Overdamping in gyroscopic systems composed of high-loss and lossless components

Alex Figotin

Department of Mathematics
University of California–Irvine
Irvine CA, USA

Collaborator: Aaron Welters (FIT)

Relevant references

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2. A. Figotin and A. Welters, *Lagrangian framework for systems composed of high-loss and lossless components*, J. Math. Phys., vol. 55, no. 6, 062902, pp. 1-39 (2014).
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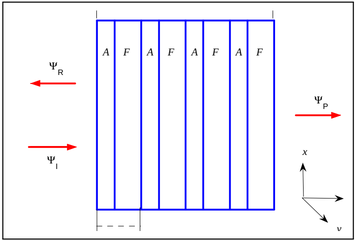
4. A. Welters, *On explicit recursive formulas in the spectral perturbation analysis of a Jordan block*, SIAM J. Matrix Anal. Appl., vol. 32, no. 1, pp. 1–22 (2011).
5. A. Figotin and I. Vitebskiy, *Absorption suppression in photonic crystals*, Phys. Rev. B, **77**, 104421 (2008).

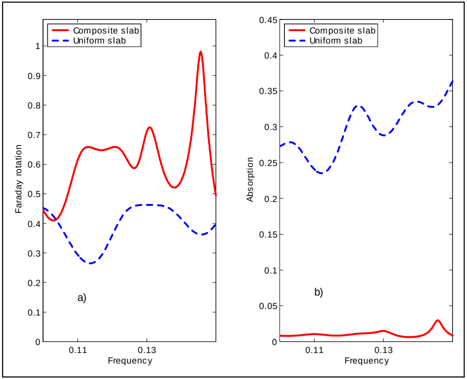
Motivation: Significant absorption suppression in magnetic materials

- Low-loss magnetic materials are crucial in many microwave, infrared, and optical devices.
- Detrimental to the performance of many such devices are the high losses or absorption associated with the magnetic materials in frequency ranges of interest.
- This is a major problem as the high losses often preclude the use of many magnetic materials with otherwise excellent physical properties or functionality that would have enhanced device performance.
- New Idea of Figotin and Vitebskiy[5] and Inui et al.[6]: reduction of losses and enhancement of magnetic properties in magnetic-dielectric composites may be more substantial with higher loss magnetic components.

[6] C. Inui, S. Ozaki, H. Kura, and T. Sato, *Enhancement of Faraday effect in one-dimensional magneto-optical photonic crystal including a magnetic layer with wavelength dependent off-diagonal elements of dielectric constant tensor*, Journal of Magnetism and Magnetic Materials, 323(18-19):2348 –2354, 2011.

- Composite structure with magnetic (*F*) and dielectric (*A*) layers.
  - (*F*) : Lossy magnetic material – useful functionality is nonreciprocal Faraday rotation.
  - (*A*) : Lossless dielectric material – significantly reduces absorption & enhances functionality of composite over broad frequency range.





Toward a theory of absorption suppression in composites

- Main objective: Realization of broadband low-loss magnetic composites with functionality comparable to bulk magnetic materials.
- Example[5] carries essential features of such a composite, so we consider composites with two components: lossless and lossy.
- To see dissipative effects in a simpler and pronounced form we introduce a scaling parameter (i.e., $\beta \geq 0$) for the dissipation and use perturbation theory[1–4].
- We use a Lagrangian framework[2] as a physically sound basis to model interplay between absorption and system properties.
- We study mechanisms of broadband absorption suppression in composites[1–3].
- We study interplay between gyroscopy/magnetism & dissipation.
- We intend to construct multiscale models of magnetic materials to study interplay between magnetism and dissipation in composites.

Main points

- Lagrangian approach–significant advantages, e.g., Lagrangian $\mathcal{L}$ yields both dynamics & system energy – the Hamiltonian $\mathcal{H}$.
- For Lagrangian systems, overdamping is a subtle phenomenon especially for composites with essential features include the modal dichotomy and "selective overdamping".
- Q-factor (quality factor) is an important figure of merit for analyzing the performance of a dissipative system.
- Applications and practical importance of selective overdamping lie in effective suppression of more dissipative (overdamped) modes with consequent enhancement of the role of the low-loss, high-Q (underdamped) modes.
- Overdamping is essentially a universal phenomena in non-gyroscopic systems[2], but only generic in gyroscopic systems[1].
- Provide estimates on amount of loss required for overdamping to occur & Q-factor estimates for eigenmodes[1,2].
- Notion of dual Lagrangian system yields improved results.

The Lagrangian framework

- Integrate loss into linear systems with gyroscopic features such as dielectric media with magnetic components and electrical networks with gyrators.
- The coordinates $Q = [q_r]_{r=1}^N \subseteq \mathbb{R}^N$ and their time derivatives $\dot{Q}$:

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix}\dot{Q}\\ Q\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\alpha & \theta\\ \theta^{\mathrm{T}} & -\eta\end{bmatrix}\begin{bmatrix}\dot{Q}\\ Q\end{bmatrix} = \mathcal{T} - \mathcal{V}, \text{ (Lagrangian)}$$

$$\mathcal{H} = \mathcal{T} + \mathcal{V} \geq 0, \quad \text{(Hamiltonian)}$$

$$\mathcal{T} = \frac{1}{2}\dot{Q}^{\mathrm{T}}\alpha\dot{Q} + \frac{1}{2}\dot{Q}^{\mathrm{T}}\theta Q,\ \mathcal{V} = \frac{1}{2}Q^{\mathrm{T}}\eta Q - \frac{1}{2}\dot{Q}^{\mathrm{T}}\theta Q,$$

where α, θ, η are $N \times N$ matrices with real entries and

$$\alpha = \alpha^{\mathrm{T}} > 0 \quad \text{(positive kinetic energy)}, \quad \eta = \eta^{\mathrm{T}} \geq 0 \quad \text{(stability)},$$

$$\theta^{\mathrm{T}} = -\theta \text{ (skew-symmetric)}.$$

Definition (gyroscopic system)

Any system with Lagrangian $\mathcal{L}$ having $\theta \neq 0$ will be called gyroscopic; otherwise, it will be called non-gyroscopic.

