hole symmetry

Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix} - \begin{pmatrix}J^D\\ J^B\end{pmatrix} \qquad \text{(dynamical eqns.)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}\rho^D\\ \rho^B\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \mathcal{W}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge*

$$\nabla\cdot J^\sharp + \rho^\sharp = 0, \quad \sharp = D, B$$

Fundamental Equations

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$$\begin{pmatrix}\nabla \cdot \mathbf{D}\\ \nabla \cdot \mathbf{B}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = W \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge* $\rightsquigarrow$ neglect sources for simplicity

$$\nabla \cdot J^\sharp + \rho^\sharp = 0, \qquad \sharp = D, B$$

Constitutive Relations

For a **linear** medium the constitutive relations maps a **trajectory**

$$(-\infty, t] \ni s \mapsto \big(\mathbf{E}(s), \mathbf{H}(s)\big)$$

onto

$$\begin{pmatrix} \mathbf{D}(t,x) \\ \mathbf{B}(t,x) \end{pmatrix} := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s,x) \begin{pmatrix} \mathbf{E}(s,x) \\ \mathbf{H}(s,x) \end{pmatrix}$$

$\rightsquigarrow$ reaction of medium to impinging em wave depends on the **past**

Constitutive Relations

$$\big(\mathbf{D}(t), \mathbf{B}(t)\big) := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s) \, \big(\mathbf{E}(s), \mathbf{H}(s)\big)$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is real,* $W = \overline{W}$*, and*
2. *satisfies the causality condition* $W(t) = 0$ *for all* $t > 0$*.*

Constitutive Relations

$$(\mathbf{D}(t), \mathbf{B}(t)) = \big(W * (\mathbf{E}, \mathbf{H})\big)(t)$$

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Rewriting the Dynamical Equations

$$\frac{\partial}{\partial t} W * \Psi = -\mathrm{i}\,\mathrm{Rot}\,\Psi := -\mathrm{i}\begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}\Psi$$

$$\Longleftrightarrow$$

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

where $\Psi = (\mathbf{E}, \mathbf{H})$ is the electromagnetic field

Rewriting the Dynamical Equations

$$\mathrm{i}\,\tfrac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

$$\Big\downarrow \mathscr{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

Reality condition implies

$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega) = \overline{\widehat{W}(\omega)}$$

Real solutions = linear combinations of complex waves of $\pm\omega(\pm k)$

$$\cos(k \cdot x - \omega t) = \frac{1}{2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} + \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Re}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

$$\sin(k \cdot x - \omega t) = \frac{1}{\mathrm{i}2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} - \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Im}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

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Approximate Maxwell Equations for $\omega \approx \omega_0$

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi = \mathrm{Rot}\,\Psi$$

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$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

1. Approximate material weights $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$
2. Undo Fourier transform

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Real solutions linear combination of $\pm\omega \rightsquigarrow$ pair of equations

$$\omega > 0 : \qquad \begin{cases} \widehat{W}(\omega_0)\, \mathrm{i}\partial_t \Psi = \mathrm{Rot}\, \Psi \\ \mathrm{Div}\, \widehat{W}(\omega_0)\, \Psi = 0 \end{cases}$$

$$\omega < 0 : \qquad \begin{cases} \widehat{W}(-\omega_0)\, \mathrm{i}\partial_t \Psi = \mathrm{Rot}\, \Psi \\ \mathrm{Div}\, \widehat{W}(-\omega_0)\, \Psi = 0 \end{cases}$$

$$W = \overline{W} \iff \widehat{W}(-\omega_0) = \overline{\widehat{W}(\omega_0)}$$

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$$W = \overline{W} \quad \Longleftrightarrow \quad \widehat{W}(-\omega_0) = \overline{\widehat{W}(\omega_0)}$$

Difficulties Controlling $\widehat{W}(\omega) \approx \widehat{W}(\omega_0)$

Making this approximation rigorous is difficult:

