Topological nature of the Fu-Kane-Mele invariants

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Outline

1. Time reversal symmetries and "Quaternionic" structures

2. The role of the (involutive) base space

3. In the search of a classifying object

4. FKMM vs. Fu-Kane-Mele

Topological Quantum Systems with odd TRS's

Let $\mathbb{B}$ a topological space, (**"Brillouin zone"**). Assume that:

- $\mathbb{B}$ is compact, Hausdorff and path-connected;
- $\mathbb{B}$ admits a CW-complex structure.

DEFINITION (*Topological Quantum System (TQS)*)

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators. A **TQS** is a self-adjoint map

$$\mathbb{B} \ni k \longmapsto H(k) = H(k)^* \in \mathcal{K}(\mathcal{H})$$

continuous with respect to the norm-topology.

☞ The **spectrum** $\sigma(H(k)) = \{E_j(k) \mid j \in \mathcal{I} \subseteq \mathbb{Z}\} \subset \mathbb{R}$, is a sequence of eigenvalues ordered according to

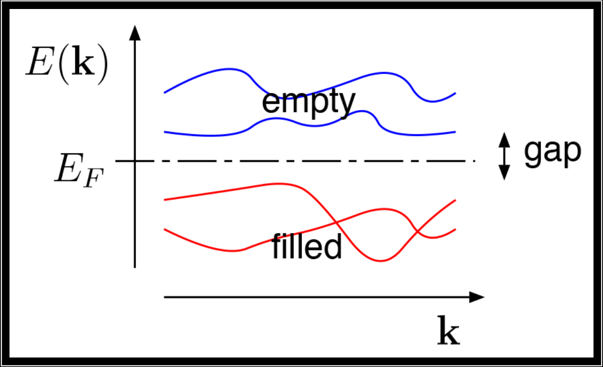
$$\ldots E_{-2}(k) \leqslant E_{-1}(k) < 0 \leqslant E_1(k) \leqslant E_2(k) \leqslant \ldots$$

☞ The maps $k \mapsto E_j(k)$ are **continuous** (energy bands) ...

Topological Quantum Systems with odd TRS's

... namely a band spectrum

$$H(k)\,\psi_j(k) \;=\; E_j(k)\,\psi_j(k)\,, \qquad\qquad k \in \mathbb{B}$$



☞ *Usually an energy* ***gap*** *separates the filled* ***valence*** *bands from the empty* ***conduction*** *bands. The* ***Fermi level*** E_F *characterizes the gap.*

Topological Quantum Systems with odd TRS's

A homeomorphism $\tau : \mathbb{B} \to \mathbb{B}$ *is called* ***involution*** *if* $\tau^2 = \mathrm{Id}_{\mathbb{B}}$. *The pair* $(\mathbb{B}, \tau)$ *is called an* ***involutive space***. *Each space* $\mathbb{B}$ *admits (at least) the* ***trivial involution*** $\tau_{\mathrm{triv}} := \mathrm{Id}_{\mathbb{B}}$.

DEFINITION (*TQS with time-reversal symmetry***)**

Let $(\mathbb{B}, \tau)$ *be an involutive space,* $\mathcal{H}$ *a separable Hilbert space endowed with a* ***complex conjugation*** C. *A TQS* $\mathbb{B} \ni k \mapsto H(k)$ *has a* ***time-reversal symmetry*** *(TRS) of parity* $\eta \in \{\pm 1\}$ *if there is a continuous unitary-valued map* $k \mapsto U(k)$ *such that*

$$U(k)\, H(k)\, U(k)^* = C\, H(\tau(k))\, C\,, \qquad C\, U(\tau(k))\, C = \eta U(k)^*\ .$$

A TQS with an ***odd*** *TRS (i.e.* $\eta = -1$*) is called of class* **AII**.

