Twisted equivariant $\mathbb{K}$-theory and topological phases

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Spectral Theory Novel Materials
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Topological phase of periodic gapped systems

- $\mathcal{H}$: Hilbert space,
- $U : G = \mathbb{Z}^d \curvearrowright \mathcal{H}$: unitary representation

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- $\mathcal{H} := L^2(\mathbb{Z}^d, \mathbb{C}^N)$: Hilbert space,
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Assumption

The Hamiltonian H has a spectral gap at $\mu \in \mathbb{R}$.

We say that H_1 and H_2 are in the same *topological phase* if $E_{\leq \mu}(H_1) \cong E_{\leq \mu}(H_2)$ as vector bundles.

The K^0-group

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Example: The first Chern number for $d = 2$;

$$c_1(E_{\leq \mu}(H)) := \frac{-1}{2\pi i} \int_{\mathbb{T}^2} \mathrm{tr}(p_x[\nabla_1, p_x][\nabla_2, p_x])dx$$

(p_x: orthogonal projection onto $E_{\leq \mu}(H)_x$).

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Rem. In 2d IQHE, it is related to the Hall conductance by the TKNN formula.

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2. (K'15) The uniform Roe algebra $C^*_u(X)$: the closure of $\{T \in \mathbb{B}(\ell^2 X) \mid \exists R > 0 \text{ s.t. } T_{xy} = 0 \text{ if } d(x,y) > R\}$

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Anyway, there is a group homomorphism

$$\mathrm{ind} \colon \mathrm{K}_0(A) \to \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$$

Generalizations

Equivariant K-theory

When $G \curvearrowright X$, G-equivariant vector bundles form a semigroup $\mathrm{Vect}_G(X)$.

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Example: The Clifford algebra $\mathrm{C}\ell_{n,m}$.

2d Quantum Spin-Hall Effect

- $\mathcal{H} := \ell^2(\mathbb{Z}^2, \mathbb{C}^n)$,
- $U : \mathbb{Z}^d \to \mathcal{U}(\mathcal{H})$: the regular representation (i.e. U_g: shift operator),
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Definition

$[E_{\leq \mu}(H)] \in \mathrm{KR}_0(C(\mathbb{T}^d, \mathbb{M}_n), \mathrm{Ad}\, T)(= \mathrm{KQ}^0(\mathbb{T}^2, \tau)) \cong \mathbb{Z}_2$ is called the *Kane-Mele invariant*.

Wigner's theorem

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The group of symmetries in quantum mechanics:

$$\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) := \{f : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H} \mid f^*\Phi = \Phi, f\gamma = \gamma f\}$$

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Theorem (Wigner's theorem)

$$\mathrm{Aut}_{\mathrm{qtm}}(\hat{\mathbb{P}}\mathcal{H}) \cong \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})/\mathbb{T}$$

where

$\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) :=$ (linear/antilinear and even/odd unitaries on $\mathcal{H}$).

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Theorem (Freed-Moore'13, K.)

The data (ϕ, c, τ) is classified by the set

$$\bigsqcup_{\phi \in \check{H}^1(G;\mathbb{Z}_2)} \check{H}^1(G;\mathbb{Z}_2) \ltimes_{\epsilon} \check{H}^2(G;\mathbb{T}).$$

The twisted equivariant K_0-group

- G: finite group, (ϕ, c, τ): twist on G,
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We define the twisted equivariant K-functor

$$ {}^{\phi}\mathrm{K}^{G}_{*,c,\tau} : {}^{\phi}\mathfrak{Calg}^{G}_{\mathbb{Z}_2} \to \mathfrak{Ab} $$

as a canonical generalization of K^G_*, KR^G_*.

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$$\mathcal{F}^G_{c,\mathcal{V}}(A) := \{s \in A \hat{\otimes} \mathbb{K}(\mathcal{V})_{\mathrm{sa}} \mid s^2 = 1, \alpha_g(s) = (-1)^{c(g)} s\}$$

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H satisfies $HU_g = (-1)^{c(g)} U_g H \Rightarrow [H|H|^{-1}] \in {}^{\phi}\mathrm{K}^G_{0,c,\tau}(A)$.

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$$G \ltimes^{\phi}_{c,\tau} A := \{\sum_{g\in G} a_g u_g \mid a_g \in A\}$$

with

$$(\sum a_g u_g)(\sum b_h u_h) = \sum \tau(g,h) a_g \alpha_g(b_h) u_{gh}$$
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(identified with the Real C^*-algebra $(G \ltimes^{\phi}_{c,\tau} A) \otimes_{\mathbb{R}} \mathbb{C}$).