1. **Behavior of** $\omega \mapsto \widehat{W}(\omega)$
   - Properties very different for different frequency regimes (lossless, reflective, resonant, etc.)
   - $\omega \mapsto \widehat{W}(\omega)$ analytic on $\mathbb{C}^+$
   - Meaningful assumptions on $\widehat{W}(\omega)$ hard to stipulate explicitly
   - **Solution:** Focus on light from a **narrow frequency window** (where $\widehat{W}$ is well-behaved)
2. **Material has "memory"**
   $\rightsquigarrow$ *constitutive relations* depend on **past trajectory**
   - Initial condition for full Maxwell equations: *trajectory* $(-\infty, t_0] \ni (E(t), H(t))$
   - Approximate equations have "no memory"
     $\rightsquigarrow$ initial condition $(E(t_0), H(t_0))$ at a single point in time
   - *How to pick and compare initial conditions?*
   - **Solution:** Use same **sources** in full and approximate equations

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Technical Assumptions

Assumption (Material weights)

$$\widehat{W}(\omega_0, x) = W_+(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$$

1. $W_+^* = W_+$ *(lossless)*
2. $0 < c\,\mathbf{1} \leq W_+ \leq C\,\mathbf{1}$ *(excludes negative index materials)*

Remark

$W_- = \overline{W_+}$ *satisfies the same assumptions if and only if* W_+ *does.*

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Gyrotropic vs. Non-Gyrotropic Materials

Definition (Gyrotropic material weights)

- If $W_+ = W_- = \overline{W_+}$ are **non-gyrotropic**.
- If $W_+ \neq W_- = \overline{W_+}$ we call **gyrotropic**.

Let us treat the non-gyrotropic case first where

$$W_+ = W_- .$$

$\Rightarrow$ Equations for $\omega > 0$ and $\omega < 0$ coincide!

$$\omega > 0 : \qquad \begin{cases} W_+ \, \mathrm{i}\partial_t \Psi_+ = \mathrm{Rot}\, \Psi_+ \\ \mathrm{Div}\, W_+ \, \Psi_+ = 0 \end{cases}$$

$$\omega < 0 : \qquad \begin{cases} W_- \, \mathrm{i}\partial_t \Psi_- = \mathrm{Rot}\, \Psi_- \\ \mathrm{Div}\, W_- \, \Psi_- = 0 \end{cases}$$

Let us treat the non-gyrotropic case first where

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E},\mathbf{H}) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

3. *No sources*

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E}, \mathbf{H}) = \mathcal{E}(\mathbf{E}(t), \mathbf{H}(t))$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\left\| (\mathbf{E}, \mathbf{H}) \right\|^2_{W_+} = \int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

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$$J_{W_+} = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\}$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

$$\big\| (\mathbf{E}, \mathbf{H}) \big\|^2_{W_+} = 2\, \mathcal{E}(\mathbf{E}, \mathbf{H})$$

2. *Dynamical equations* ⇝ »Schrödinger equation«

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

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1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$ with energy scalar product

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2. *Dynamical equations* $\rightsquigarrow$ »Schrödinger equation«

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\varepsilon & \chi \\ \chi^* & \mu\end{pmatrix}^{-1}\begin{pmatrix}0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

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2. *Dynamical equations* $\rightsquigarrow$ »Schrödinger equation«

$$\mathrm{i} \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}}_{= \Psi} = \underbrace{\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i} \nabla^{\times} \\ -\mathrm{i} \nabla^{\times} & 0 \end{pmatrix}}_{= \widetilde{M}_+} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

3. *No sources*

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$$\mathrm{i}\,\tfrac{\partial}{\partial t}\Psi = \widetilde{M}_+\Psi$$

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3. *No sources* $\rightsquigarrow$ implements $\omega \neq 0$

$$J_{W_+} = G^{\perp_{W_+}}, \qquad G = \text{gradient fields}$$

The Maxwell Operator

$$\widetilde{M}_+ = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^\times \\ -\mathrm{i}\nabla^\times & 0 \end{pmatrix}$$
$$= W_+^{-1} \, \mathrm{Rot}$$
$$\mathcal{D}(M_+) = \bigl(H^1(\mathbb{R}^3, \mathbb{C}^6) \cap \ker \mathrm{Div} \bigr) \widehat{\oplus} \mathrm{ran}\, \mathrm{Grad}$$

$\widetilde{M}_+ = \widetilde{M}_+^*$ selfadjoint on *weighted* Hilbert space $L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$

$$\bigl\langle \Psi, \widetilde{M}_+ \Phi \bigr\rangle_{W_+} = \bigl\langle \Psi, W_+ W_+^{-1} \, \mathrm{Rot} \, \Phi \bigr\rangle = \bigl\langle \mathrm{Rot} \, \Psi, \Psi \bigr\rangle$$
$$= \bigl\langle W_+ \widetilde{M}_+ \Psi, \Phi \bigr\rangle = \bigl\langle \widetilde{M}_+ \Psi, W_+ \Phi \bigr\rangle = \bigl\langle \widetilde{M}_+ \Psi, \Phi \bigr\rangle_{W_+}$$