The Serre-Swan construction

- *An **isolated family** of energy bands is any (finite) collection* $\{E_{j_1}(\cdot), \ldots, E_{j_m}(\cdot)\}$ *of energy bands such that*

$$\min_{k \in \mathbb{B}} \operatorname{dist}\left(\bigcup_{s=1}^{m} \{E_{j_s}(k)\} \,,\, \bigcup_{j \in \mathcal{I} \setminus \{j_1, \ldots, j_m\}} \{E_j(k)\} \right) \;=\; C_g \;>0\,.$$

*This is usually called **gap condition**.*

- *An isolated family is described by the **Fermi projection***

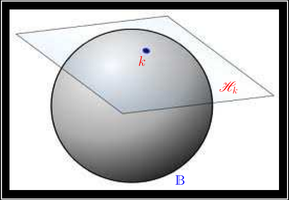
$$P_F(k) \;:=\; \sum_{s=1}^{m} |\psi_{j_s}(k)\rangle\langle\psi_{j_s}(k)|\,.$$

*This is a **continuous** projection-valued map*

$$\mathbb{B} \;\ni\; k \;\longmapsto\; P_F(k) \;\in\; \mathcal{K}(\mathcal{H})\,.$$

The Serre-Swan construction

- *For each* $k \in \mathbb{B}$
$$\mathcal{H}_k \ := \ \mathrm{Ran}\ P_F(k) \ \subset \ \mathcal{H}$$
is a subspace of $\mathcal{H}$ *of* ***fixed*** *dimension* m.
- *The collection*
$$\mathcal{E}_F \ := \ \bigsqcup_{k \in \mathbb{B}} \mathcal{H}_k$$
is a topological space (said ***total space****) and the* ***map***
$$\pi : \ \mathcal{E}_F \ \longrightarrow \ \mathbb{B}$$
defined by $\pi(k, \mathrm{v}) = k$ *is continuous (and open).*



This is a ***complex*** *vector bundle (of rank* m*) called* ***Bloch-bundle****.*

The Serre-Swan construction

☞ *An **odd** TRS induces a **"Quaternionic"** structure on the Bloch-bundle.*

DEFINITION (*Atiyah, 1966 - Dupont, 1969***)**

Let $(\mathbb{B}, \tau)$ *be an involutive space and* $\mathscr{E} \to \mathbb{B}$ *a **complex** vector bundle. Let* $\Theta : \mathscr{E} \to \mathscr{E}$ *an **homeomorphism** such that*

$$\Theta \; : \; \mathscr{E}|_k \longrightarrow \mathscr{E}|_{\tau(k)} \qquad \text{is } \textbf{anti}\text{-linear} .$$

$[\mathcal{R}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a **"Real"**-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \; : \; \mathscr{E}|_k \xrightarrow{+1} \mathscr{E}|_k \qquad \forall \; k \in \mathbb{B} \, ;$$

$[\mathcal{Q}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a **"Quaternionic"**-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \; : \; \mathscr{E}|_k \xrightarrow{-1} \mathscr{E}|_k \qquad \forall \; k \in \mathbb{B} \, .$$

The classification problem

DEFINITION (*Topological phases*)

Let $\mathbb{B} \ni k \longmapsto H(k)$ *be an* ***odd TR-symmetric*** *TQS with an isolated family of* ***m*** *energy bands and associated* ***"Quaternionic"*** *Bloch bundle* $\mathcal{E}_F \longrightarrow \mathbb{B}$. *The* ***topological phase*** *of the system is specified by*

$$[(\mathcal{E}_F, \Theta)] \in \mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau) .$$

⇓⇓⇓

Main Question:

How to classify $\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$ *at least for* ***low-dimensional*** $\mathbb{B}$*?*

The classification problem

Known results for $\dim(\mathbb{B}) \leqslant 3$

- $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \overset{c_1}{\simeq} H^2(\mathbb{B}, \mathbb{Z})$ *(Peterson, 1959)*
- $\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau) \overset{c_1^{\mathcal{R}}}{\simeq} H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1))$ *(Kahn, 1987 - D. & Gomi, 2014)*

CAZ	TRS	Category	VB
A	0	*complex*	$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B})$
AI	$+$	*"Real"*	$\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau)$
AII	$-$	*"Quaternionic"*	$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$

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Electrons in a periodic environment

- **Periodic** quantum systems (e.g. absence of **disorder**):
  - $\mathbb{R}^d$-translations $\Rightarrow$ **free** (Dirac) fermions;
  - $\mathbb{Z}^d$-translations $\Rightarrow$ **crystal** (Bloch) fermions.

