Asymptotic behaviour of the spectra of systems of Maxwell equations in periodic composite media with high contrast

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Joint work with Shane Cooper

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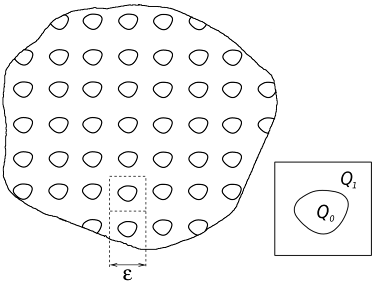
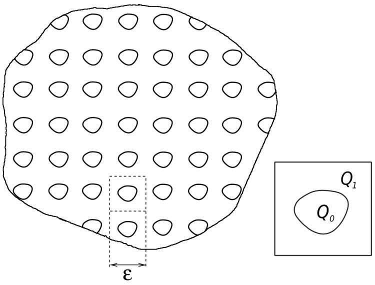
Homogenisation setting

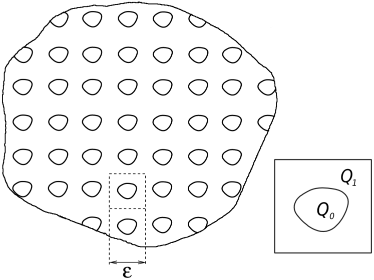




"Classical" homogenisation

$$-\mathrm{div}\Bigl(A\Bigl(\frac{x}{\varepsilon}\Bigr)\nabla u\Bigr) + u = f, \qquad f \in L^2(\mathbb{R}^d),$$

$$A \geq \nu I > 0$$



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Convergence (two-scale expansions, compensated compactness, two-scale convergence, periodic unfolding, Bloch decomposition):

$$u = u_\varepsilon \rightharpoonup u_0 \ \text{ in } \ H^1(\mathbb{R}^d), \qquad -\mathrm{div}A^{\mathrm{hom}}\nabla u_0 + u_0 = f.$$

Problem under study: high contrast

We study the problem

$$\mathrm{curl}\,\left(\epsilon_\eta^{-1}\left(\frac{x}{\eta}\right)\mathrm{curl}\,H^\eta\right)=\omega_\eta^2 H^\eta,\qquad \epsilon_\eta(y)=\begin{cases}1, & y\in Q_1,\\ \eta^{-2}, & y\in Q_0,\end{cases}$$

$\eta\in(0,1)$ period, $Q_1:=Q\backslash\overline{Q_0}$ simply connected Lipschitz set

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Define: "Homogenised matrix'

$$A^{\text{hom}} := \int_{Q_1} \left(\operatorname{curl} N(y) + I\right) \mathrm{d}y,$$

$$\operatorname{curl}\left(\operatorname{curl} N(y) + I\right) = 0 \quad \text{in } Q_1, \quad \left(\operatorname{curl} N(y) + I\right) \times n = 0 \ \text{ on } \partial Q_0, \quad N \text{ is } Q-\text{periodic.}$$

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Consider: $\omega \in \mathbb{R}_+$, $H^0(x,y) := u(x) + \nabla_y v(x,y) + z(x,y)$,
$(u,v,z) \in \big[H^1_{\#\mathrm{curl}}(\mathbb{T})\big]^3 \times L^2\big(\mathbb{R}^3; H^1_\#(Q)\big) \times \big[L^2\big(\mathbb{T}; H^1_0(Q_0)\big)\big]^3$, solution to

$$\mathrm{curl}_x\big(A^{\mathrm{hom}}\mathrm{curl}_x u(x)\big) = \omega^2\Big(u(x) + \int_{Q_0} z(x,y)dy\Big), \qquad x \in \mathbb{T},$$

$$\mathrm{div}_y\big(\nabla_y v(x,y) + z(x,y)\big) = 0, \qquad (x,y) \in \mathbb{T} \times Q,$$

$$\mathrm{curl}_y\big(\mathrm{curl}_y z(x,y)\big) = \omega^2\big(u(x) + \nabla_y v(x,y) + z(x,y)\big), \qquad (x,y) \in \mathbb{T} \times Q_0.$$

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Our result: A) $\exists$ at least one eigenfrequency ω_η such that $|\omega_\eta - \omega| < C\eta$,
B) $\operatorname{dist}\left(H^0, X_\eta\right) < \widehat{C}\eta$, where $\widehat{C} > 0$, $X_\eta := \operatorname{span}\left\{H^\eta : \omega_\eta \text{ as above}\right\}$.