$\Rightarrow \mathrm{e}^{-\mathrm{i}t \widetilde{M}_+}$ **unitary**, yields **conservation of energy**

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$$\begin{aligned}\widetilde{M}_+ &= \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^\times \\ -\mathrm{i}\nabla^\times & 0 \end{pmatrix} \\ &= W_+^{-1} \, \mathrm{Rot} \\ \mathcal{D}(M_+) &= \mathcal{D}(\mathrm{Rot})\end{aligned}$$

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Fundamental Constituents

Non-Gyrotropic Media

1. *"Hamilton" operator* $M = W_+^{-1} \, \mathrm{Rot} \big|_{J_{W_+}}$ where
   - $W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1}$ are the *material weights* and
   - $\mathrm{Rot} = -\sigma_2 \times \nabla^\times$ is the *free Maxwell operator*
2. *Hilbert space* $J_{W_+} \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$ where

$$\langle \phi, \psi \rangle_{W_+} = \langle \phi, W_+ \psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \phi(x) \cdot W_+(x) \psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \phi$$

4. *Even particle-hole symmetry*: Complex conjugation C

Restriction to Real States

$(C\Psi)(x) := \overline{\Psi(x)}$ complex conjugation

$$W_- = \overline{W_+} = C\, W_+\, C = W_+$$
$$\Longrightarrow$$
$$C\, \widetilde{M}_+\, C = \, - \, \widetilde{M}_- = -\widetilde{M}_+$$

$\Longrightarrow C$ is an **even particle-hole symmetry**

Thus, $C\, \mathrm{e}^{-\mathrm{i}t\, \widetilde{M}_+} = \mathrm{e}^{-\mathrm{i}t\, \widetilde{M}_-}\, C = \mathrm{e}^{-\mathrm{i}t\, \widetilde{M}_+}\, C$ and

$$\mathrm{Re}\; \mathrm{e}^{-\mathrm{i}t\, \widetilde{M}_+} = \mathrm{e}^{-\mathrm{i}t\, \widetilde{M}_+}\, \mathrm{Re}$$

where $\mathrm{Re}\; := \frac{1}{2}(\mathrm{id} + C)$ is the real part operator

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Reconsidering the Case of *Gyrotropic* Materials

$W_+ \neq W_- \implies$ two **different** dynamical equations

$$\omega > 0: \quad \begin{cases} \mathrm{i}\partial_t \Psi_+ = \widetilde{M}_+ \Psi_+ \\ \mathrm{Div}\, W_+ \Psi_+ = 0 \end{cases}$$

$$\omega < 0: \quad \begin{cases} \mathrm{i}\partial_t \Psi_- = \widetilde{M}_- \Psi_- \\ \mathrm{Div}\, W_- \Psi_- = 0 \end{cases}$$

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Use spectral projections

$$P_\pm := 1_{\{\omega > 0\}}\bigl(\pm \widetilde{M}_\pm\bigr)$$

to restrict Maxwell operators to positive/negative frequencies

$$M_\pm := \widetilde{M}_\pm \big|_{\mathrm{ran}\, P_\pm} = P_\pm \, \widetilde{M}_\pm \, P_\pm \big|_{\mathrm{ran}\, P_\pm}$$

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$W_+ \neq W_- \implies$ two *different* dynamical equations

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$$\omega < 0 : \qquad \begin{cases} \mathrm{i}\partial_t \Psi_- = M_- \Psi_- \\ \mathrm{Div}\, W_- \, \Psi_- = 0 \end{cases}$$

Use spectral projections

$$P_\pm := 1_{\{\omega > 0\}}(\pm \widetilde{M}_\pm)$$

to restrict Maxwell operators to positive/negative frequencies

$$M_\pm := \widetilde{M}_\pm \big|_{\mathrm{ran}\, P_\pm} = P_\pm \, \widetilde{M}_\pm \, P_\pm \big|_{\mathrm{ran}\, P_\pm}$$

Reconsidering the Case of *Gyrotropic* Materials

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Maxwell Operator for Gyrotropic Media

Definition (Maxwell Operator)

$$M := M_+ \oplus M_-$$
$$\mathcal{D}(M) := \big(P_+ \mathcal{D}(\mathrm{Rot})\big) \oplus \big(P_- \mathcal{D}(\mathrm{Rot})\big)$$

seen as an operator on

$$\mathcal{H} := \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_- \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^3, \mathbb{C}^6).$$

Theorem

$M = M^*$ on $\mathcal{H}$.

