Ground state energy of Robin Laplacians in corner domains

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Evolution equations in singular spaces
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Plan

An eigenvalue problem: the Robin Laplacian

Context:

- $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain.
- Eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{on} \quad \Omega, \\ \partial_n u - \beta u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- $\beta \in \mathbb{R}$ and ∂_n the outward derivative.

Associated operator:

$$\mathcal{Q}_\beta[\Omega](u) = \int_\Omega |\nabla u|^2 - \beta \int_{\partial\Omega} |u|^2 \mathrm{d}S \equiv \mathcal{Q}_\beta(u), \quad u \in H^1(\Omega).$$

- Compactness of $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$: $\mathcal{Q}_\beta[\Omega]$ is lower semibounded.
- $L_\beta[\Omega]$ the associated self-adjoint operator.
- $\lambda_j(\beta, \Omega) \equiv \lambda_j(\beta)$ the j-th eigenvalue of $L_\beta[\Omega]$.

Basic properties for Lipschitz domains:

- If $\beta > 0$, then $\lambda_1(\beta) < 0$.
- $\beta \mapsto \lambda_1(\beta)$ is concave, decreasing on $\mathbb{R}$.
- $\lambda_j(\beta) \to -\infty$ when $\beta \to +\infty$ (to be continued).

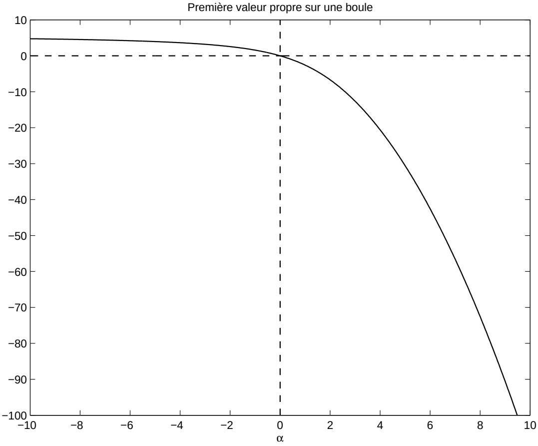


Figure: $\lambda_1(\beta)$ as a function of β on the unit disk.

Problematics and applications

Problematics: Behavior of $\lambda_j(\beta)$ when $\beta \to +\infty$.

- Influence of the geometry of Ω:
  Singularities of the boundary, curvature, symmetries.
  Localization of the eigenfunctions as $\beta \to +\infty$.
- To go further: reduction to the boundary, precise asymptotics, WKB expansion.

Some applications:

- Reaction-diffusion with competition between boundary source and absorption.
- Enhanced surface superconductivity with zero magnetic field:
  The critical temperature is linked to $\lambda_1(\beta)$.
  The associated eigenfunctions described the superconducting electrons.
- Trace inequality from the compacity of $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (Ehrling's Lemma):

$$\|u\|^2_{L^2(\partial\Omega)} \leq \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C(\epsilon)\|u\|^2_{L^2(\Omega)}, \quad C(\epsilon) > 0.$$

  Best constant : $C(\epsilon) = -\epsilon\lambda_1(\frac{1}{\epsilon}, \Omega)$.

Semi-classical reformulation and homogeneity:

- Set $h = \beta^{-2}$, so that $h^2 L_\beta[\Omega]$ writes

$$-h^2\Delta \quad \text{with boundary condition} \quad h^{1/2}\partial_n u - u = 0$$

- Dilation invariant domain Π (unbounded) are important:

$$\text{Scaling } X = \beta x : \quad L_\beta[\Pi] \equiv \beta^2 L[\Pi] \quad \text{with} \quad L[\Pi] = L_1[\Pi].$$

Define the ground state energy of $L[\Pi]$:

$$E(\Pi) = \inf_{u \in H^1(\Pi)} \frac{\|\nabla u\|^2_{L^2(\Pi)} - \|u\|^2_{L^2(\partial\Pi)}}{\|u\|^2_{L^2(\Pi)}}. \text{ Then, either } \begin{cases} E(\Pi) = -\infty \\ \text{or } E(\Pi) \text{ is an eigenvalue} \\ \text{or } E(\Pi) \text{ is essential spectrum} \end{cases}$$

(Elementary) model case

The regular case:

- Separation of variables on a half-space $\mathbb{R}^+_n := \mathbb{R}^{n-1} \times \mathbb{R}^+$:

$$L[\mathbb{R}^+_n] = \mathrm{Id} \otimes L[\mathbb{R}^+] + (-\Delta_{\mathbb{R}^{n-1}}) \otimes \mathrm{Id}.$$

- The case of $\mathbb{R}^+$ is explicit: solve

$$\begin{cases} -u''(x) = Eu(x), & x > 0 \\ -u'(0) - u(0) = 0 \end{cases}$$

  Only one eigenpair: $E = -1$ and $u(x) = Ce^{-x}$. The spectrum is $\{-1\} \cup \mathbb{R}^+$.
- Therefore $E(\Pi) = -1$ if Π is a half-space. It is essential spectrum for $L[\Pi]$.