Spectral decomposition of the limit problem

Write $v(x,\cdot) = G * (\mathrm{div_y}\, z)(x,\cdot)$, then

$$\mathrm{curl}_y\,\mathrm{curl}_y\, z(x,y) = \omega^2 \left(u(x) + \nabla_y \int_{Q_0} G(y-y')\,\mathrm{div}_{y'} z(x,y')\,dy' + z(x,y) \right), \quad (x,y) \in \mathbb{T} \times Q_0.$$

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Note that

$$\nabla_y \int_{Q_0} G(y-y')\,\mathrm{div}_{y'} z(x,y')\,dy' + z(x,y) = \omega^2 B(y) u(x),$$

B is 3×3 matrix function with columns $B^j [H^1_{\#}(Q)]^3$, $j = 1,2,3$:

$$\mathrm{curl}\,\mathrm{curl}\, B^j = e_j + \omega^2 B^j \quad \text{in } Q_0,$$

$$\mathrm{curl}\, B^j(y) = 0, \quad y \in Q_1, \tag{1}$$

$$\mathrm{div}\, B^j(y) = 0, \quad y \in Q, \tag{2}$$

$$a(B^j) = 0, \tag{3}$$

$a(B^j)$ "circulation" of B^j :
H^1-continuous extension of $a(\phi)_i = \int_0^1 \phi_i(te_i)dt$, $\phi \in [C^\infty(Q)]^3$.

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Equivalent variational formulation

$$\int_{Q_0} \mathrm{curl}\, B^j \cdot \mathrm{curl}\,\varphi = \int_Q e_j \cdot \varphi + \omega^2 \int_Q B^j \cdot \varphi, \qquad \forall \varphi \in [H^1_\#(Q)]^3 \text{ subject to (1)–(3).}$$

"Macroscopic" equation

Operator-pencil spectral problem

$$\operatorname{curl}\left(A^{\mathsf{hom}}\operatorname{curl} u(x)\right) = \Gamma(\omega)u(x), \qquad x \in \mathbb{T}, \tag{4}$$

where Γ is a matrix-valued function that vanishes at $\omega = 0$, and for $\omega \neq 0$ has elements

$$\Gamma_{ij}(\omega) = \omega^2\left(\delta_{ij} + \omega^2\int_Q B_i^j\right), \quad i, j = 1, 2, 3. \tag{5}$$

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Alternative representation for $\Gamma(\omega)$:
Consider $\phi^k \in [H_0^1(Q_0)]^3$, $k \in \mathbb{N}$, solutions to non-local problems

$$\operatorname{curl}\operatorname{curl}\phi^k(y) = \alpha_k\left(\nabla \int_{Q_0} G(y - y')\operatorname{div}\phi^k(y')\,dy' + \phi^k(y)\right), \quad y \in Q_0, \tag{6}$$

subject to orthonormality conditions

$$\int_{Q_0}\int_{Q_0}\left(\nabla^2 G(y - y') + I\right)\phi_j(y) \cdot \overline{\phi^k(y')}\,dy\,dy' = \delta_{jk}, \quad j, k = 1, 2, , \dots,$$

where $\nabla^2 G$ is the Hessian matrix of G. Then

$$\Gamma_{ij}(\omega) = \omega^2\delta_{ij} + \omega^4 \sum_{k=1}^{\infty} \frac{\left(\int_{Q_0}\phi_i^k\right)\left(\int_{Q_0}\phi_j^k\right)}{\alpha_k - \omega^2}, \quad i, j = 1, 2, 3, \qquad \omega^2 \notin \{0\} \cup \{\alpha_k\}_{k=1}^{\infty}. \tag{7}$$

Structure of the limit spectrum

$$u(x)=\sum_{m\in\mathbb{Z}^3}\exp(2\pi\mathrm{i}m\cdot x)\hat{u}(m),\qquad \hat{u}(m):=\int_{\mathbb{T}}\exp(-2\pi\mathrm{i}m\cdot x)u(x)\,dx,$$

$$u(x) = \sum_{m\in\mathbb{Z}^3} \exp(2\pi \mathrm{i} m\cdot x)\hat{u}(m), \qquad \hat{u}(m) := \int_{\mathbb{T}} \exp(-2\pi \mathrm{i} m\cdot x) u(x)\, dx,$$