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Theorem

$M = M^*$ *on* $\mathcal{H}$.

Fundamental Constituents

Gyrotropic Media

1. *"Hamilton" operator*
$$M = \Bigl(W_+^{-1} \, \mathrm{Rot} \big\vert_{\mathrm{ran}\, P_+} \Bigr) \oplus \Bigl(W_-^{-1} \, \mathrm{Rot} \big\vert_{\mathrm{ran}\, P_-} \Bigr)$$
2. *Hilbert space*
$$\mathcal{H} = \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_- \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \oplus L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6)$$
3. Dynamics given by *Schrödinger equation*
$$\mathrm{i}\partial_t \psi(t) = M \psi(t), \qquad \psi(0) = \bigl(P_+(\mathbf{E}, \mathbf{H}) \, , \, P_-(\mathbf{E}, \mathbf{H}) \bigr)$$
4. *Even particle-hole symmetry*: Complex conjugation
$K = \sigma_1 \otimes C$

Even Particle-Hole Symmetry $K = \sigma_1 \otimes C$

$M = M_+ \oplus M_-$ has permanent symmetry $K = \sigma_1 \otimes C$:

$$C\, P_+ \, C = 1_{\{\omega > 0\}} (C\, \widetilde{M}_+ \, C) = 1_{\{\omega > 0\}} (-\widetilde{M}_-) = P_-$$

because $W_- = C\, W_+ \, C$ and $C \,\mathrm{Rot}\, C = -\mathrm{Rot}$
$\Longrightarrow C\, M_+ \, C = -M_-$

$$\begin{aligned} K\, M\, K &= (\sigma_1 \otimes C)\,(M_+ \oplus M_-)\,(\sigma_1 \otimes C) \\ &= (C\, M_- C) \oplus (C\, M_+ \, C) = (-M_+) \oplus (-M_-) = -M \end{aligned}$$

$\Longrightarrow K$ is an **even particle-hole symmetry**

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Model Supports Real States

Theorem

$$\left.\begin{aligned} K\,M\,K &= -M \\ \mathrm{Re}_{\,K} &= \tfrac{1}{2}(\mathrm{id} + K) \end{aligned}\right\} \Longrightarrow \mathrm{Re}_{\,K}\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}_{\,K}$$

Model Supports Real States

Theorem

$$\left.\begin{aligned} K\,M\,K = -M \\ \mathrm{Re}_K = \tfrac{1}{2}(\mathrm{id} + K) \end{aligned}\right\} \implies \mathrm{Re}_K\, \mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\, \mathrm{Re}_K$$

Corollary

$$\mathrm{Re}_K \left(\mathrm{e}^{-\mathrm{i}t\,M_+}\, P_+ \oplus 0 \right) = \mathrm{e}^{-\mathrm{i}t\,M}\, \mathrm{Re}_K$$

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$$K = \sigma_1 \otimes C$$

Simplified Point of View

Is there an **easier approach**?

Reduction to Complex Fields with $\omega > 0$

A *complex* plane wave with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-it\omega(k)}\,\mathrm{e}^{+ik\cdot x}\,(\mathbf{E}_0,\mathbf{H}_0), \quad \omega(k) = |k|\,,\ \mathbf{E}_0,\mathbf{H}_0 \perp k,$$

defines *two* linearly independent *real* waves:

$$\begin{aligned} (\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) &= \mathrm{Re}\,\Psi_+ = \cos(k\cdot x - \omega t)\,(\mathbf{E}_0,\mathbf{H}_0) \\ (\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) &= \mathrm{Im}\,\Psi_+ = \sin(k\cdot x - \omega t)\,(\mathbf{E}_0,\mathbf{H}_0) \end{aligned}$$