Integrating the dissipation into the framework

- The dissipation is introduced using Rayleigh's method through the general Euler-Lagrange equations

$$\alpha \ddot{Q} + 2\theta \dot{Q} + \eta Q = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) - \frac{\partial \mathcal{L}}{\partial Q} = -\frac{\partial \mathcal{R}}{\partial \dot{Q}} + F = -\beta R \dot{Q} + F,$$

where $F = F(t)$ is an external force, $\frac{\partial \mathcal{R}}{\partial \dot{Q}}$ are the dissipative forces, and $\mathcal{R}$ is the Rayleigh dissipation function

$$\mathcal{R} = \frac{1}{2}\dot{Q}^{\mathrm{T}} \beta R \dot{Q}, \quad R = R^{\mathrm{T}} \geq 0, \quad N_R := \operatorname{rank} R > 0,$$

$$0 \leq \beta \text{ (dimensionless loss parameter)}$$

- The Rayleigh dissipation function accounts for energy losses according to the energy balance equation

$$\partial_t \mathcal{H} = -2\mathcal{R} + \operatorname{Re}\left(\dot{Q}^{\mathrm{T}} F\right).$$

- Modeling of a two-component composite with a lossy and a lossless component is taken into account with the loss fraction condition:

$$0 < \delta_R < 1, \text{ where } \delta_R := \frac{\operatorname{rank} R}{\#\text{-DOF}} = \frac{N_R}{N} \text{ (loss fraction).}$$

Electric circuit example[1]: two-component composite

An electric circuit involving a gyrator $G_{12} \geq 0$, three capacitances C_1, C_2, $C_{12} > 0$, two inductances L_1, $L_2 > 0$, a resistor $R_2 = \beta\ell$ (fixed $\ell > 0$) and two sources E_1, E_2.

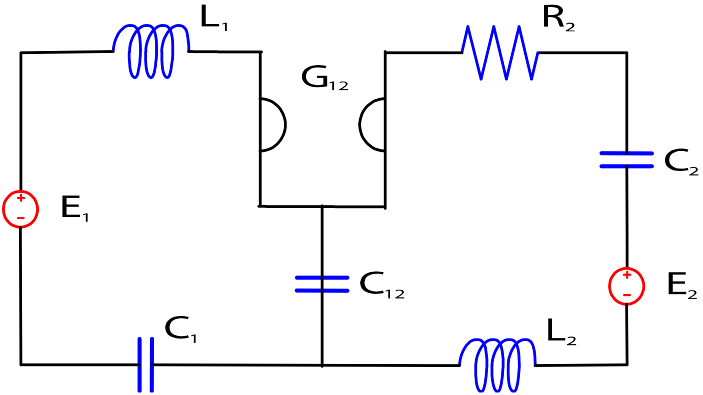


Figure: Two-component composite with gyroscopic element (i.e., gyrator G_{12}): Lossy component - right, losses $\beta = \frac{R_2}{\ell}$; Lossless component - left.

- The 2-DOF ($N = 2$) electric circuit with currents $I_1 = \dot{q}_1$, $I_2 = \dot{q}_2$,

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \quad \mathcal{H} = \mathcal{T} + \mathcal{V} \geq 0, \text{ (Lagrangian \& Hamiltonian)}$$

$$\mathcal{T} = \frac{L_1}{2}\dot{q}_1^2 + \frac{L_2}{2}\dot{q}_2^2 + \frac{G_{12}}{4}(q_1\dot{q}_2 - \dot{q}_1 q_2),$$

$$\mathcal{V} = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_{12}}(q_1 - q_2)^2 + \frac{1}{2C_2}q_2^2 - \frac{G_{12}}{4}(q_1\dot{q}_2 - \dot{q}_1 q_2),$$

$$\mathcal{R} = \frac{R_2}{2}\dot{q}_2^2. \qquad \text{(Rayleigh dissipation function)}$$

- Dimensionless loss parameter $\beta = \frac{R_2}{\ell}$, fixed $\ell > 0$;

$$\alpha = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \; \eta = \begin{bmatrix} \frac{1}{C_1} + \frac{1}{C_{12}} & -\frac{1}{C_{12}} \\ -\frac{1}{C_{12}} & \frac{1}{C_2} + \frac{1}{C_{12}} \end{bmatrix}, \; \theta = \begin{bmatrix} 0 & -\frac{G_{12}}{2} \\ \frac{G_{12}}{2} & 0 \end{bmatrix},$$

$$\beta R = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix}, \; N_R := \operatorname{rank} R = 1, \; 0 < \delta_R := \frac{N_R}{N} = \frac{1}{2} < 1.$$

- Euler-Lagrange equations with dissipative & external forces $F = [E_1, E_2]^{\mathrm{T}}$:

$$L_1 \frac{\partial}{\partial t}\dot{q}_1 - G_{12}\dot{q}_2 + \frac{1}{C_1}q_1 + \frac{1}{C_{12}}(q_1 - q_2) = E_1,$$

$$L_2 \frac{\partial}{\partial t}\dot{q}_2 + G_{12}\dot{q}_1 + \frac{1}{C_2}q_2 - \frac{1}{C_{12}}(q_1 - q_2) = -R_2\dot{q}_2 + E_2.$$

