Asymptotic analysis of stratified elastic media in the space of functions with bounded deformation.

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Introduction

We study the asymptotic behavior of the solution $\boldsymbol{u}_\varepsilon$ to

$$\mathcal{P}_\varepsilon : \begin{cases} -\,\textbf{div}\,(\lambda_\varepsilon \operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))\boldsymbol{I} + 2\mu_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon)) = \boldsymbol{f} \quad \text{in } \Omega, \\ \boldsymbol{u} \in H_0^1(\Omega;\mathbb{R}^3), \quad \boldsymbol{e}(\boldsymbol{u}_\varepsilon) := \frac{1}{2}(\nabla \boldsymbol{u}_\varepsilon + \nabla^T \boldsymbol{u}_\varepsilon), \\ \Omega := (0,L) \times \Omega' \subset \mathbb{R}^3, \qquad \boldsymbol{f} \in L^\infty(\Omega;\mathbb{R}^3) \end{cases}$$

under the assumption

$$\mu_\varepsilon \ \text{ only depends on } x_1, \quad \lambda_\varepsilon = l\mu_\varepsilon \ (l \geq 0)$$

$$\mu_\varepsilon, \quad \frac{1}{\mu_\varepsilon} \quad \text{bounded in } \ L^1(0,L).$$

Scalar case studied by Bouchitté and Picard 1996.

A priori estimates. Space of functions with bounded deformation

$\boldsymbol{u}_\varepsilon$, $\boldsymbol{e}(\boldsymbol{u}_\varepsilon)$ "only" bounded in $L^1(\Omega)$

$$\Rightarrow$$

$\boldsymbol{u}_\varepsilon$ weakly* relatively compact in

$$BD(\Omega) := \left\{\boldsymbol{\varphi} \in L^1(\Omega;\mathbb{R}^3) : \ \boldsymbol{E}\boldsymbol{\varphi} \in \mathcal{M}(\Omega;\mathbb{S}^3)\right\},$$
$$\boldsymbol{E}\boldsymbol{\varphi} := \frac{1}{2}\left(\boldsymbol{D}\boldsymbol{\varphi} + \boldsymbol{D}\boldsymbol{\varphi}^T\right), \quad \boldsymbol{D}\boldsymbol{\varphi} := \text{ distributional gradient of } \boldsymbol{\varphi}.$$

$\Rightarrow$ up to a subsequence,

$$\begin{array}{lll} \boldsymbol{u}_\varepsilon \to \boldsymbol{u} & & \text{strongly in } L^1(\Omega;\mathbb{R}^3), \\ \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \overset{\star}{\rightharpoonup} & \underbrace{\boldsymbol{E}\boldsymbol{u}}_{\text{vector measure}} & \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{S}^3), \end{array}$$

Question: what is the limit problem satisfied by *u*?

Weak* convergence of the coefficients and extra assumptions

$$\mu_\varepsilon, \quad \frac{1}{\mu_\varepsilon} \quad \text{bounded in } L^1(0,L)$$

$$\Rightarrow$$

$$\mu_\varepsilon \overset{\star}{\rightharpoonup} m, \qquad \frac{1}{\mu_\varepsilon} \overset{\star}{\rightharpoonup} \nu \quad \text{weakly* in } \mathcal{M}([0,L]).$$

Extra assumptions:

$$m(\{t\})\nu(\{t\}) = 0 \quad \forall t \in [0,L], \qquad \text{(no common atom)}$$
$$m(\{0\}) = m(\{L\}) = \nu(\{0\}) = \nu(\{L\}) = 0.$$

Limit problem

Under these assumptions, $\boldsymbol{u}$ unique solution to

$$\mathcal{P}^{lim}: \qquad \min_{\boldsymbol{\varphi} \in BD_0^{\nu,m}(\Omega)} \frac{1}{2} a(\varphi, \varphi) - \int_\Omega \boldsymbol{f} \cdot \varphi dx.$$

Limit space

$$BD_0^{\nu,m}(\Omega) := \left\{ \varphi \in BD(\Omega) \,\middle|\, \begin{array}{l} \boldsymbol{E}\varphi \ll \nu \otimes \mathcal{L}^2,\ \frac{\boldsymbol{E}\varphi}{\nu \otimes \mathcal{L}^2} \in L^2_{\nu \otimes \mathcal{L}^2}(\Omega; \mathbb{S}^3) \\ \varphi^\star_\alpha \in L^2_m(0, L; H^1_0(\Omega')) \quad \alpha \in \{2, 3\} \\ \varphi = 0 \text{ on } \partial\Omega \end{array} \right\},$$