Then

$$\mathcal{M}(m)\hat{u}(m) = \Gamma(\omega)\hat{u}(m), \qquad m\in\mathbb{Z}^3,$$
$$\mathcal{M}_{lp}(m) = 4\pi^2 \varepsilon_{ils} m_s A^{\mathrm{hom}}_{ij} \varepsilon_{jpt} m_t = 4\pi^2 (e_l\times m)\cdot A^{\mathrm{hom}}(e_p\times m).$$

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$$\hat{u}(m) = C(\tilde{m})^\top \tilde{u}(\tilde{m}) + \alpha(\tilde{m})\tilde{m},$$
$$\tilde{u}(\tilde{m}) \in \mathbb{R}^2,\ \alpha(\tilde{m}) \in \mathbb{R},\ \ C(\tilde{m}) = \begin{pmatrix} \tilde{e}_{11}(\tilde{m}) & \tilde{e}_{12}(\tilde{m}) & \tilde{e}_{13}(\tilde{m}) \\ \tilde{e}_{21}(\tilde{m}) & \tilde{e}_{22}(\tilde{m}) & \tilde{e}_{23}(\tilde{m}) \end{pmatrix}.$$

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Determine $\big(\tilde{u}(\tilde{m}), \alpha(\tilde{m})\big) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$\begin{aligned} &|m|^2 \Lambda(\tilde{m})\tilde{u}(\tilde{m}) = C(\tilde{m})\Gamma(\omega)C(\tilde{m})^\top \tilde{u}(\tilde{m}) + \alpha(\tilde{m})C(\tilde{m})\Gamma(\omega)\tilde{m}, \\ &\Gamma(\omega)C(\tilde{m})^\top \tilde{u}(\tilde{m}) \cdot \tilde{m} = -\alpha(\tilde{m})\Gamma(\omega)\tilde{m} \cdot \tilde{m}, \end{aligned} \tag{8}$$

where

$$\Lambda(\tilde{m}) := \begin{pmatrix} \lambda_1(\tilde{m}) & 0 \\ 0 & \lambda_2(\tilde{m}) \end{pmatrix}.$$

Different admissible propagation regimes

If the inclusion Q_0 is symmetric under a rotation by π around at least two of the three coordinate axes, then matrices $A^{\rm hom}$ and $\Gamma(\omega)$ are diagonal:
$A^{\rm hom} = {\rm diag}(a_1, a_2, a_3)$, $\Gamma(\omega) = {\rm diag}\bigl(\beta_1(\omega), \beta_2(\omega), \beta_3(\omega)\bigr)$.
The eigenvalues $\lambda_{1,2}(\tilde{m})$ of $\mathcal{M}(\tilde{m})$ are the solutions to the quadratic equation

$$\lambda^2 - \lambda\bigl\{(a_2+a_3)\tilde{m}_1^2 + (a_1+a_3)\tilde{m}_2^2 + (a_1+a_2)\tilde{m}_3^2\bigr)\bigr\} + \bigl(a_1a_2\tilde{m}_3^2 + a_2a_3\tilde{m}_1^2 + a_1a_3\tilde{m}_2^2\bigr) = 0. \tag{9}$$

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Suppose Q_0 is symmetric by a $\pi/2$ rotation around at least two of the three axes, say x_1 and x_2, then $a = a_1 = a_2 = a_3$ and $\beta(\omega) = \beta_1(\omega) = \beta_2(\omega) = \beta_3(\omega)$.
If $\beta(\omega) \neq 0$, then $\tilde{u}(\tilde{m})$ is an arbitrary element of $\mathbb{R}^2$ and $\hat{u}(m) = C(\tilde{m})^\top \tilde{u}(\tilde{m})$ is an arbitrary vector of the (2-dimensional) eigenspace spanned by the vectors $\tilde{e}_1(\tilde{m})$ and $\tilde{e}_2(\tilde{m})$.
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Isotropic propagation (no "weak" band gaps)

Suppose Q_0 is symmetric by a $\pi/2$ rotation around one of the three coordinate axis, say x_1, and by a π rotation around another axis, say x_2, one has $a = a_1$, $b = a_2 = a_3$ and $\beta_2(\omega) = \beta_3(\omega)$.
Propagation is restricted solely to the direction of $\tilde{e}_1(\tilde{m})$ (*resp.* $\tilde{e}_2(\tilde{m})$) which is orthogonal to the eigenvector(s) corresponding to the negative eigenvalue of $\Gamma(\omega)$.
Directional propagation (existence of "weak" band gaps)

Proposition

The spectrum of the limit problem is the union of the following sets.

