Uniqueness for an inverse boundary value problem in electromagnetism

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Global uniqueness with partial data

Habib Ammari and Gunther Uhlmann, Indiana Univ. Math. J. 53 (2004)

Consider the Schrödinger equation on a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, with $\partial\Omega \in C^2$, for a real-valued potential $q \in L^\infty(\Omega)$: $(\Delta - q)u = 0$ in Ω, where $u \in H^1(\Omega)$.

Let $\Gamma \subset \partial\Omega$ and $\Gamma_c := \partial\Omega \setminus \bar{\Gamma}$. Define the partial Cauchy data set associated to q as

$$C_q := \{(u|_{\partial\Omega}, \partial_\nu u|_\Gamma) \,:\, u \in H^1(\Omega),\, (\Delta - q)u = 0 \text{ in } \Omega,\, u|_{\Gamma_c} = 0\}.$$

Theorem (Ammari and Uhlmann). If $q_1 = q_2$ a.e. near $\partial\Omega$, and $C_{q_1} = C_{q_2}$ then $q_1 = q_2$ a.e. in Ω.

Proof

- Integration by parts. Assume that $(\Delta - q_j)u_j = 0$ in Ω, $\mathrm{supp}(q_1 - q_2) \subset \Omega' \subset\subset \Omega$, $u_j|_{\Gamma_c} = 0$, and $C_{q_1} = C_{q_2}$. Then $\int_{\Omega'}(q_1 - q_2)u_1 u_2 dx = 0$.
- Density argument. Assume that $\Omega \setminus \overline{\Omega'}$ is connected. Then the set of solutions to $(\Delta - q)v = 0$ in Ω s.t. $v = 0$ on Γ_c, is dense in the set of all the solutions, in $L^2(\Omega')$.
- CGO solutions. Use the special solutions $v_j(x) = e^{x\cdot\rho_j}(1 + \psi_j(x))$ (with $\rho_j \in \mathbb{C}^d$) to $(\Delta - q_j)v_j = 0$ in $\mathbb{R}^d$, so that $e^{x\cdot\rho_j}$ is harmonic, i.e. $\rho_j \cdot \rho_j = 0$, and ψ_j tends to zero in a weighted L^2 space for large $|\rho_j|$.
- Extract information on the Fourier transform. Thanks to the density result, $\int_{\Omega'}(q_1 - q_2)v_1 v_2 dx = 0$. Fix $k \in \mathbb{R}^d$. For certain ρ_j depending on $\tau >> 1$, get

$$\mathcal{F}(q_1 - q_2)(k) = -\int_{\mathbb{R}^d}(q_1 - q_2)e^{-ik\cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2)dx = 0 \quad \text{as } \tau \to \infty$$

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Let μ, ε, σ be positive functions on a $C^{1,1}$ bounded domain Ω in $\mathbb{R}^3$, describing the permeability, permittivity and conductivity, respectively, of an inhomogeneous, isotropic medium Ω.

Consider the electric and magnetic fields, E, H, satisfying the so-called time-harmonic Maxwell equations at a frequency $\omega > 0$, namely

$$\begin{cases} \nabla \times H + i\omega\gamma E = 0, \\ \nabla \times E - i\omega\mu H = 0, \end{cases} \tag{1}$$

in Ω, where $\gamma = \varepsilon + i\sigma/\omega$, and $\nabla\times$ denotes the *curl* operator.

There exist positive ω's for which there are nontirivial solutions to (1) in $H(curl;\Omega)$ such that $N \times E|_{\partial\Omega} = 0$ or $N \times H|_{\partial\Omega} = 0$. These ω's are called *resonant frequencies*.

The impedance map Λ^{im} and the admittance map Λ^{ad} given by $\Lambda^{\text{im}} : N \times H|_{\partial\Omega} \mapsto N \times E|_{\partial\Omega}$, $\Lambda^{\text{ad}} : N \times E|_{\partial\Omega} \mapsto N \times H|_{\partial\Omega}$ are not well-defined for resonant frequencies.