- The Bloch-Floquet (or Fourier) theory exploits the invariance under translations of a periodic structure to describe the state of the system in terms of the **quasi-momentum** k on the **Brillouin zone** $\mathbb{B}$.

- Complex conjugation (TRS) endows $\mathbb{B}$ with an involution τ.

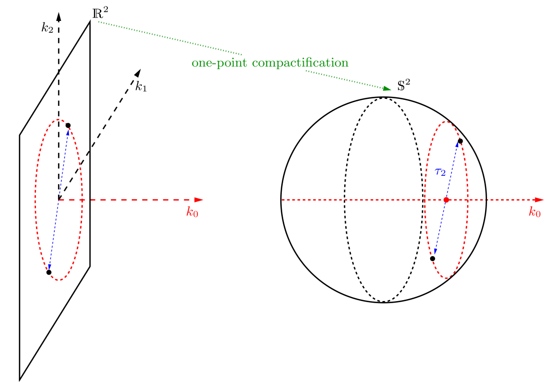
- Examples are:
  - Gapped electronic systems,
  - BdG superconductors,
  - **Photonic crystals** (M. Lein talk).

Continuous case $\mathbb{B} \equiv \mathbb{S}^{1,d}$

$$\mathbb{S}^d \xrightarrow{\theta_{1,d}} \mathbb{S}^d$$

$$\mathbb{S}^{1,d} := (\mathbb{S}^d, \theta_{1,d})$$

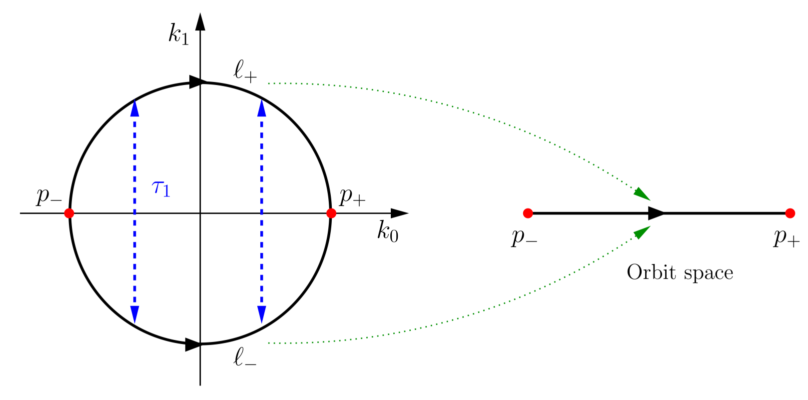
$$(+k_0, +k_1, \ldots, +k_d) \xmapsto{\theta_{1,d}} (+k_0, -k_1, \ldots, -k_d)$$



Periodic case $\mathbb{B} \equiv \mathbb{T}^{0,d,0}$

$$\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1} \xrightarrow{\tau_d := \theta_{1,1} \times \ldots \times \theta_{1,1}} \mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}$$

$$\mathbb{T}^{0,d,0} := \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{d\text{ - times}} = (\mathbb{T}^d, \tau_d)$$



Topological states for Bloch electrons

	$d=1$	$d=2$	$d=3$	$d=4$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	*Free*
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}$	*Periodic*

☞ *The first proof (for the case $d = 1, 2$) is due to* ***Fu***, ***Kane*** *and* ***Mele (2005 - 2007)****. They introduced the notion of* ***Fu-Kane-Mele indices*** *(values of a Pfaffian on the fixed points) and the distinction between* ***strong*** *and* ***weak*** *invariants.*

$$\textbf{Fu-Kane-Mele index} \; := \prod_{k_i \in \mathbb{B}^\tau} \frac{\sqrt{\det[W(x_i)]}}{\mathrm{Pf}[W(k_i)]}$$

with $\mathbb{B}^\tau \ni k \mapsto W(k_i)$ an antisymmetric matrix built with the Bloch functions.