1. *The elements of* $\{\alpha_k : k \in \mathbb{Z}\}$ *such that at least one of the corresponding* ϕ^k *has zero mean over* Q*. These are eigenvalues of infinite multiplicity and the corresponding eigenfunctions* $H^0(x,y)$ *are of the form*

$$w(x)\left(\nabla \int_{Q_0} G(y-y')\,\mathrm{div}\,\phi^k(y')\,dy' + \phi^k(y)\right), \qquad w \in L^2(\mathbb{T}).$$

2. *The set* $\{\omega^2 : \exists m \in \mathbb{Z}^3 \text{ such that (8) holds}\}$*, with eigenfunctions* $H^0(x,y)$ *of the limit problem having the form* $u(x) + \nabla_y v(x,y) + z(x,y)$*, where* $u(x) = \exp(2\pi \mathrm{i} m \cdot x)\hat{u}(m)$ *is an eigenfunction of macroscopic problem and*

$$\nabla_y v(x,y) + z(x,y) = \omega^2 B(y) u(x,y) \quad \text{a.e. } (x,y) \in \mathbb{T} \times Q,$$

that is $H^0(x,y) = \big(I + \omega^2 B(y)\big)\exp(2\pi \mathrm{i} m \cdot x)\hat{u}(m)$.

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Corollary

If the matrix $\Gamma(\omega)$ *is negative-definite, the value* $\lambda = \omega^2$ *does not belong to the spectrum of the limit problem.*

Proof of main result

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A) We seek solutions in the form of an asymptotic expansion

$$H^\eta(x) = H^0\big(x, \tfrac{x}{\eta}\big) + \eta H^1\big(x, \tfrac{x}{\eta}\big) + \eta^2 H^2\big(x, \tfrac{x}{\eta}\big) + \ldots, \qquad (10)$$

where $H^j(x, y)$, $j = 0, 1, 2, \ldots,$ are Q-periodic in the variable y.

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where $H^j(x, y)$, $j = 0, 1, 2, \ldots$, are Q-periodic in the variable y.

B) $\mathcal{A}_\eta$ operator in $L^2_{\#\mathrm{sol}}(\mathbb{T})$ defined by the form

$$\int_{\mathbb{T}} \varepsilon_\eta^{-1}\big(\tfrac{\cdot}{\eta}\big)\,\mathrm{curl}\,u \cdot \mathrm{curl}\,v, \qquad u, v \in [H^1_\#(\mathbb{T})]^3 \cap L^2_{\#\mathrm{sol}}(\mathbb{T}) =: \mathcal{H}.$$

For fixed ω in the spectrum of limit problem let H^0 be a corresponding eigenfunction. Consider solution $\widetilde{H}^\eta \in \mathcal{H}$ to

$$(\mathcal{A}_\eta + I)\widetilde{H}^\eta = (\omega^2 + 1)H^0(\cdot, \tfrac{\cdot}{\eta}). \qquad (11)$$

and

$$H^{(2)}(\cdot, \eta) := H^0\big(\cdot, \tfrac{\cdot}{\eta}\big) + \eta H^1\big(\cdot, \tfrac{\cdot}{\eta}\big) + \eta^2 H^2\big(\cdot, \tfrac{\cdot}{\eta}\big), \qquad (12)$$

Proof of main result

Ingredients of proof:

A) We seek solutions in the form of an asymptotic expansion

$$H^\eta(x) = H^0\left(x, \tfrac{x}{\eta}\right) + \eta H^1\left(x, \tfrac{x}{\eta}\right) + \eta^2 H^2\left(x, \tfrac{x}{\eta}\right) + \dots, \tag{10}$$

where $H^j(x, y)$, $j = 0, 1, 2, \dots$, are Q-periodic in the variable y.