For any $\omega > 0$, one can consider the (global) Cauchy data set $C(\mu, \gamma)$ as boundary measurements defined by

$$C(\mu, \gamma) := \{(N \times E|_{\partial\Omega}, N \times H|_{\partial\Omega}) : (E, H) \in H(\text{curl}; \Omega)^2 \text{ solves (1) in } \Omega\}.$$

For partial data restricted to a smooth, open subset Γ of $\partial\Omega$, define

$$C(\mu, \gamma; \Gamma) := \{(N \times E|_{\partial\Omega}, N \times H|_{\Gamma}) : (E, H) \in H(\text{curl}; \Omega)^2 \text{ solves (1) in } \Omega, \\ \text{and } \operatorname{supp}(N \times E|_{\partial\Omega}) \subset \overline{\Gamma}\}.$$

Definition. Fix $M > 0$. The pair of coefficients (μ, γ) is called *admissible* if $\mu, \gamma \in C^{1,1}(\overline{\Omega})$ and

- $\mathrm{Re}\gamma \geq M^{-1}, \quad \mu \geq M^{-1}$ in Ω,
- $\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \leq M$.

Theorem (B, Marletta, Reyes). Assume that (μ_j, γ_j) is an admissible pair of coeffcients for $j = 1, 2$, $\mathrm{supp}(\mu_1 - \mu_2)$, $\mathrm{supp}(\gamma_1 - \gamma_2) \subset \Omega$ and $C(\mu_1, \gamma_1; \Gamma) = C(\mu_2, \gamma_2; \Gamma)$. Then $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$ in Ω.

Some references:

Global data

- E. Somersalo, D. Isaacson and M. Cheney; J. Comp. Appl. Math. 42 (1992).
- P. Ola, L. Päivärinta and E. Somersalo; Duke Math. J. 70 (1993).
- P. Ola and E. Somersalo; SIAM J. Appl. Math. 56 (1996).
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Partial data

- P. Caro, P. Ola and M. Salo; Comm. PDE. 34 (2009).
- P. Caro; Inverse Probl. Imaging 5 (2011).
- F. J. Chung, P. Ola, M. Salo and L. Tzou; ArXiv:1502.01618.

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The equations

E, H solve the Maxwell system $\Leftrightarrow X = (h \;\; H^t \,|\, e \;\; E^t)^t$ solves the *augmented system* $(P+V)X = 0$ and $e = h = 0$, where

$$P := \left(\begin{array}{cc|cc} & & & D\cdot \\ & & D & -D\times \\ \hline & D\cdot & & \\ D & D\times & & \end{array}\right), \quad V := \left(\begin{array}{cc|cc} \omega\mu & & & D\alpha\cdot \\ & \omega\mu I_3 & D\alpha & \\ \hline & D\beta\cdot & \omega\gamma & \\ D\beta & & & \omega\gamma I_3 \end{array}\right),$$

$$D := (1/i)\nabla, \qquad \alpha := \log\gamma, \qquad \beta := \log\mu.$$

Note that $P^2 = -\Delta I_8$. Further, X solves $(P+V)X = 0$ if and only if

$$Y = \mathrm{diag}(\mu^{1/2} I_4\,,\, \gamma^{1/2} I_4)\, X$$

solves the *rescaled system* $(P+W)Y = 0$.

The opertor $-PW^t + WP$ is zeroth-order. Thus, the matrix operator $(P + W)(P - W^t)$ is Schrödinger-type, since

$$(P + W)(P - W^t) = -\Delta I_8 + Q,$$

where $Q := -PW^t + WP - WW^t$.

We can get solutions to the Maxwell system via solutions to a Schrödinger-type system as follows:

If $(-\Delta + Q)Z = 0$ and the scalar fields of

$$X := \mathrm{diag}(\mu^{-1/2} I_4\,,\, \gamma^{-1/2} I_4)\,(P - W^t)Z$$

vanish, then the vector fields of X satisfy the Maxwell system.