☞ *It makes sense* ***only*** *when $\mathbb{B}^\tau$ is* ***finite !!***

Topological states for Bloch electrons

	$d=1$	$d=2$	$d=3$	$d=4$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	*Free*
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}$	*Periodic*

An afterwards ...

- *Computed by* ***Kitaev (2009)*** *for all* d *by* ***K-theory*** *(stable range).*
- *"Handmade"* ***frame construction*** *for the case* $\mathbb{T}^{0,2,0}$ *by* ***Graf*** *and* ***Porta (2013)*** *and for the case* $\mathbb{T}^{0,3,0}$ *by* ***Fiorenza***, ***Monaco*** *and* ***Panati (2016)***.
- ***Kennedy*** *and* ***Zirnbauer (2015)*** *by the calculation of the* ***equivariant homotopy*** *(very general but* ***hard*** *to compute).*
- ***D.*** *and* ***Gomi (2015)*** *by the introduction of the* ***FKMM-invariant*** *(a characteristic class) and the computation of the* ***equivariant cohomology*** *(very general and* ***not so hard*** *to compute).*

Why more general involutive spaces?

- $\mathbb{B}$ *can be interpreted as the space of* ***control parameters*** *for a quantum system* ***adiabatically perturbed****. In this sense* $(\mathbb{B}, \tau)$ *can be very general. In particular the* ***fixed-point set*** $\mathbb{B}^\tau$ *could be empty (free action) or a sub-manifold of whatever co-dimension (and not necessary a discrete set of points).*

- *Recently* ***Gat*** *and* ***Robbins (arXiv:1511.08994)*** *considered the cases* $\mathbb{B} = \mathbb{S}^{0,3}$ *(****rigid rotor****) and* $\mathbb{B} = \mathbb{T}^{1,1,0}$ *(****phase space of slow dynamic of a 1D periodic particle****). In the first case* $\mathbb{B}^\tau = \emptyset$ *and in the second* $\mathbb{B}^\tau = \mathbb{S}^1 \sqcup \mathbb{S}^1$*. The classification is obtained by a "handmade" frame construction (****11*** *pages for* ***2*** *cases ...):*

$$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{S}^{0,3}) \simeq \begin{cases} 2\mathbb{Z}+1 & m \text{ odd} \\ 2\mathbb{Z} & m \text{ even} \end{cases}, \qquad \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{1,1,0}) \simeq 2\mathbb{Z}\,.$$

- *Many of the previous approaches just fail when* $\mathbb{B}^\tau$ *is not a discrete set: e. g.* ***which is the meaning of the Fu-Kane-Mele indices when*** $\mathbb{B}^\tau$ ***is not a discrete set?***

The spirit of the Born-Oppenheimer approximation

- *Many quantum systems are described by Hamiltonians which depend by* ***slow*** *and* ***fast*** *degrees of freedom (e.g. the Molecular Dynamics).*

- *Under certain spectral conditions the slow and fast degrees of freedom* ***decouple adiabatically****.*

- *This means that the fast degrees of freedom* ***adjust instantly*** *to changes of the slow degrees of freedom. As a consequence, the fast degree evolve according to an* ***effective Hamiltonian*** *which depends parametrically by the slow degrees of freedom. Moreover, the slow degrees of freedom can be considered* ***as classical****.*

- *"De facto" we are in a situation described by a* ***TQS***

$$X \ni (q,p) \longmapsto H_{\mathrm{fast}}(q,p)$$

with X *the* ***classical phase space****.*

- *The* ***TR symmetry*** *acts on the classical variables and induces an involution on the space* X*.*

More general involutive spheres

$\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$ *with* $\theta_{p,q}$ *defined by*

$$(k_0, k_1, \ldots, k_{p-1}, k_p, \ldots, k_{p+q-1}) \overset{\theta_{p,q}}{\mapsto} (k_0, k_1, \ldots, k_{p-1}, -k_p, \ldots, -k_{p+q-1})$$

$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}+1$	?
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}$	?
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{2,q})$	0	?	?	...	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{3,q})$	0	?	...		
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{4,q})$	0	...			