B) $\mathcal{A}_\eta$ operator in $L^2_{\#\mathrm{sol}}(\mathbb{T})$ defined by the form

$$\int_{\mathbb{T}} \varepsilon_\eta^{-1}\left(\tfrac{\cdot}{\eta}\right) \operatorname{curl} u \cdot \operatorname{curl} v, \qquad u, v \in [H^1_\#(\mathbb{T})]^3 \cap L^2_{\#\mathrm{sol}}(\mathbb{T}) =: \mathcal{H}.$$

For fixed ω in the spectrum of limit problem let H^0 be a corresponding eigenfunction. Consider solution $\widetilde{H}^\eta \in \mathcal{H}$ to

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C) There exists a constant $\widehat{C} > 0$ such that the estimate

$$\mathfrak{b}_\eta\left(\widetilde{H}^\eta - H^{(2)}(\cdot, \eta), \varphi\right) \le \widehat{C}\eta\sqrt{\mathfrak{b}_\eta(\varphi, \varphi)} \qquad \forall \varphi \in [H^1_\#(\mathbb{T})]^3,$$

where

$$\mathfrak{b}_\eta(u, v) := \int_{\mathbb{T}} \varepsilon_\eta^{-1}\left(\tfrac{\cdot}{\eta}\right) \operatorname{curl} u \cdot \operatorname{curl} v + \int_{\mathbb{T}} u \cdot v$$

Proof of main result

Theorem

$\exists C > 0$ *such that* $\left\| \widetilde{H}^\eta - H^0(\cdot, \cdot/\eta) \right\|_{L^2(\mathbb{T})} \leq C\eta \qquad \forall \eta.$

Proof of main result

Theorem

$\exists C > 0$ *such that* $\left\|\widetilde{H}^\eta - H^0(\cdot,\cdot/\eta)\right\|_{L^2(\mathbb{T})} \le C\eta \qquad \forall \eta.$

Proof

Setting $\varphi = \widetilde{H}^\eta - H^{(2)}(\cdot,\eta)$

$$\widehat{C}^2\eta^2 \ge \mathfrak{b}_\eta\big(\widetilde{H}^\eta - H^{(2)}(\cdot,\eta),\, \widetilde{H}^\eta - H^{(2)}(\cdot,\eta)\big) \ge \left\|\widetilde{H}^\eta - H^{(2)}(\cdot,\eta)\right\|^2_{L^2(\mathbb{T})}.$$

Note that

$$\left\|H^{(2)}(\cdot,\eta) - H^0(\cdot,\cdot/\eta)\right\|_{L^2(\mathbb{T})} \le \widetilde{C}\eta, \qquad \widetilde{C} > 0,$$

and hence

$$\left\|\widetilde{H}^\eta - H^0(\cdot,\cdot/\eta)\right\|_{L^2(\mathbb{T})} \le \left\|\widetilde{H}^\eta - H^{(2)}(\cdot,\eta)\right\|_{L^2(\mathbb{T})}$$

$$+\left\|H^{(2)}(\cdot,\eta) - H^0(\cdot,\cdot/\eta)\right\|_{L^2(\mathbb{T})} \le (\widehat{C} + \widetilde{C})\eta.$$

Symmetry of $A^{\rm hom}$ and $\Gamma(\omega)$ under rotations

Proposition

Suppose that σ is a rotation such that $\sigma Q = Q$ and assume that

$$a(y) = \sigma^{-1} a(\sigma y)\sigma, \qquad y \in Q. \tag{13}$$

Then , $A^{\rm hom}$ inherits the same symmetry, i.e. one has

$$A^{\rm hom} = \sigma^{-1} A^{\rm hom} \sigma.$$

In particular, $A^{\rm hom}_{kl} = A^{\rm hom}_{lk} = 0$, for all $l \neq k$.

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Corollary

1. If (13) holds for $\sigma = \sigma_k$, where σ_k is the rotation by π around the x_k-axis, then $A^{\rm hom}_{kl} = 0$, for all $l \neq k$.

2. If (13) holds for $\sigma = \sigma_k$, where σ_k is the rotation by $\pi/2$ around the x_k-axis, then $A^{\rm hom}_{kl} = 0$, for all $l \neq k$ and $A^{\rm hom}_{ii} = A^{\rm hom}_{jj}$, $i, j \neq k$.

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Similar results can be demonstrated for $\Gamma(\omega)$.