Identification $\mathbb{R}$-VS $L^2_{\mathrm{trans}}(\mathbb{R}^3,\mathbb{R}^6)$ with $\mathbb{C}$-VS $\mathcal{H}_+ = \mathrm{ran}\,P_+$:

$$\alpha_{\mathrm{Re}}\,(\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) + \alpha_{\mathrm{Im}}\,(\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) = \mathrm{Re}\,\Big(\big(\alpha_{\mathrm{Re}} - i\alpha_{\mathrm{Im}}\big)\,\Psi_+\Big)$$

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A *complex* plane wave with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-\mathrm{i}t\omega(k)} \, \mathrm{e}^{+\mathrm{i}k\cdot x} \, (\mathbf{E}_0, \mathbf{H}_0), \quad \omega(k) = |k| \, , \; \mathbf{E}_0, \mathbf{H}_0 \perp k,$$

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Reduction to Complex Fields with $\omega > 0$

Bloch waves with $\omega > 0$

$$\Psi_+(t,k,x) = \mathrm{e}^{-\mathrm{i}t\omega_n(k)}\,\varphi_n(k,x), \quad M_+(k)\,\varphi_n(k) = \omega_n(k)\,\varphi_n(k),$$

defines *two* linearly independent *real* waves: Still true?

$$\begin{aligned}(\mathbf{E}_{\mathrm{Re}}\,,\mathbf{H}_{\mathrm{Re}}\,) &= \mathrm{Re}\,\Psi_+ \\ (\mathbf{E}_{\mathrm{Im}}\,,\mathbf{H}_{\mathrm{Im}}\,) &= \mathrm{Im}\,\Psi_+\end{aligned}$$

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Lemma

The $\mathbb{R}$-vector space of transversal, real vector fields $L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6)$ can be canonically identified with the $\mathbb{C}$-vector space of complex positive frequency fields $\mathcal{H}_+ = \mathrm{ran} \, P_+$. The vector space isomorphisms are

$$\begin{aligned} P_+ \, &: L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6) \longrightarrow \mathcal{H}_+ , \\ \mathrm{Re} \, &: \mathcal{H}_+ \longrightarrow L^2_{\mathrm{trans}}(\mathbb{R}^3, \mathbb{R}^6) . \end{aligned}$$

Reduction to Complex Fields with $\omega > 0$

Identification of $L^2_{\mathrm{trans}}(R^3, \mathbb{R}^6)$ with $\mathcal{H}_+ = \mathrm{ran}\, P_+$

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{Re}\,\left(\mathrm{e}^{-\mathrm{i}t\,M_+}\Psi_+\right)$$

where $\mathrm{Re} := \frac{1}{2}(\mathrm{id} + C)$ is the real part operator

Real states $\Longleftrightarrow$ Complex states with $\omega > 0$

Maxwell equations $\Longleftrightarrow$ $\mathrm{i}\partial_t \Psi_+ = M_+ \Psi_+$

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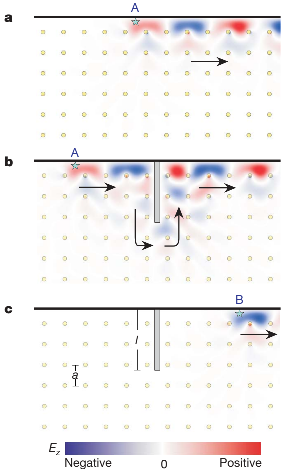
A Novel Class of Materials: *Photonic Topological Insulators*

$$\left.\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \quad \right\} \Longrightarrow$$

symmetry breaking



Joannopoulos, Soljačić et al (2009)

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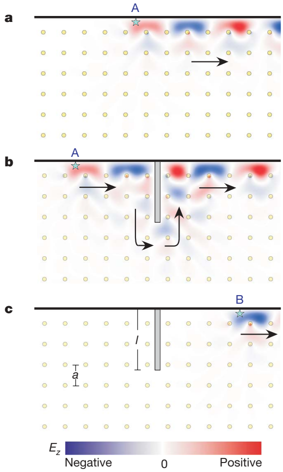

Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

$\downarrow$

- Identify topological observables
  $O = T +$ error

$\downarrow$

- Find all topological invariants T

$\downarrow$

- Classification of PhCs by symmetries

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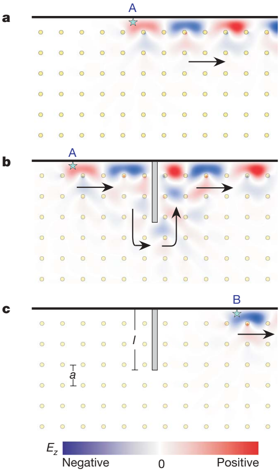

Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

↓

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↓

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↓

- **Classification of PhCs by symmetries**

Which Symmetries *Are* Broken?