$\frac{\boldsymbol{E}\varphi}{\nu \otimes \mathcal{L}^2}$: Radon-Nikodým density of $\boldsymbol{E}\varphi$ with respect to $\nu \otimes \mathcal{L}^2$.

$$\varphi^\star(x) = \begin{cases} \lim_{r \to 0} \fint_{B_r(x)} \varphi(y)\, dy & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Structure of the limit space

$BD_0^{\nu,m}(\Omega)$ Hilbert space

$$||\boldsymbol{\varphi}||_{BD_0^{\nu,m}(\Omega)} := \left(\int_\Omega |\frac{\boldsymbol{E\varphi}}{\nu\otimes\mathcal{L}^2}|^2 \, d\nu \otimes \mathcal{L}^2\right)^{\frac{1}{2}} + \left(\int_\Omega |\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)|^2 dm \otimes \mathcal{L}^2\right)^{\frac{1}{2}},$$

$$\boldsymbol{e}_{x'}(\boldsymbol{\varphi}) := \begin{pmatrix} \partial_1\varphi_1 & \frac{1}{2}(\partial_1\varphi_2 + \partial_2\varphi_1) & 0 \\ \frac{1}{2}(\partial_1\varphi_2 + \partial_2\varphi_1) & \partial_2\varphi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Limit energy

$$a(\psi,\varphi) := \int_\Omega \boldsymbol{a}^\perp \frac{\boldsymbol{E}\boldsymbol{\psi}}{\nu\otimes\mathcal{L}^2} : \frac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2} d\nu\otimes\mathcal{L}^2 + \int_\Omega \boldsymbol{a}^\| \boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star) : \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star) dm\otimes\mathcal{L}^2,$$

$$\boldsymbol{a}^\perp\boldsymbol{\Xi} := \begin{pmatrix} l\,\mathrm{tr}\,\boldsymbol{\Xi} + 2\Xi_{11} & 2\Xi_{12} & 2\Xi_{13} \\ 2\Xi_{12} & \frac{l^2}{l+2}\,\mathrm{tr}\,\boldsymbol{\Xi} + \frac{2l}{l+2}\Xi_{11} & 0 \\ 2\Xi_{13} & 0 & \frac{l^2}{l+2}\,\mathrm{tr}\,\boldsymbol{\Xi} + \frac{2l}{l+2}\Xi_{11} \end{pmatrix},$$

$$\boldsymbol{a}^\|\boldsymbol{\Gamma} := \frac{2l}{l+2}\sum_{\beta=2}^{3}\Gamma_{\beta\beta}\sum_{\alpha=2}^{3}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha + 2\sum_{\alpha,\beta=2}^{3}\Gamma_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta.$$

Main result

Theorem

$BD_0^{\nu,m}(\Omega)$ is a Hilbert space. The form $a(\cdot,\cdot)$ is coercive and continuous on $BD_0^{\nu,m}(\Omega)$. The solution $\boldsymbol{u}_\varepsilon$ to $\mathcal{P}_\varepsilon$ weakly converges in $BD(\Omega)$ to the unique solution $\boldsymbol{u}$ to $\mathcal{P}^{lim}$.*

Decomposition of $E\varphi$ for $\varphi \in BD(\Omega)$

$$\boldsymbol{E}\varphi = \underbrace{\boldsymbol{E}^a\varphi}_{\ll \mathcal{L}^3} + \underbrace{\boldsymbol{E}^c\varphi}_{\text{Cantor part } \perp \mathcal{L}^3} + \underbrace{\boldsymbol{E}^j\varphi}_{\text{jump part } \perp \mathcal{L}^3}$$

$\boldsymbol{E}^c\varphi$ is singular w.r.t. $\mathcal{L}^3$ and vanishes on σ-finite subsets w.r.t. $\mathcal{H}^2$,

$\boldsymbol{E}^a\varphi = \boldsymbol{e}(\varphi)\mathcal{L}^3$, $\quad \boldsymbol{e}(\varphi)$ approximative symmetric gradient of φ,