More general involutive spheres

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$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}+1$	$\emptyset$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}$	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\dots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{2,q})$	0	$2\mathbb{Z}$	0	$\dots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{3,q})$	0	0	$\dots$		
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{4,q})$	0	$\dots$			

More general involutive tori (fixed-point case)

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-\text{times}} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-\text{times}} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-\text{times}}$$

$a + b \leqslant 3,\ c = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	?	...
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	?	...	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	...		

More general involutive tori (fixed-point case)

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$a + b \leqslant 3,\ c = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	$(2\mathbb{Z})^2$	$\ldots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\ldots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	$\ldots$		

More general involutive tori (free-involution case)

PROPOSITION (D. - Gomi, 2016)

$$\mathbb{T}^{a,b,c} \simeq \mathbb{T}^{a+c-1,b,1} \qquad \forall\, c \geqslant 2$$

$a+b \leqslant 2,\ c=1$	$a=0$	$a=1$	$a=2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,0,1})$	0	?	?
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,1,1})$	?	?	...
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,2,1})$	?	...	

For all $m \in \mathbb{N}$ *odd or even!*

More general involutive tori (free-involution case)

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$a+b \leqslant 2,\ c=1$	$a=0$	$a=1$	$a=2$
$\mathrm{Vec}^m_{\mathfrak{Q}}(\mathbb{T}^{a,0,1})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathrm{Vec}^m_{\mathfrak{Q}}(\mathbb{T}^{a,1,1})$	$2\mathbb{Z}$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\dots$
$\mathrm{Vec}^m_{\mathfrak{Q}}(\mathbb{T}^{a,2,1})$	$(2\mathbb{Z})^2$	$\dots$	

For all $m \in \mathbb{N}$ *odd or even!*

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Relative equivariant cohomology

In **[D. - Gomi, 2016]** we classified

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0}) \qquad \textit{and} \qquad \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d-1})\,, \quad d \leqslant 4$$

by a ***characteristic class*** with values in $H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1))$: the ***FKMM-invariant***.

$$\begin{array}{ccccccc} H^1_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_1} & H^2(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) & \xrightarrow{r} & H^2_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) \\ \wr\mid & & & & \wr\mid & & \wr\mid \\ [\mathbb{B}^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} & & & & \mathrm{Pic}_{\mathcal{R}}(\mathbb{B}, \tau) & & \mathrm{Pic}_{\mathbb{R}}(\mathbb{B}^\tau) \end{array}$$

☞ *Our previous results only apply to the case*

$$\mathbb{B}^\tau \;=\; \{\textit{finite collection of points}\}.$$

To consider more general involutive spaces we need more generality !

The (generalized) FKMM-invariant

THEOREM *(D. - Gomi, 2016)*

Given $(\mathbb{B},\tau)$ *let*

$$\mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big) \;:=\; \big\{[(\mathscr{L},\mathsf{s})] \mid \mathscr{L}\in\mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)\,,\quad \mathsf{s}:\mathscr{L}|_{\mathbb{B}^{\tau}}\to\mathbb{U}(1)\big\}\;.$$

The choice of s *is* ***canonical*** *and the group structure is given by the* ***tensor product****. Then*

$$\mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big) \;\overset{\tilde{\kappa}}{\simeq}\; H^2\big(\mathbb{B}|\mathbb{B}^{\tau},\mathbb{Z}(1)\big)\;.$$