Non-Gyrotropic Materials

$$W_+ = \overline{W_+}$$

1 Relevant Symmetry of Complexified Equation

$T : (\psi^E, \psi^H) \mapsto \big(\overline{\psi^E}, -\overline{\psi^H}\big)$ with $T\, M_+\, T = +M_+$ (+TR)

reverses arrow of time: $T\, \mathrm{e}^{-\mathrm{i}t\, M_+} = \mathrm{e}^{-\mathrm{i}(-t)\, M_+}\, T$

$\Longrightarrow$ *Needs to be broken to have unidirectional edge modes!*

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The Topology of Light States in Periodic Media

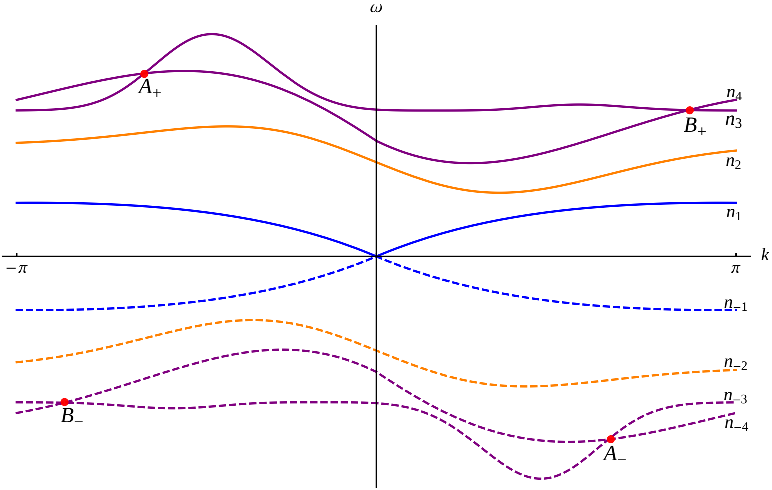
$$\left.\begin{array}{c}\text{Existence of topological}\\ \text{boundary states}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Topology of the (bulk)}\\ \text{Bloch bundle}\\ \mathcal{E}_{\mathrm{Bloch}} = \big(\xi_{\mathrm{Bloch}} \xrightarrow{\pi} \mathbb{T}^3\big)\end{array}\right.$$

where $\xi_{\mathrm{Bloch}} = \bigsqcup_{k \in \mathbb{T}^3} \mathrm{span}\{\varphi_n(k)\}_{n \in \mathcal{J}}$ is associated to finitely many frequency bands* separated by a spectral gap from the others. $\mathcal{E}_{\mathrm{Bloch}}$ may be endowed with symmetries.

What is the correct bundle here?

* Not ground state bands!

The Topology of Light States in Periodic Media



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Which Schrödinger Framework to Choose?

$$M = M_+ \oplus M_-$$

Choose bands symmetrically: $\{\omega_n(k), -\omega_n(-k)\}$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+ \oplus \mathcal{E}_- \cong \mathcal{E}_+ \oplus \mathcal{E}_+^*$$

But: $\mathcal{E}_+ \oplus \mathcal{E}_-$ is **always trivial** complex vector bundle as

$$c_1(\mathcal{E}_-) = -c_1(\mathcal{E}_+)$$

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Reason: $\mathcal{E}_+ \oplus \mathcal{E}_-$ too big, contains many unphysical states

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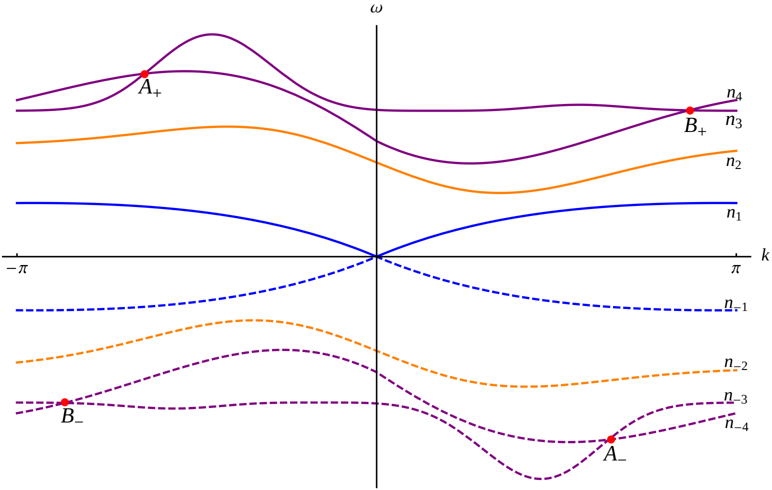