An orthogonality identity

Proposition. Assume that (μ_j, γ_j) are admissible $(j = 1, 2)$ such that

- $C(\mu_1, \gamma_1; \Gamma) = C(\mu_2, \gamma_2; \Gamma)$,
- $\mu_1 = \mu_2$, $\gamma_1 = \gamma_2$, $\nabla\mu_1 = \nabla\mu_2$, $\nabla\gamma_1 = \nabla\gamma_2$ on Γ,
- $Z_1 \in H^1(\Omega; \mathbb{C}^8)$ solves $(-\Delta I_8 + Q_1)Z_1 = 0$ in Ω and "gives solutions" (E_1, H_1) to the Maxwell system, with $N \times E_1|_{\partial\Omega} = 0$ on $\Gamma_c := \partial\Omega \setminus \overline{\Gamma}$.
- $Y_2 \in H^1(\Omega; \mathbb{C}^8)$ solves $(P + W_2^*)Y_2 = 0$ in Ω and $Y_2|_{\partial\Omega} = 0$ on Γ_c.

Then $\langle (Q_1 - Q_2)Z_1 | Y_2 \rangle_\Omega = 0$.

The idea of this kind of identity relating a solution to a Schrödinger-type equation corresponding to the Maxwell system and a solution to a Dirac-type operator not related to the Maxwell system comes from the paper

C. E. Kenig, M. Salo and G. Uhlmann; Duke Math. J. (2011).

Density results (inspired by [H. Ammari and G. Uhlmann (2004)])

Remarks:

- We proved that if $(P + W^*)Y = 0$ then Y also satisfies a Schrödinger-type equation $(-\Delta I_8 + \widetilde{Q})Y = 0$, where $\widetilde{Q}$ is zeroth-order.
- If $(E, H) \in H(\mathrm{curl}; \Omega)^2$ solves the Maxwell system in Ω then E is a solution to $LE = 0$ in Ω, where $LE := \nabla \times (\mu^{-1} \nabla \times E) - \omega^2 \gamma E$.

Results:

Let $\Omega' \subset\subset \Omega$ with $\partial\Omega' \in C^2$ such that $\Omega \setminus \overline{\Omega'}$ is connected.

- $\widetilde{K}(\Omega) := \{\widetilde{Y} \in H^2(\Omega; \mathbb{C}^8) \,:\, (-\Delta I_8 + \widetilde{Q})\widetilde{Y} = 0 \text{ in } \Omega,\ \widetilde{Y}|_{\partial\Omega} = 0 \text{ on } \Gamma_c\}$ is dense in the set $K(\Omega) := \{Y \in H^2(\Omega; \mathbb{C}^8) \,:\, (-\Delta I_8 + \widetilde{Q})Y = 0 \text{ in } \Omega\}$ with respect to the topology in $L^2(\Omega'; \mathbb{C}^8)$.

- Let $\widetilde{N}(\Omega)$ be the set of functions $\widetilde{E} \in H(\mathrm{curl}; \Omega)$ with $\nabla \times (\nabla \times \widetilde{E}) \in L^2(\Omega; \mathbb{C}^3)$ solving $L\,\widetilde{E} = 0$ in Ω such that $N \times \widetilde{E}|_{\partial\Omega} = 0$ on Γ_c. Then $\widetilde{N}(\Omega)$ is dense in the set $N(\Omega) := \{E \in H(\mathrm{curl}; \Omega) \,:\, \nabla \times (\nabla \times E) \in L^2(\Omega; \mathbb{C}^3),\ L\,E = 0 \text{ in } \Omega\}$ with respect to the topology in $L^2(\Omega'; \mathbb{C}^3)$.

Proof of the theorem

Let $\Omega' \subset\subset \Omega$ with $\partial\Omega' \in C^2$ and $\mathrm{supp}(\mu_1 - \mu_2)$, $\mathrm{supp}(\gamma_1 - \gamma_2) \subset \Omega'$.

Let Z_1, Y_2 be certain special solutions of almost exponential growth (Faddeev-Calderón-Sylvester-Uhlmann) satisfying $(-\Delta I_8 + Q_1)Z_1 = 0$, $(P + W_2^*)Y_2 = 0$ in $\mathbb{R}^3$, where Z_1 "gives solutions" to the Maxwell system with μ_1, γ_1. Here the coefficients are extended to the whole Euclidean space.