$\boldsymbol{E}^j\varphi = (\varphi^+_{J_\varphi} - \varphi^-_{J_\varphi}) \odot \boldsymbol{n}_{J_\varphi} \mathcal{H}^2_{\lfloor J_\varphi}$ $\quad J_\varphi$ "jump set" countably $\mathcal{H}^2$-rectifiable ,

$$\varphi^\pm_{J_\varphi}(x) = \begin{cases} \lim\limits_{r\to 0} ⨍_{B_r^\pm(x,\boldsymbol{n}_{J_\varphi})} \varphi(y)\,dy & \text{if this limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

$$B_r^\pm(x,\boldsymbol{n}_{J_\varphi}) := \{y \in B(x,r), \pm(\boldsymbol{y}-\boldsymbol{x})\cdot \boldsymbol{n}_{J_\varphi}(x) > 0\},$$

$$\boldsymbol{a} \odot \boldsymbol{b} := \frac{1}{2}(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a}).$$

(Ambrosio, Coscia, Dal Maso 1997)

Decomposition of ν and m

$$\nu = \nu^a + \nu^c + \nu^{at}, \qquad \nu^{at} = \sum_{t \in \mathcal{A}_\nu} \nu(\{t\})\delta_t, \qquad \nu^a = \frac{\nu}{\mathcal{L}^1}\mathcal{L}^1,$$

$$m = m^a + m^c + m^{at}, \quad m^{at} = \sum_{t \in \mathcal{A}_m} m(\{t\})\delta_t, \quad m^a = \frac{m}{\mathcal{L}^1}\mathcal{L}^1.$$

$$\mathcal{A}_\nu := \{t \in [0, L]; \nu(\{t\}) > 0\},$$
$$\mathcal{A}_m := \{t \in [0, L]; m(\{t\}) > 0\},$$
$$(\mathcal{A}_\nu \cap A_m = \emptyset).$$

Specific decomposition of $\boldsymbol{E}\varphi$ for $\varphi \in BD_0^{\nu,m}(\Omega)$

$$\boldsymbol{E}(\boldsymbol{\varphi}) \ll \nu \otimes \mathcal{L}^2 \Rightarrow \begin{array}{l} \boldsymbol{E}^a\boldsymbol{\varphi} \ll \nu^a \otimes \mathcal{L}^2 \\ \boldsymbol{E}^c\boldsymbol{\varphi} \ll \nu^c \otimes \mathcal{L}^2 \\ \boldsymbol{E}^j\boldsymbol{\varphi} \ll \mathcal{H}^2_{\lfloor \Sigma_\nu}. \end{array}$$

$$\Rightarrow$$

$$\boldsymbol{E}\boldsymbol{\varphi} = \boldsymbol{e}(\boldsymbol{\varphi})\mathcal{L}^3 + \tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu^c \otimes \mathcal{L}^2}\nu^c \otimes \mathcal{L}^2 + \sum_{t \in \mathcal{A}_\nu} (\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-) \odot \boldsymbol{e}_1 \mathcal{H}^2_{\lfloor \Sigma_t}$$

where

$$\Sigma_t := \{t\} \times \Omega', \quad \Sigma_\nu = \bigcup_{t \in \mathcal{A}_\nu} \Sigma_t.$$

Ensuing decomposition of the limit form $a(.,.)$

$$
\begin{aligned}
a(\psi,\varphi)=&\int_\Omega \boldsymbol{a}\boldsymbol{e}(\psi):\boldsymbol{e}(\varphi)dx+\sum_{t\in\mathcal{A}_\nu}\nu(\{t\})^{-1}\int_{\Sigma_t}(\psi^+-\psi^-)\cdot\boldsymbol{A}(\varphi^+-\varphi^-)d\mathcal{H}^2\\
&+\sum_{t\in\mathcal{A}_m} m(\{t\})\int_{\Sigma_t}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\psi^\star):\boldsymbol{e}_{x'}(\varphi^\star)d\mathcal{H}^2\\
&+\int_\Omega \boldsymbol{a}^{\perp}\frac{\boldsymbol{E}\psi}{\nu^c\otimes\mathcal{L}^2}:\frac{\boldsymbol{E}\varphi}{\nu^c\otimes\mathcal{L}^2}d\nu^c\otimes\mathcal{L}^2\\
&+\int_\Omega \boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\psi^\star):\boldsymbol{e}_{x'}(\varphi^\star)dm^c\otimes\mathcal{L}^2.
\end{aligned}
$$

where

$$
\boldsymbol{a}:=\left(\frac{\nu}{\mathcal{L}^1}\right)^{-1}\boldsymbol{a}^{\perp}+\frac{m}{\mathcal{L}^1}\boldsymbol{a}^{\|},\qquad \boldsymbol{A}:=\begin{pmatrix} l+2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.
$$