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$$M_+$$

Choose only $\omega_n(k) > 0$

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_+$$

$\mathcal{E}_+$ **can be non-trivial**

$$c_1(\mathcal{E}_+) \neq 0$$

Complex sections in $\mathcal{E}_+$ in 1-to-1 correspondence with **real** states

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Comparison Between Photonics and Quantum Mechanics

Theorem (Classification via $\mathcal{E}_+$, De Nittis-L., 2016)

Material	Photonics	Quantum Mechanics
ordinary	class AI +TR	class AI +TR
exhibiting edge currents	class A none	class A/AII none/-TR
vacuum & dual-symmetric	*New!* 2 anticommuting +TR	
metals (non-rigorous, $W \not> 0$)	*New!* commuting +TR & -TR	

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Idea of using *positive frequency bundle* in classification of topological insulators should extend to other classical wave equations!

Quantum Analogies Investigated in the Past

- Schrödinger formalism of Maxwell equations
  Physics: *in vacuo* $\rightsquigarrow$ Dirac, Wigner, ... (1920s)
  Mathematics: *non-gyrotropic* $\rightsquigarrow$ Birman & Solomyak (1987)
- Random Maxwell & acoustic operators
  Figotin & Klein (1997)
- Derivation of non-linear Schrödinger equation from non-linear Maxwell equations
  Babin & Figotin (early 2000s)
- Adiabatic perturbation theory for photonic crystals
  De Nittis & L. (2014)
- Ray optics in photonics $\longleftrightarrow$ semiclassics in quantum mechanics
  De Nittis & L. (2015) for photonic crystals
- Classification of Photonic Topological Insulators
  De Nittis & L. (2014 & 2016)

Work in Progress

- Unified mathematical framework for operators $M = W\, D$
  with Giuseppe De Nittis and Carlos Villegas
- Magnons (test case to undestand $W \not> 0$)
  with Koji Satō and Kei Yamamoto
- Non-standard topological classes with two $\pm$TR
  with Giuseppe De Nittis and Kiyonori Gomi
- Non-linear photonic topological insulators
  with Giuseppe De Nittis and Kiyonori Gomi

Open Problems

For operators of product form

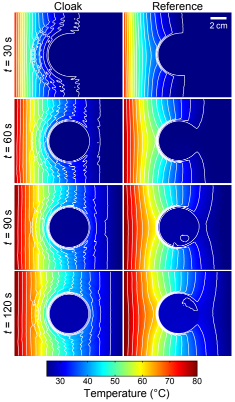
$$M = W\,D$$

- Scattering theory
  $\leadsto$ technical conditions on W and D?
- What if $W \not> 0$ (e. g. in metals or for magnons)
  $\leadsto$ Theory of Krein spaces
- *Non-linear* topological insulators (e. g. in photonic or magnonic crystals)
  $\leadsto$ Existence of topological solitons?
- Dispersion
- Spectral problems
  e. g. M periodic Maxwell operator, $W \in L^\infty$
  $\Longrightarrow \sigma(M) \setminus \{0\} = \sigma_{\mathrm{ac}}(M) \setminus \{0\}$

Source for Inspiration: *Wave-Wave* Analogies

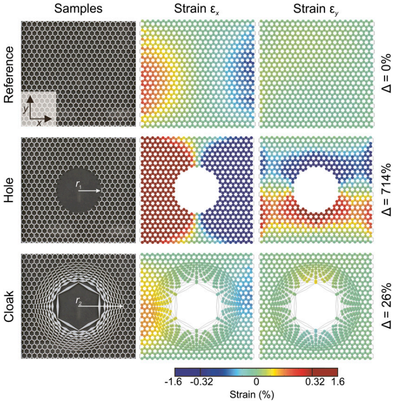
Cloaks

Thermal



Schittny, Wegener et al (2013)

Mechanical



Bückmann, Wegener et al (2015)

Thank you for your attention!