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Theorem

$${}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(A) \cong \mathrm{KR}(G \ltimes^{\phi}_{c,-\overline{\tau}} A)$$

(Here $\overline{\tau} = \tau + \epsilon(c,c)$.)

Index for topological phases

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we get the group homomorphism

$$\mathrm{ind}\colon {}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(A) \to {}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(\mathrm{C}\ell_{0,d}).$$

Example: CT-symmetries

We consider the case that $(\phi, c) : \mathcal{A} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is injective. Choices of $(\mathcal{A}, \tau)$ are classified by

$$C^1 = \pm 1 \text{ and } T^2 = \pm 1$$

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There are 10 choices and ${}^\phi C^*_{c,\tau}\mathcal{A} := \mathcal{A} \ltimes^\phi_{c,\tau} \mathbb{R}$ is classified by

$\mathcal{A}$	1	$\mathcal{P}$	$\mathcal{T}$		$\mathcal{C}$		$\mathcal{G}$			
C^2					1	-1	1	1	-1	-1
T^2			1	-1			1	-1	1	-1
${}^\phi C^*_{c,\tau}\mathcal{A}$	$\mathbb{C}$	$\mathbb{C}\ell_1$	$\mathbb{M}_2(\mathbb{R})$	$\mathbb{H}$	$\mathrm{C}\ell_{0,2}$	$\mathrm{C}\ell_{2,0}$	$\mathrm{C}\ell_{1,2}$	$\mathrm{C}\ell_{0,3}$	$\mathrm{C}\ell_{2,1}$	$\mathrm{C}\ell_{3,0}$
${}^\phi\mathrm{K}^{\mathcal{A}}_{0,c,\tau}$	K_0	K_1	KR_0	KR_4	KR_2	KR_6	KR_1	KR_3	KR_7	KR_5
Cartan	A	AIII	AI	AII	D	C	BDI	DIII	CI	CII

Table: The 10-fold way and Clifford algebras

dim	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
2	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
3	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Table: Kitaev's periodic table

cf. Bott periodicity

$$\pi_i(U) \cong \begin{cases} \mathbb{Z} & i = 2n+1 \\ 0 & i = 2n \end{cases}, \pi_i(O) \cong \begin{cases} \mathbb{Z} & i = 8n-1, 8n+3 \\ \mathbb{Z}_2 & i = 8n, 8n+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: reflection-invariant systems

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It is not difficult to determine the finite-dimensional algebras $G \ltimes^{\phi}_{c,\tau} \mathrm{C}\ell_{0,d}$ and we get

$$ {}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(\mathrm{C}\ell_{0,d}) \cong \begin{cases} {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d-1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,+), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d+1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,-), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d,c,\tau}(\mathbb{R})^2 & \text{if } (\epsilon,\nu) = (-,+), \\ \mathrm{K}_{d,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (-,-). \end{cases} $$

where $RP = \epsilon PR$ and $RT = \nu TR$.

Reflection	Class	C_q or R_q	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
R	A	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^+	AIII	C_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R^-	AIII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	R_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	D	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^+, R^{++}	DIII	R_4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	AII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	R_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	AI	R_7	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$
	D	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"
R^-, R^{--}	DIII	R_2	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	AII	R_3	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	R_4	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	R_5	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	R_6	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0
R^{+-}	BDI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
R^{-+}	DIII	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^{+-}	CII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
R^{-+}	CI	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
R^{-+}	BDI, CII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+-}	DIII, CI	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Classification of reflection invariant topological phases
Takahiro Morimoto and Akira Furusaki, Topological classification with additional symmetries from Clifford algebras, Phys. Rev. B 88, 125129.

Example: $1D$ type A reflection invariant systems

$$\mathrm{ind}\colon \mathrm{K}^0_{\mathcal{R}}(\mathbb{T}^1) \to \mathrm{K}^{\mathcal{R}}_0(\mathrm{C}\ell_{0,1}) \cong \mathbb{Z}.$$

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The simplest vector bundle with nontrivial index is $E \to \mathbb{T}^1$ s.t. $E|_0 \cong V_+$ and $E|_\pi \cong V_-$ ($V_\pm \cong \mathbb{C}$ with the $\mathbb{Z}_2$-action given by ± 1).

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$$H := \frac{1}{2}\begin{pmatrix} s+s^* & i(s-s^*) \\ i(s-s^*) & -(s+s^*) \end{pmatrix} \in \mathbb{B}(\ell^2(\mathbb{Z}; V_+ \oplus V_-)),$$

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cf.) the clean Kitaev chain (a $1D$ type BDI systems):

$$H = \frac{1}{2}\begin{pmatrix} s+s^*+2\mu & -i(s-s^*) \\ -i(s-s^*) & -(s+s^*+2\mu) \end{pmatrix},$$

(μ: chemical potential).