2D infinite sectors:

- Let S_θ be an infinite sector with opening angle $\theta \in (0, 2\pi)$:

$$\sigma(L[S_\theta]) = \begin{cases} -\sin^{-2}(\frac{\theta}{2}) \cup \{\mu_k,\ k \leq 2 \leq N\} \cup [-1, +\infty) & \text{if } \theta < \pi \\ [-1, +\infty) & \text{if } \theta \geq \pi \end{cases}$$

- $E(S_\theta)$ is a discrete eigenvalue iff $\theta < \pi$.

Known results on the asymptotics

Particular cases:

- Let Ω be a C^1 bounded domain, then

$$\forall j \geq 1, \quad \lambda_j(\beta, \Omega) = -\beta^2 + o(\beta^2).$$

- Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with n vertices of angle $(\theta_k)_{1 \leq k \leq n}$.

$$\lambda_1(\beta, \Omega) = -\max_{\theta_k \in (0,\pi)} \left(1, \sin^{-2} \tfrac{\theta_k}{2}\right)\right)\beta^2 + o(\beta^2).$$

General cases, from a short article of Levitin and Parnovski (08'):

- Assume that $\Omega \subset \mathbb{R}^n$ is bounded, has corners, and for all $x \in \overline{\Omega}$, denotes by $\Pi_x \subset \mathbb{R}^n$ the tangent geometry (a cone).

Theorem [Levitin and Parnovski 08]

Assume that Ω satifies the uniform interior cone condition. Denote by $E(\Pi_x)$ the bottom of the spectrum of $L[\Pi_x]$. Then, as $\beta \to +\infty$:

$$\lambda_1(\beta, \Omega) = \inf_{x \in \partial\Omega} E(\Pi_x)\beta^2 + o(\beta^2).$$

Some questions

Definition: Infimum of the local energies

We define $\mathscr{E}(\Omega) = \inf_{x\in\partial\Omega} E(\Pi_x)$.

On the asymptotics:

- $\boxed{\text{Is } \mathscr{E}(\Omega) \text{ finite?}}$ Is it a minimum?
- Can you estimate $\lambda(\beta,\Omega) - \beta^2\mathscr{E}(\Omega)$ as $\beta \to +\infty$?
- Can you construct "good" quasi-modes?

Analysis in corner domains:

- Asymptotics very close to the first eigenvalue of the semiclassical magnetic Laplacian in corner domain (Bonnaillie-Dauge-Popoff 16).
- The local energy $\overline{\Omega} \ni x \mapsto E(\Pi_x)$ is clearly discontinuous. Can you say something?
- Can you describe the essential spectrum of $L[\Pi]$?
- Can you go further in the asymptotics, with the minimum assumptions on the geometry?

Plan

The corner domains

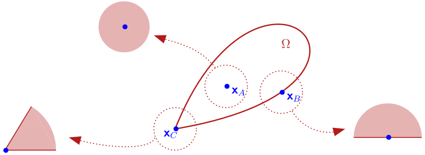
Definition of corner domains $\mathfrak{D}(M)$ with $M=\mathbb{R}^n$ of $M=\mathbb{S}^n$ and admissible cones $\mathfrak{P}_n$ in $\mathbb{R}^n$:

Initialisation:

- $\mathfrak{P}_0=\{0\}$ and $\mathfrak{D}(\mathbb{S}^0)=\{-1,1,\{-1,1\}\}$.

Recursion:

- An open set Ω is in $\mathfrak{D}(M)$ iif for all $x\in\overline{\Omega}$, there exists a tangent cone $\Pi_x\subset\mathbb{R}^n$ to Ω at x with $\Pi_x\in\mathfrak{P}_n$:



- $\Pi_x\in\mathfrak{P}_n$ iif the section $\Pi_x\cap\mathbb{S}^{n-1}$ belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$.

Dimension 2 : polygonal domains with finite number of vertices (opening angles $\neq 0,\pi,2\pi$).