More precisely, for some $\zeta_j \in \mathbb{C}^3$ with $\zeta_j \cdot \zeta_j = \omega^2 \epsilon_0 \mu_0$, depending on a large free parameter τ ($|\zeta_j| \geq \tau$),

$$Z_1(x, \zeta_1) = e^{i\zeta_1 \cdot x}(L_1(\zeta_1) + R_1(x, \zeta_1)),$$
$$Y_2(x, \zeta_2) = e^{i\zeta_2 \cdot x}(M_2(\zeta_2) + S_2(x, \zeta_2)),$$

where R_1, S_2 tend to zero in some sense when $\tau \to \infty$.

Then, thanks to the density results in $L^2(\Omega')$ and the bounded invertibility of $P - W_1^t$ with certain boundary conditions, it follows that

$$\langle (Q_1 - Q_2) Z_1 | Y_2 \rangle_\Omega = 0. \tag{2}$$

For fixed $\xi \in \mathbb{R}^3$, we take $\zeta_1 - \overline{\zeta_2} = -\xi$ and have

$$\begin{aligned} \langle (Q_1 - Q_2) Z_1 | Y_2 \rangle_\Omega &= \int_\Omega (Q_1 - Q_2) Z_1 \cdot \overline{Y_2} dx \\ &= \int_\Omega e^{-i\xi\cdot x} (Q_1 - Q_2)(L_1 + R_1)(\overline{M_2} + \overline{S_2}) dx \\ &= \begin{cases} \widehat{f}(\xi) + \mathcal{O}(\tau^{-1}), & \text{for certain choice of } L_1, M_2, \\ \widehat{g}(\xi) + \mathcal{O}(\tau^{-1}), & \text{for certain choice of } L_1, M_2. \end{cases} \end{aligned}$$

Thus, from (2) we obtain

$$|\widehat{f}(\xi)| + |\widehat{g}(\xi)| \le \frac{C}{\tau},$$

where

$$f = \chi_\Omega \cdot \left(\frac{1}{2}\Delta(\alpha_1 - \alpha_2) + \frac{1}{4}\left(\nabla\alpha_1 \cdot \nabla\alpha_1 - \nabla\alpha_2 \cdot \nabla\alpha_2\right) + \left(\kappa_2^2 - \kappa_1^2\right) \right),$$

$$g = \chi_\Omega \cdot \left(\frac{1}{2}\Delta(\beta_1 - \beta_2) + \frac{1}{4}\left(\nabla\beta_1 \cdot \nabla\beta_1 - \nabla\beta_2 \cdot \nabla\beta_2\right) + \left(\kappa_2^2 - \kappa_1^2\right) \right),$$

with $\alpha_j := \log\gamma_j, \qquad \beta_j := \log\mu_j, \qquad \kappa_j := \omega\mu_j^{1/2}\gamma_j^{1/2}.$

Deduce that $f = g = 0$. Using a Carleman estimate, Pedro Caro proves that

$$e^{d_1/h} \sum_{j=1,2} \left(h\|\phi_j\|^2_{L^2(\Omega)} + h^3\|\nabla\phi_j\|^2_{L^2(\Omega)}\right) \le C\, e^{d_2/h}$$

$$\times \left(h^4\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}\right) + \sum_{j=1,2} \left(h\|\phi_j\|^2_{L^2(\partial\Omega)} + h^3\|\nabla\phi_j\|^2_{L^2(\partial\Omega)}\right) \right),$$

where $\phi_1 := \gamma_1^{1/2} - \gamma_2^{1/2}$, $\phi_2 := \mu_1^{1/2} - \mu_2^{1/2}$, $C = C(\Omega, M)$,
$0 < h < C^{-1/3} \le 1$, and

$$d_1 := \inf\{|x - x_0|^2 \,:\, x \in \Omega\}, \qquad d_2 := \sup\{|x - x_0|^2 \,:\, x \in \Omega\},$$

for certain point $x_0 \notin \overline{\Omega}$. Thus, we are done.