There is a group homomorphism

$$\kappa \;:\; \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau)\;\longrightarrow H^2\big(\mathbb{B}|\mathbb{B}^{\tau},\mathbb{Z}(1)\big)$$

called the ***FKMM-invariant****.*

- *If* $(\mathscr{E},\Theta)\in\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau)$ *then* $(\det\mathscr{E},\det\Theta)\in\mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$*;*
- *It exists a* ***canonical*** $\mathsf{s}_{\mathscr{E}}:\mathbb{B}^{\tau}\to\det\mathscr{E}|_{\mathbb{B}^{\tau}}$
- $(\det\mathscr{E},\mathsf{s}_{\mathscr{E}})\in\mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big)$

$$\kappa(\mathscr{E},\Theta)\;:=\;\tilde{\kappa}(\det\mathscr{E},\mathsf{s}_{\mathscr{E}})\;.$$

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General properties

$$\kappa \ : \ \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \longrightarrow H^2(\mathbb{B}|\mathbb{B}^\tau,\mathbb{Z}(1))$$

- ***Isomorphic*** *$\mathcal{Q}$-bundles have the same FKMM-invariant;*
- *If $(\mathscr{E},\Theta)$ is $\mathcal{Q}$-**trivial** then $\kappa(\mathscr{E},\Theta)=0$;*
- *κ is **natural** under the pullback induced by equivariant maps;*
- $\kappa(\mathscr{E}_1\oplus\mathscr{E}_2,\Theta_1\oplus\Theta_2) \ = \ \kappa(\mathscr{E}_1,\Theta_1) \ + \ \kappa(\mathscr{E}_2,\Theta_2)$;
- *κ is the image of a **universal class** $\mathfrak{h}_{univ}$;*
- *When $\mathbb{B}^\tau \ = \ \{$**finite** collection of points$\}$*

  $$\kappa(\mathscr{E},\Theta) \ \simeq \ \textit{Fu-Kane-Mele invariants} \ ;$$

- *When $\mathbb{B}^\tau \ = \ \emptyset$*

  $$\kappa(\mathscr{E},\Theta) \ \simeq \ c_1^{\mathcal{R}}(\det\mathscr{E},\det\Theta) \ ;$$

- *When $\mathbb{B}^\tau \ = \ \emptyset$ and $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \ \neq \ \emptyset$ then:*
  $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau)$ *is a **torsor** over* $\mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$. *Hence*

  $$\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \ \simeq \ \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau) \ \simeq \ H^2(\mathbb{B}|\mathbb{B}^\tau,\mathbb{Z}(1)) \ .$$

Low dimension $(\dim(\mathbb{B}) \leqslant 3)$

- *If* $\dim(\mathbb{B}) \leqslant 3$ *the map*

$$\kappa \ : \ \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \ \hookrightarrow \ H^2\big(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)\big)$$

  is ***injective****;*

- *In many situation (e.g over low dimensional* $\mathbb{S}^{p,q}$ *and* $\mathbb{T}^{a,b,c}$*) the map* κ *is* ***bijective****;*

- *If* $\dim(\mathbb{B}) \leqslant 3$ *and* $\mathbb{B}^\tau = \emptyset$ *then*

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \ \simeq H^2\big(\mathbb{B}, \mathbb{Z}(1)\big)$$

  and

$$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{B},\tau) \ \simeq \begin{cases} H^2\big(\mathbb{B}, \mathbb{Z}(1)\big) & \textit{if} \ \ \mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \neq \emptyset \\ \emptyset & \textit{if} \ \ \mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) = \emptyset \end{cases}$$

- *If* $\dim(\mathbb{B}) \leqslant 2$ *and* $\mathbb{B}^\tau \neq \emptyset$ *then*

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \ \simeq \ H^2\big(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)\big)$$

- *However* κ *is* ***not bijective*** *in* $\dim(\mathbb{B}) = 3$ *if* $\mathbb{B}^\tau \neq \emptyset$

Thank you for your attention