The tangent cones in dimension 3

With each point $x \in \overline{\Omega}$ is associated its tangent cone Π_x whose section by $\mathbb{S}^2$ is a curvilinear polygon.

Situation of $x \in \overline{\Omega}$	Model geometry Π_x
Interior point	Space $\mathbb{R}^3$
Regular boundary	Half-space $\mathbb{R}^3_+$
Edge	Infinite wedge $S_\theta \times \mathbb{R}$
Corner	3d cone $\mathfrak{C}$

- Ω polyhedral: all the tangent cone are straight (no curvature).
- In general corner domains: the regular boundary of a tangent cone has non-zero curvature. It blows up at the origin! Example: circular cone.

Examples

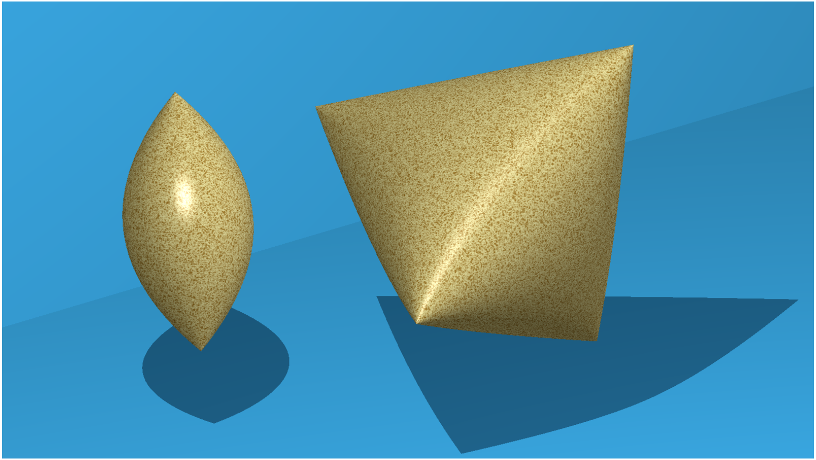


Figure: Domains with conical points

Examples

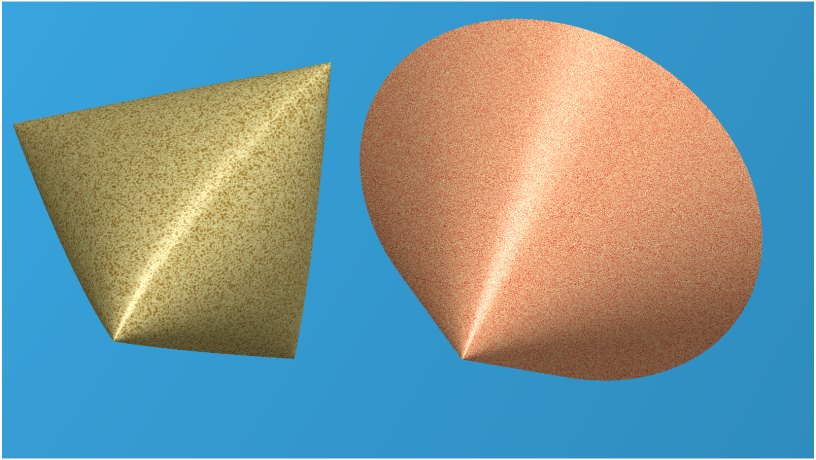


Figure: Tangent cone at a conical point

Examples

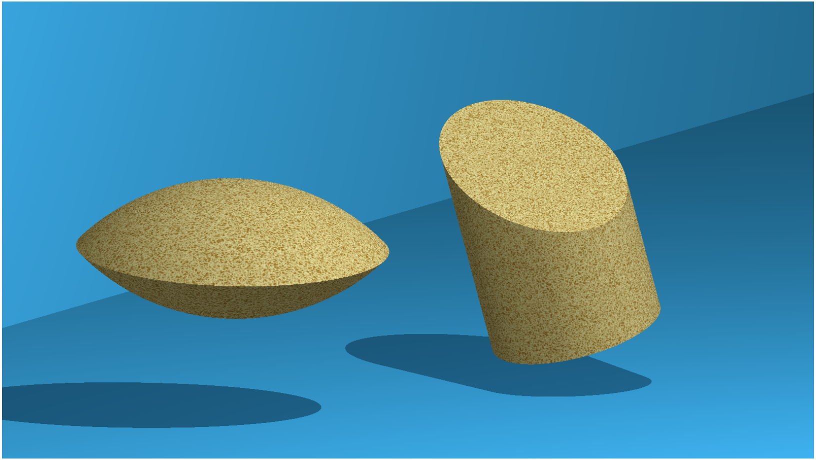


Figure: Domains with edges

Examples

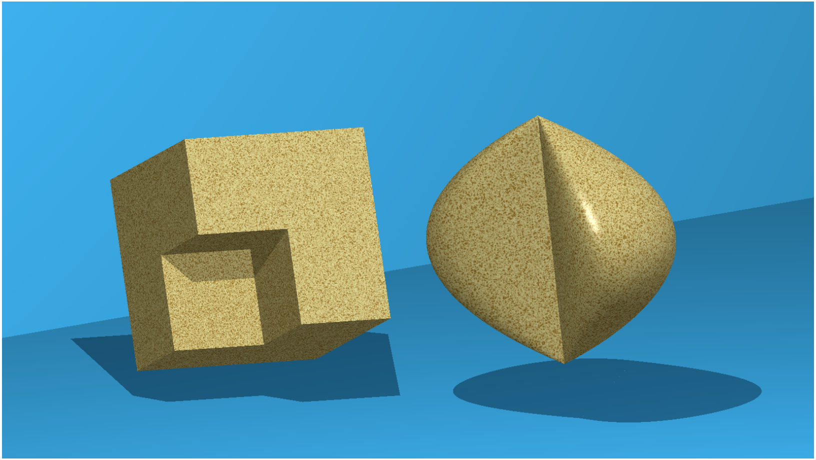


Figure: Domains with corners and edges

Examples

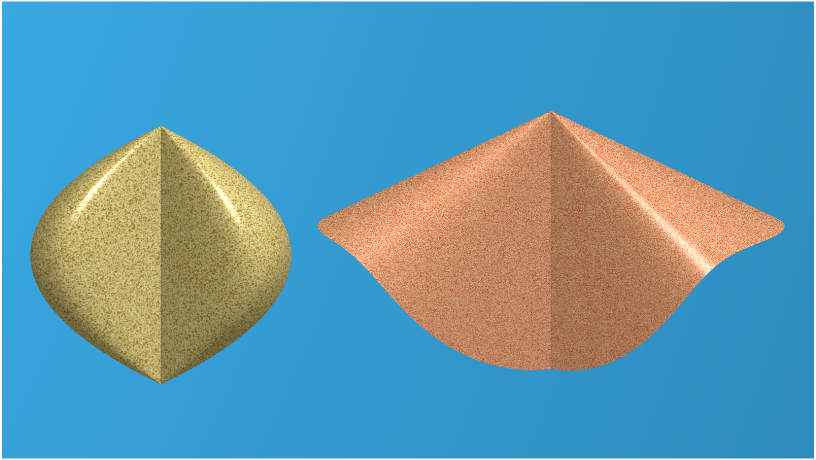


Figure: Tangent cone at the top of the seed

Structure of corner domains

Reduced cone:

- Up to a rotation, $\Pi \in \mathfrak{P}_n$ writes

$$\Pi = \mathbb{R}^{n-d} \times \Gamma \quad \text{with} \quad \Gamma \in \mathfrak{P}_d$$

- If d is maximal, then Γ is the reduced cone and d is the reduced dimension of Π.