Lifting to canonical evolution equations

- In the Hilbert space $H = \mathbb{C}^{2N}$ with the standard scalar product $(\cdot,\cdot)$, the states v are the solutions of the canonical evolution equations

$$\partial_t v = -\mathrm{i}A(\beta)\,v + f, \quad \text{where } A(\beta) = \Omega - \mathrm{i}\beta B, \quad \Omega^* = \Omega, \quad B \geq 0,$$
$$\operatorname{rank} B = N_R \leq N \text{ (rank deficiency)}.$$

- Change-of-variables from Q, $\dot{Q}$ to v using conjugate momentum P:

$$v = Ku,\ u = \begin{bmatrix} P \\ Q \end{bmatrix},\ P = \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) = \alpha \dot{Q} + \theta Q$$

- The system energy and dissipated power:

$$\mathcal{H} = U[v(t)] = \frac{1}{2}(v(t), v(t)),\ 2\mathcal{R} = W_{\mathrm{dis}}[v(t)] = \beta(v(t), Bv(t)).$$

- Formulas for Ω, B, f, K in terms of α, η, θ, F:

$$\Omega = \mathrm{i}KJK^{\mathrm{T}}, \quad B = \begin{bmatrix} \sqrt{\alpha}^{-1} R \sqrt{\alpha}^{-1} & 0 \\ 0 & 0 \end{bmatrix},\ f = \begin{bmatrix} \sqrt{\alpha}^{-1} F \\ 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix},\ K = \begin{bmatrix} \sqrt{\alpha}^{-1} & 0 \\ 0 & \sqrt{\eta} \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\theta \\ 0 & \mathbf{1} \end{bmatrix}$$

Eigenmodes, quality factor, spectral problem

- An eigenmode $v(t) = w\mathrm{e}^{-\mathrm{i}\zeta t}$, $w \neq 0$ ($f = 0$) is a state with
  - $\mathrm{e}^{-\mathrm{i}\zeta t} = \mathrm{e}^{-\gamma t}\mathrm{e}^{\mathrm{i}\omega t}$, where
  - $\omega = \operatorname{Re}\zeta$ is frequency,
  - $\gamma = -\operatorname{Im}\zeta \geq 0$ is damping factor,
  - *Standard spectral problem* :

$$A(\beta)\, w = \zeta w.$$

- The quality factor is

$$Q_\zeta = 2\pi \frac{\text{energy stored}}{\text{energy lost per cycle}} = |\omega| \frac{U[v(t)]}{W_{\mathrm{dis}}[v(t)]} = \frac{1}{2}\frac{|\omega|}{\gamma}. \quad \text{(Q-factor)}$$

- As figure of merit, Q-factor is an important quantity which characterizes the performance of the dissipative composite system.
- The spectral perturbation analysis near $\beta = \infty$ has subtleties since $A(\beta) = \Omega - \mathrm{i}\beta B$, $\beta > 0$ is non-self-adjoint and B is rank deficient.

Overdamped and underdamped modes

Definition (overdamped, underdamped)

Any eigenmode with time-dependency $e^{-i\zeta(\beta)t}$ for which there exists a $\beta' \geq 0$ such that its frequency $\omega = \operatorname{Re}\zeta(\beta)$ has the property

$$\omega = \operatorname{Re}\zeta(\beta) = 0, \quad \text{for all } \beta > \beta' \quad \text{(overdamped)},$$

or

$$\omega = \operatorname{Re}\zeta(\beta) \neq 0, \quad \text{for all } \beta > \beta' \quad \text{(underdamped)},$$

will be called an overdamped mode (and is said to be overdamped) or underdamped mode (and is said to be underdamped), respectively.

Example[3]: modal dichotomy and overdamping

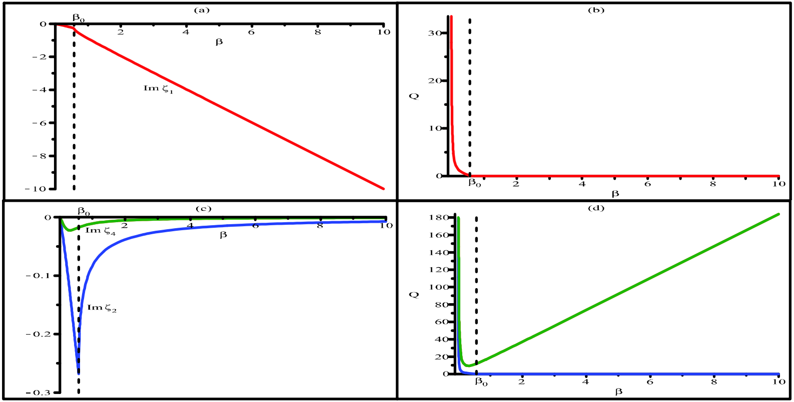


Figure: Eigenmodes of an electric circuit with time-dependency $\mathrm{e}^{-\mathrm{i}\zeta_j(\beta)t}$, $j = 1, 2, 3, 4$ and overdamped regime $\beta > \beta_0$. Underdamped (high-Q) modes $j = 3, 4$ in green (curves identical due to spectral symmetry $\zeta_4(\beta) = -\overline{\zeta_3(\beta)}$). Overdamped modes $j = 1, 2$ in red (high-loss) and blue (low-loss). (a) Imaginary part of $\zeta_1(\beta)$; (b) Q-factor of the high-loss mode. (c) Imaginary part of $\zeta_j(\beta)$, $j = 2, 3, 4$; (d) Q-factor of the low-loss modes $j = 2, 3, 4$.

Example: spectral symmetries and selective overdamping

An animation using previous circuit example, showing the spectral symmetries and selective overdamping phenomenon.

Spectral symmetries and clustering

Lemma (Spectral clustering)

If $M = M_0 + E$ is a square matrix with M_0 normal then

$$\sigma(M) \subseteq \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(M_0)) \leq \|E\|\}$$

where $\|E\|$ is operator norm of E.