Comments

Emergence of jumps $\boldsymbol{u}^+ - \boldsymbol{u}^-$ at the interfaces $\Sigma_t = \{t\} \times \Omega'$ corresponding to atoms $\{t\}$ of ν, giving rise to the following concentrations of elastic energy

$$\frac{1}{2}\nu(\{t\})^{-1}\int_{\Sigma_t}(\boldsymbol{u}^+ - \boldsymbol{u}^-)\cdot\boldsymbol{A}(\boldsymbol{u}^+ - \boldsymbol{u}^-)d\mathcal{H}^2.$$

Comments (2)

Concentrations of elastic energy also appear on the planes Σ_t corresponding to atoms of m. These extra terms are similar to membrane stretching energy and take the form

$$\frac{1}{2} m(\{t\}) \int_{\Sigma_t} \boldsymbol{a}^{\|} \boldsymbol{e}_{x'}(\boldsymbol{u}^\star) : \boldsymbol{e}_{x'}(\boldsymbol{u}^\star) d\mathcal{H}^2.$$

Comments (3)

A concentration of elastic energy also emerges on a set of fractal Hausdorf dimension comprised between 2 and 3. It is given in terms of the Cantor parts ν^c and m^c of the measures ν and m by

$$\frac{1}{2}\int_\Omega \boldsymbol{a}^\perp \frac{\boldsymbol{Eu}}{\nu^c\otimes\mathcal{L}^2} : \frac{\boldsymbol{Eu}}{\nu^c\otimes\mathcal{L}^2} d\nu^c \otimes \mathcal{L}^2 + \frac{1}{2}\int_\Omega \boldsymbol{a}^\| \boldsymbol{e}_{x'}(\boldsymbol{u}^\star) : \boldsymbol{e}_{x'}(\boldsymbol{u}^\star) dm^c \otimes \mathcal{L}^2.$$

Comments (4)

The bulk effective energy takes the form of a classical linear elastic energy defined, in terms of its approximate symmetric gradient $\boldsymbol{e}(\boldsymbol{u})$ by

$$\frac{1}{2}\int_{\Omega} \boldsymbol{a}\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{u})dx.$$

The total elastic energy is the sum of the terms mentioned above.

Case when ν and m have vanishing Cantor parts and a finite number of atoms.

If $\nu^c = m^c = 0$ and $\mathcal{A}_\nu$, $\mathcal{A}_m$ are finite, $\mathcal{P}^{lim}$ is equivalent to

$$\begin{cases} -\,\mathbf{div}\boldsymbol{ae}(\boldsymbol{u}) = \boldsymbol{f} \;\text{ in }\; \Omega \setminus \Sigma, \quad \boldsymbol{u} \in BD_0^{\nu,m}(\Omega), \\ \nu(\{t\})^{-1}\boldsymbol{A}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{ae}(\boldsymbol{u}^-)\boldsymbol{e}_1 = \boldsymbol{ae}(\boldsymbol{u}^+)\boldsymbol{e}_1 & \text{on } \Sigma_t,\ \forall t \in \mathcal{A}_\nu, \\ \boldsymbol{ae}(\boldsymbol{u}^-)\boldsymbol{e}_1 - \boldsymbol{ae}(\boldsymbol{u}^+)\boldsymbol{e}_1 - m(\{t\})\mathbf{div}_{x'}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{u}^\star) = 0 & \text{on } \Sigma_t,\ \forall t \in \mathcal{A}_m, \end{cases}$$

where

$$\Sigma := \Sigma_\nu \cup \Sigma_m, \qquad \Sigma_\nu := \bigcup_{t\in\mathcal{A}_\nu} \Sigma_t, \quad \Sigma_m := \bigcup_{t\in\mathcal{A}_m} \Sigma_t.$$

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