- Exemple: for half spaces, $d = 1$. For wedges, $d = 2$.

Strata:

For a corner domain $\Omega \in \mathfrak{D}(M)$ and $0 \leq k \leq n$, we define

$$\mathcal{A}_k(\Omega) := \{x \in \overline{\Omega}, \text{ the reduced dimension of } d(\Pi_x) \text{ is } k\}.$$

The strata are the connected componant of $\mathcal{A}_k$.

Proposition (structure of a stratum)

Each stratum of $\mathcal{A}_k$ is a submanifold of codimension k.

Said differently: a corner domain admits a Whitney stratification.

Singular chains $\mathbb{X}$

The idea of singular chains is inspired by

V. G. Maz'ya, B. A. Plamenevskii.
Elliptic boundary value problems on manifolds with singularities
Probl. Mat. Anal. **6** (1977) 85–142.

and extensively used in

M. Dauge. *Elliptic boundary value problems on corner domains*
Lecture Notes in Mathematics **1341** Springer-Verlag (1988)

Singular chains $\mathbb{X}$. Typical singular chain starting from a vertex:

$$\mathbb{X} = \begin{cases} (x_0) & x_0 \in \overline{\Omega},\ \Gamma_{x_0} \text{ the reduced tangent cone} \\ (x_0, x_1) & x_1 \in \overline{\Omega}_0 \quad \text{where} \quad \Omega_0 = \Gamma_{x_0} \cap \mathbb{S}^{d_0-1} \\ (x_0, x_1, x_2) & x_2 \in \overline{\Omega}_1 \quad \text{where} \quad \Omega_1 = \Gamma_{x_0,x_1} \cap \mathbb{S}^{d_1-1} \end{cases}$$

Denote by $\mathfrak{C}(\Omega)$ the set of singular chains in Ω.