Fundamental properties of $A(\beta) = -\mathrm{i}\beta B + \Omega$:

$$\operatorname{Re} A(\beta) = \Omega, \quad -\operatorname{Im} A(\beta) = \beta B \geq 0, \quad A(\beta)^* = -A(\beta)^{\mathrm{T}},$$

(Modal Symm.) If $v(t) = w e^{-\mathrm{i}\zeta t}$ is an eigenmode then so is $\overline{v(t)} = \overline{w} e^{-\mathrm{i}(-\overline{\zeta})t}$.

(Spectral Symm.) Spectrum $\sigma(A(\beta))$ is symmetric with respect to the imaginary axis and lies in the closed lower half-plane, i.e., if $\zeta \in \sigma(A(\beta))$ then $-\overline{\zeta} \in \sigma(A(\beta))$ and $\operatorname{Im}\zeta \leq 0$.

(Clustering) If $\zeta \in \sigma(A(\beta))$ then, with $\omega_{\max} := \max \sigma(\Omega)$, we have

$$|\operatorname{Re}\zeta| \leq \omega_{\max}, \quad \min_{b \in \sigma(B)} |-\operatorname{Im}\zeta - \beta b| \leq \omega_{\max}.$$

Theorem (modal dichotomy I[2])

Let $b_{\min} := \min\left[\sigma\left(B\right) \setminus \{0\}\right]$ *and* $\omega_{\max} := \max \sigma\left(\Omega\right)$*. If* $\beta > 2\frac{\omega_{\max}}{b_{\min}}$ *then there exists unique invariant subspaces of* $A\left(\beta\right)$ *with:*

$$\text{(i) } H = H_{h\ell}\left(\beta\right) \oplus H_{\ell\ell}\left(\beta\right);$$

$$\text{(ii) } \sigma\left(A\left(\beta\right)|_{H_{\ell\ell}(\beta)}\right) = \left\{\zeta \in \sigma\left(A\left(\beta\right)\right) : 0 \leq |\zeta| \leq \omega_{\max}\right\};$$

$$\text{(iii) } \sigma\left(A\left(\beta\right)|_{H_{h\ell}(\beta)}\right) = \left\{\zeta \in \sigma\left(A\left(\beta\right)\right) : -\operatorname{Im}\zeta \geq \beta b_{\min} - \omega_{\max}\right\}.$$

Furthermore, these subspaces have dimensions

$$\dim H_{h\ell}\left(\beta\right) = N_R, \quad \dim H_{\ell\ell}\left(\beta\right) = 2N - N_R \geq N.$$

Moreover, the maximum Q-factor of the modes in $H_{h\ell}\left(\beta\right)$ *satisfy*

$$\max_{\zeta \in \sigma\left(A(\beta)|_{H_{h\ell}(\beta)}\right)} Q_\zeta \leq \frac{1}{2}\frac{\omega_{\max}}{\beta b_{\min} - \omega_{\max}} < \frac{1}{2},$$

and the damping factors of the modes in $H_{\ell\ell}\left(\beta\right)$ *satisfy*

$$\lim_{\beta \to \infty} \max_{\zeta \in \sigma\left(A(\beta)|_{H_{\ell\ell}(\beta)}\right)} -\operatorname{Im}\zeta = 0.$$

Dual Lagrangian system

- Duality condition: $\eta > 0$.
- The "dual" Lagrangian system has Lagrangian and dissipation function (i.e., just interchange $\alpha \leftrightarrow \eta$ in $\mathcal{L}$)

$$\mathcal{L}^\flat\left(Q, \dot{Q}\right) := -\mathcal{L}\left(\dot{Q}, Q\right) = \frac{1}{2}\begin{bmatrix}\dot{Q}\\ Q\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\eta & \theta\\ \theta^{\mathrm{T}} & -\alpha\end{bmatrix}\begin{bmatrix}\dot{Q}\\ Q\end{bmatrix}, \text{ (dual Lagrangian)}$$

$$\mathcal{R}^\flat\left(\dot{Q}\right) := \mathcal{R}\left(\dot{Q}\right) = \frac{1}{2}\dot{Q}^{\mathrm{T}}\beta R\dot{Q}. \text{ (dual Rayleigh dissipation function)}$$

- Notation: X in Lagrangian system $\leftrightarrow$ $X^\flat$ in dual Lagrangian system, e.g., the dual of the system operator $A\left(\beta\right) = \Omega - \mathrm{i}\beta B$ is the system operator $A^\flat\left(\beta\right) = \Omega^\flat - i\beta B^\flat$.

* Introduction of the dual yields significant improvements on the results on modal dichotomy and overdamping via "duality."

Proposition (spectral duality[1])

The following statements are true:

1. *The spectrums of* $A(\beta)$, $A^{\flat}(\beta)$ *are related by*
$$\sigma(A(\beta)) = -\sigma\left(A^{\flat}(\beta)\right)^{-1}.$$
2. *If* ζ *is an eigenvalue of* $A(\beta)$ *then* $-\zeta^{-1}$ *is an eigenvalue of* $A^{\flat}(\beta)$ *and they have the same geometric multiplicity, algebraic multiplicity, and partial multiplicities (i.e., same Jordan normal form). Moreover, the associated Q-factors are equal, i.e.,*
$$Q_{-\zeta^{-1}} = Q_{\zeta} = \frac{1}{2}\frac{|\mathrm{Re}\,\zeta|}{-\mathrm{Im}\,\zeta}.$$
3. *The maximum eigenvalue* $\omega^{\flat}_{\max}$ *of* $\Omega^{\flat}$ *is given by*
$$\omega^{\flat}_{\max} = \omega^{-1}_{\min}, \quad \textit{where } \omega_{\min} := \min_{\omega\in\sigma(\Omega)} |\omega| > 0.$$

* Duality connects high-loss $H^{\flat}_{h\ell}(\beta)$, $\sigma\left(A^{\flat}(\beta)|_{H^{\flat}_{h\ell}(\beta)}\right)$ of $A^{\flat}(\beta)$ to a same dimensional low-loss invariant subspace $H_{\ell\ell,0}(\beta) \subseteq H_{\ell\ell}(\beta)$ of $A(\beta)$ via $\sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right) = -\sigma\left(A^{\flat}(\beta)|_{H^{\flat}_{h\ell}(\beta)}\right)^{-1}$.

Theorem (modal dichotomy I-duality[1])

If $\eta > 0$ *and* $\beta > \max\left\{2\frac{\omega_{\max}}{b_{\min}}, 2\frac{\omega^{\flat}_{\max}}{b^{\flat}_{\min}}\right\}$ *then there exists unique invariant subspaces* $H_{\ell\ell,0}(\beta)$, $H_{\ell\ell,1}(\beta)$ *of* $A(\beta)$ *with the properties*

$$\text{(i)}\quad H_{\ell\ell}(\beta) = H_{\ell\ell,0}(\beta) \oplus H_{\ell\ell,1}(\beta);$$

$$\text{(ii)}\quad \sigma\left(A(\beta)|_{H_{\ell\ell,1}(\beta)}\right) = \{\zeta \in \sigma(A(\beta)) : \omega_{\min} \leq |\zeta| \leq \omega_{\max}\};$$

$$\text{(iii)}\ \sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right) = -\sigma\left(A^{\flat}(\beta)|_{H^{\flat}_{h\ell}(\beta)}\right)^{-1}.$$

Furthermore, these subspaces have dimensions

$$\dim H_{\ell\ell,0}(\beta) = N_R, \quad \dim H_{\ell\ell,1}(\beta) = 2(N - N_R).$$

Moreover, the Q-factor, frequencies, and damping factors of the modes in $H_{\ell\ell,0}(\beta)$ *are bounded above by*

$$\max_{\zeta\in\sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right)} Q_\zeta \leq \frac{1}{2}\frac{\omega^{\flat}_{\max}}{\beta b^{\flat}_{\min} - \omega^{\flat}_{\max}} < \frac{1}{2},$$

$$\max_{\zeta\in\sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right)} |\zeta| \leq \frac{1}{\beta b^{\flat}_{\min} - \omega^{\flat}_{\max}} < \omega_{\min}.$$

Finally, if $N_R < N$ *(rank deficiency) then the Q-factor of the modes in* $H_{\ell\ell,1}(\beta)$ *satisfy*

$$\lim_{\beta\to\infty} \min_{\zeta\in\sigma\left(A(\beta)|_{H_{\ell\ell,1}(\beta)}\right)} Q_\zeta = \infty.$$

Theorem (modal dichotomy II[1])

Suppose $N_R < N$ *and* $\eta > 0$. *Let* $\rho_{\min} := \min_{0 \neq \rho \in \sigma(\Omega_1)} |\rho|$, *where* $\Omega_1 := P_B^{\perp} \Omega P_B^{\perp}\big|_{\operatorname{Ker} B}$ *and* $P_B^{\perp}$ *orthogonal projection onto* $\operatorname{Ker} B$. *Denote by* $c(\beta)$, $\beta > 2\frac{\omega_{\max}}{b_{\min}}$ *and* $c^{-1}(y)$, $y > 0$, *the inverse functions*

$$c(\beta) := \left(2\frac{\omega_{\max}^2}{b_{\min}}\right)\frac{1}{\beta - 2\frac{\omega_{\max}}{b_{\min}}},\ c^{-1}(y) := 2\frac{\omega_{\max}^2}{b_{\min}}\frac{1}{y} + 2\frac{\omega_{\max}}{b_{\min}}.$$

If $\beta > \max\left\{c^{-1}\left(\frac{\rho_{\min}}{2}\right), 2\frac{\omega_{\max}^{\flat}}{b_{\min}^{\flat}}\right\}$ *then* $c(\beta) < \frac{\rho_{\min}}{2} \leq \frac{\omega_{\max}}{2}$,

$$\sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right) = \{\zeta \in \sigma(A(\beta)) : 0 \leq -\operatorname{Im}\zeta \leq c(\beta) \text{ and } |\operatorname{Re}\zeta| \leq c(\beta)\}$$

$$\sigma\left(A(\beta)|_{H_{\ell\ell,1}(\beta)}\right) = \{\zeta \in \sigma(A(\beta)) : 0 \leq -\operatorname{Im}\zeta \leq c(\beta) \text{ and } |\operatorname{Re}\zeta| \geq \rho_{\min} - c(\beta)\},$$

and the minimum Q-factor of the modes in $H_{\ell\ell,1}(\beta)$ *are bounded below by*

$$\min_{\zeta \in \sigma\left(A(\beta)|_{H_{\ell\ell,1}(\beta)}\right)} Q_\zeta \geq \frac{1}{2}\frac{\rho_{\min} - c(\beta)}{c(\beta)} > \frac{1}{4}\frac{\rho_{\min}}{c(\beta)} \geq \frac{1}{2}.$$

Subtlety of overdamping phenomenon

Theorem (non-gyroscopic: partial overdamping[2])

Suppose $\theta = 0$ *(i.e., a non-gyroscopic system). If* $\beta > 2\frac{\omega_{\max}}{b_{\min}}$ *then all eigenmodes in* $H_{h\ell}(\beta)$ *are overdamped, i.e.,*

$$\sigma\left(A(\beta)\,|_{H_{h\ell}(\beta)}\right) \subseteq \{\zeta \in \sigma(A(\beta)) : \mathrm{Re}\,\zeta = 0\}.$$

Theorem (non-gyroscopic & non-composite: complete overdamping[2])

Suppose $\theta = 0$ *and* R *has full rank (i.e., a non-composite model of dissipation). If* $\beta > 2\frac{\omega_{\max}}{b_{\min}}$ *then all eigenmodes are overdamped, i.e.,*

$$\sigma(A(\beta)) = \{\zeta \in \sigma(A(\beta)) : \mathrm{Re}\,\zeta = 0\}.$$

Example (gyroscopic & non-generic case: no overdamping[1])

If $\alpha = \eta = R = \mathbf{1}$ and $0 \notin \sigma(\theta)$ then no overdamping occurs and, in particular, for all $\beta > 0$, i.e.,

$$\sigma(A(\beta)) \cap \{\zeta \in \sigma(A(\beta)) : \mathrm{Re}\,\zeta = 0\} = \varnothing.$$

Theorem (selective overdamping[1]: generic case)

If $N_R < N$, $\eta > 0$, and the nonzero eigenvalues of B and $B^\flat$ are simple (generic case), then there exists β_j, $j = 1, 2, 3$ with $2\frac{\omega_{\max}}{b_{\min}} \leq \beta_0 \leq \beta_1 \leq \beta_2$ such that

1. *If $\beta > \beta_0$ then, for the decomposition $H = H_{h\ell}(\beta) \oplus H_{\ell\ell}(\beta)$ of invariant subspaces of $A(\beta)$, all modes in the N_R-dimensional subspace $H_{h\ell}(\beta)$ are overdamped, i.e.,*
$$\sigma\left(A(\beta)|_{H_{h\ell}(\beta)}\right) \subseteq \{\zeta \in \sigma(A(\beta)) : \operatorname{Re}\zeta = 0\}.$$
2. *If $\beta > \beta_1$ then, for the decomposition $H_{\ell\ell}(\beta) = H_{\ell\ell,0}(\beta) \oplus H_{\ell\ell,1}(\beta)$ of invariant subspaces of $A(\beta)$, all modes in the $2N_R$-dimensional subspace $H_{h\ell}(\beta) \oplus H_{\ell\ell,0}(\beta)$ are overdamped, i.e.,*
$$\sigma\left(A(\beta)|_{H_{h\ell}(\beta)}\right) \cup \sigma\left(A(\beta)|_{H_{\ell\ell,0}(\beta)}\right) \subseteq \{\zeta \in \sigma(A(\beta)) : \operatorname{Re}\zeta = 0\}.$$
3. *If $\beta > \beta_2$ then all the modes in the $2(N - N_R)$-dimensional subspace $H_{\ell\ell,1}(\beta)$ are underdamped, i.e.,*
$$\sigma\left(A(\beta)|_{H_{\ell\ell,1}(\beta)}\right) \cap \{\zeta \in \sigma(A(\beta)) : \operatorname{Re}\zeta = 0\} = \emptyset.$$

Estimating the loss required for selective overdamping

Theorem (selective overdamping[1]: generic case)

Assume the hypotheses in previous theorem, let $b_0 := b_0^\flat := 0$*, and denote the nonzero eigenvalues of* B *and* $B^\flat$ *by* b_i, $1 \leq j \leq N_R$ *and* $b_i^\flat$, $1 \leq j \leq N_R$*, respectively. Then the previous theorem is true with*

$$\beta_0 := \frac{2\omega_{\max}}{d}, \text{ where } d = \min_{0 \leq i,j \leq N_R, i \neq j} |b_i - b_j|,$$

$$\beta_1 := \max\left\{\beta_0, \frac{2\omega_{\max}^\flat}{d^\flat}\right\}, \text{ where } d^\flat = \min_{0 \leq i,j \leq N_R, i \neq j} \left|b_i^\flat - b_j^\flat\right|,$$

$$\beta_2 := \max\left\{\beta_1, \min\left\{c^{-1}\left(\frac{\rho_{\min}}{2}\right), \left(c^\flat\right)^{-1}\left(\frac{\rho_{\min}^\flat}{2}\right)\right\}\right\}.$$

Modal dichotomy in the high-loss regime

Theorem (Modal dichotomy III[1])

Assume the hypotheses in the previous theorem. Let $\Omega_1 := P_B^{\perp}\Omega P_B^{\perp}\big|_{\operatorname{Ker} B}$, $r := \operatorname{rank}(\Omega_1)$. *Then* $r = 2(N - N_R) > 0$ *and there exists an eigenvector basis* $\{w_j(\beta)\}_{j=1}^{2N}$ *of* $A(\beta)$ *with eigenvalues*

$$\textit{high-loss: } \zeta_j(\beta) = -\mathrm{i} b_j \beta + O\left(\beta^{-1}\right), \ \operatorname{Re}\zeta_j(\beta) = 0,\ 1 \leq j \leq N_R,$$

$$\textit{low-loss: } \zeta_j(\beta) = \frac{-\mathrm{i}}{b_{j-N_R}^{\flat}\beta} + O\left(\beta^{-3}\right), \ \operatorname{Re}\zeta_j(\beta) = 0,\ N_R + 1 \leq j \leq 2N_R,$$

$$\zeta_j(\beta) = \rho_j - \mathrm{i} d_j \beta^{-1} + O\left(\beta^{-2}\right), \ 0 \neq \rho_j \in \mathbb{R},\ d_j \geq 0,\ 2N_R + 1 \leq j \leq 2N,$$

as $\beta \to \infty$ *with* $w_j(\beta) = \mathring{w}_j + O\left(\beta^{-1}\right)$, *where* $\{\mathring{w}_j\}_{j=1}^{2N}$ *is basis of* H *such that* $\{\mathring{w}_j\}_{j=1}^{N_R}$ *is a basis for* $\operatorname{Ran} B$, $\{\mathring{w}_j\}_{j=N_R+1}^{2N}$ *is a basis for* $\operatorname{Ker} B$, *and*

$$B\mathring{w}_j = b_j\mathring{w}_j,\ 1 \leq j \leq N_R;\ \Omega_1\mathring{w}_j = 0,\ N_R + 1 \leq j \leq 2N_R;$$

$$\Omega_1\mathring{w}_j = \rho_j\mathring{w}_j,\ 2N_R + 1 \leq j \leq 2N.$$

Electric circuit example[1]: revisited

✓ Rank deficiency ($H = \mathbb{C}^4$): $1 = N_R < N = 2$, $N_R = \operatorname{rank} R = \operatorname{rank} B$.

✓ Duality condition: $\eta > 0$.

✓ Generic case: $\sigma(B) = \{0, b_1\}$, $\sigma\left(B^\flat\right) = \left\{0, b_1^\flat\right\}$, where

$$b_{\min} = b_1 = \ell L_2^{-1}, \quad b_{\min}^\flat = b_1^\flat = \ell \left[C_2^{-1} + (C_1 + C_{12})^{-1}\right]^{-1}.$$

✓ Modal dichotomy I-duality: $\frac{2\omega_{\max}}{b_{\min}}$, $\frac{2\omega_{\max}^\flat}{b_{\min}^\flat}$, where

$$\sigma(\Omega) = \{\pm\omega_-, \pm\omega_+\},$$

$$0 < \omega_{\min} = \omega_- \leq \omega_+ = \omega_{\max}, \; \omega_{\max}^\flat = \frac{1}{\omega_{\min}},$$

$$\omega_\pm = \sqrt{\frac{a}{2} \pm \sqrt{\frac{a}{2} - \frac{\det \eta}{L_1 L_1}}}, \; \det \eta = \frac{1}{C_1 C_2} + \frac{1}{C_1 C_{12}} + \frac{1}{C_2 C_{12}},$$

$$a = \frac{1}{L_2 C_{12}} + \frac{1}{L_1 C_{12}} + \frac{1}{L_2 C_2} + \frac{1}{L_1 C_1} + \frac{G_{12}^2}{L_1 L_2}.$$

✓ Modal dichotomy II, III: $\sigma\left(\Omega_1\right)=\left\{\rho_3,\rho_4\right\}$, where

$$\begin{aligned}
0 &< \rho_3=-\rho_4=\rho_{\min}=\frac{1}{\rho^\flat_{\min}}=\sqrt{\frac{1}{L_1}\left(\frac{1}{C_1}+\frac{1}{C_{12}}\right)}, \\
\text{high-loss} &: \quad \zeta_1\left(\beta\right)=\zeta_1^a\left(\beta\right)+O\left(\beta^{-1}\right),\ \zeta_1^a\left(\beta\right):=-\mathrm{i}b_1\beta, \\
\text{low-loss} &: \quad \zeta_2\left(\beta\right)=\zeta_2^a\left(\beta\right)+O\left(\beta^{-3}\right),\ \zeta_2^a\left(\beta\right):=\frac{-\mathrm{i}}{b_1^\flat\beta}, \\
\zeta_j\left(\beta\right) &= \zeta_j^a\left(\beta\right)+O\left(\beta^{-2}\right),\ \zeta_j^a\left(\beta\right):=\rho_j-\mathrm{i}d_j\beta^{-1},\ j=3,4, \\
d_3 &= d_4=\frac{1}{2}\frac{G_{12}^2}{\ell L_1}+\frac{1}{2}\frac{1}{\ell}\frac{C_1}{\left(C_1+C_{12}\right)C_{12}}.
\end{aligned}$$

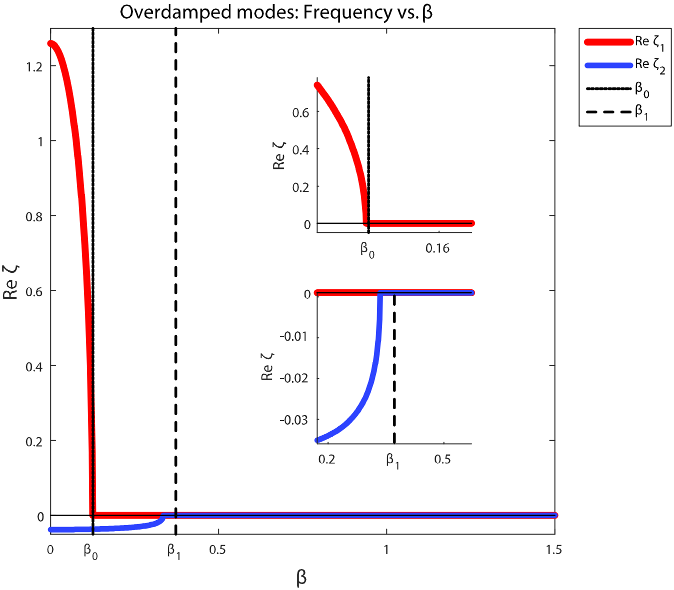
✓ Selective overdamping: $2\frac{\omega_{\max}}{b_{\min}}\leq\beta_0\leq\beta_1\leq\beta_2$, where

$$\beta_0=\frac{2\omega_{\max}}{d},\quad \beta_1=\max\left\{\beta_0,\frac{2\omega^\flat_{\max}}{d^\flat}\right\},$$

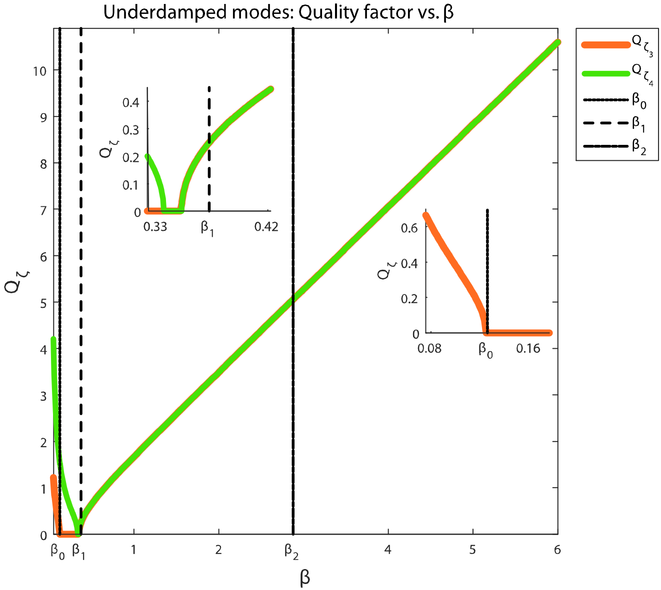
$$\beta_2:=\max\left\{\beta_1,\min\left\{c^{-1}\left(\frac{\rho_{\min}}{2}\right),\left(c^\flat\right)^{-1}\left(\frac{\rho^\flat_{\min}}{2}\right)\right\}\right\},$$

$$d=\min_{0\leq i,j\leq N_R,i\neq j}\left|b_i-b_j\right|=b_{\min},\ d^\flat=\min_{0\leq i,j\leq N_R,i\neq j}\left|b_i^\flat-b_j^\flat\right|=b^\flat_{\min}.$$

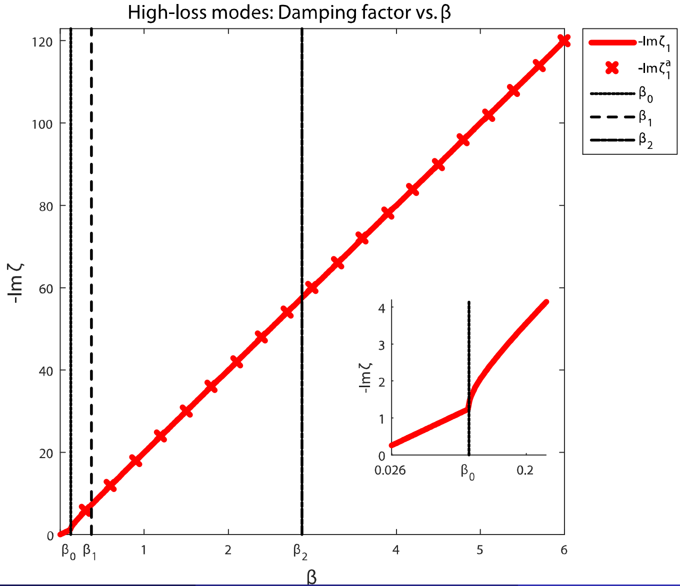
Circuit example[3]: overdamped modes & frequencies



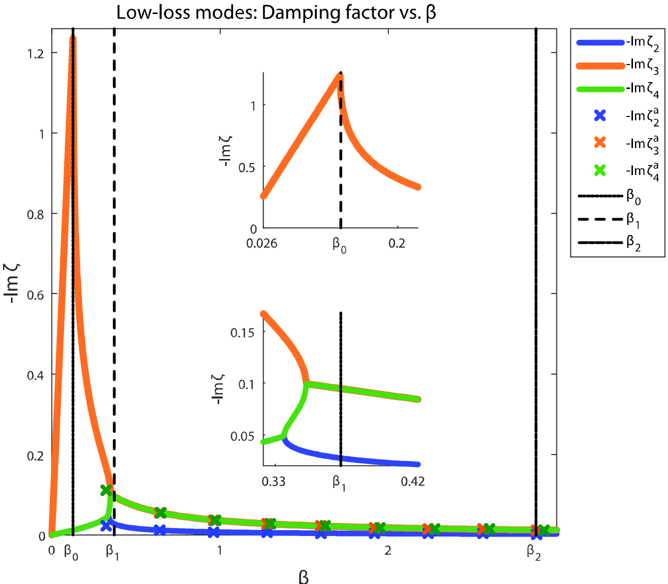
Circuit example[3]: underdamped modes & Q-factor



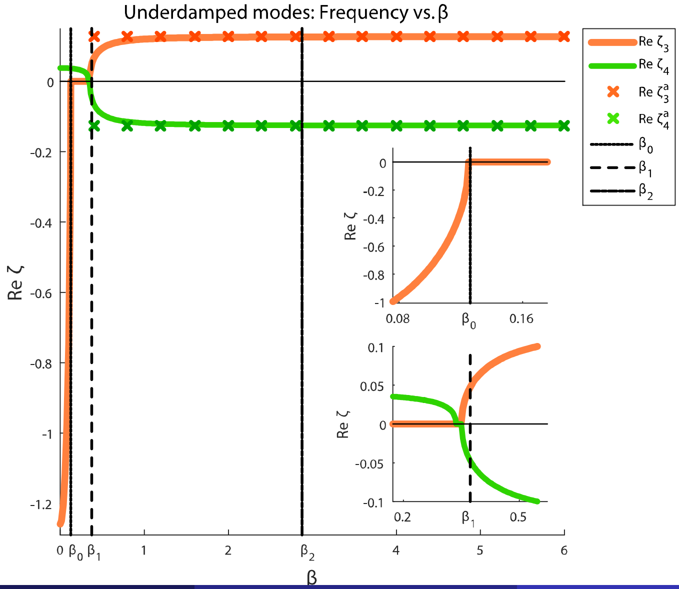
High-loss modes: damping factors and asymptotics



Low-loss modes: damping factors and asymptotics



Underdamped modes: frequencies and asymptotics



Auxiliary Slides