Singular chains $\mathbb{X}$

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ЭЛЛИПТИЧЕСКИЕ КРАЕВЫЕ ЗАДАЧИ НА МНОГООБРАЗИЯХ С ОСОБЕННОСТЯМИ

В этой работе изучаются краевые задачи для эллиптических систем уравнений в частных производных на многообразиях с особенностями довольно общей природы. В качестве особенностей допускаются "ребра" различных размерностей и их всевозможные пересечения (под "ненулевыми углами"). Такие же множества рассматриваются как носители разрывов коэффициентов.

Краевые задачи в областях с изолированными особыми точками на границе достаточно хорошо изучены (см. [1–5] и приведенную там литературу). В работах [6–20] с различных точек зрения изучались краевые задачи с особенност ми коэффициентов или границы, сосредоточенными на гладком многообразии. Задачи типа задач С.Л.Соболева в случае подмногообразий с многомерными особенностями рассматривались в [21].

Основной результат работы – теорема о нетеровости (об однозначной разрешимости) для краевой задачи (задачи с параметром). Задачи рассматриваются в некоторых гильбертовых функциональных пространствах с весовыми нормами. Грубо говоря, вес представляет собой произведение степеней расстояний до особых подмногообразий границы. Указана процедура построения такого пространства, в котором краевая задача нетерова (однозначно разрешима при наличии параметра). Эта процедура сводится к последовательной проверке тривиальности ядра и коядра любой "модельной" задачи, возникающей в результате "замораживания" коэффициентов в точке особого подмногообразия.

По-видимому, неизвестно, можно ли в общей ситуации найти содержательные условия тривиальности ядра и коядра таких задач. В частных случаях (для сильно эллиптических задач [14, 22] в случае общей краевой задачи для эллиптического уравнения второго порядка [17,18], для задачи с "косой" производной [13,16]) такие условия найдены.

Краевые задачи рассматриваются на n-мерных многообразиях класса $\mathcal{D}^l$, введенного авторами в [24]. Определения и основные свойства класса $\mathcal{D}^l$ приводятся в § 1. Там же даны примеры многообразий из $\mathcal{D}^l$. В § 2 вводится некоторый класс дифференциальных операторов на указанных многообразиях. Функциональным пространствам посвящен § 3. В § 4 содержатся постановки исходной и модельных краевых задач. Свойства последних изучаются в § 5. Основные результаты сформулированы и доказаны в § 6. В заключение рассмотрена в качестве примера задача Дирихле.

Настоящ я статья является подробным изложением результатов, анонсированных в [23].

§ 1. Многообразия классов $\mathcal{D}^l_n$

Для полноты изложения напомним определения и основ ые свойства многообразий класса $\mathcal{D}^l_n$, введенных в [24].

1°. Отображения класса $\mathcal{D}^l$. Пусть $\mathcal{R}^d$ – d-мерное эвклидово пространство, $\mathcal{R}^1_+=\{\rho\in\mathcal{R}^1,\ \rho\geq 0\}$. Будем рассматривать функции χ, определенные в прямом произведении $\mathcal{P}=\mathcal{R}^{d_0}\times\mathcal{R}^1_+\times\ldots\times\mathcal{R}^{d_q}\times\mathcal{R}^1_+$, $d_j\geq 0$, $j=1,\ldots,q$. Точки множества $\mathcal{P}$ обозначим через $(\zeta_0,\rho_0,\ldots,\zeta_q,\rho_q)$, $\zeta_j\in\mathcal{R}^{d_j}$, $\rho_j\geq 0$.

Введем обозначения: μ_j – мультииндекс размерности d_j, ν_j – неотрицательное целое число, $M_k=(\mu_0,\ldots,\mu_k)$, $N_k=(\nu_0,\ldots,\nu_k)$, $M=M_q$, $N=N_q$; $S(M,N)=(|M_0|+|N_0|,\ldots,|M_{q-1}|+|N_{q-1}|,0)$, $\zeta_j=(\zeta_{1,j},\ldots,\zeta_{d_j,j})$,

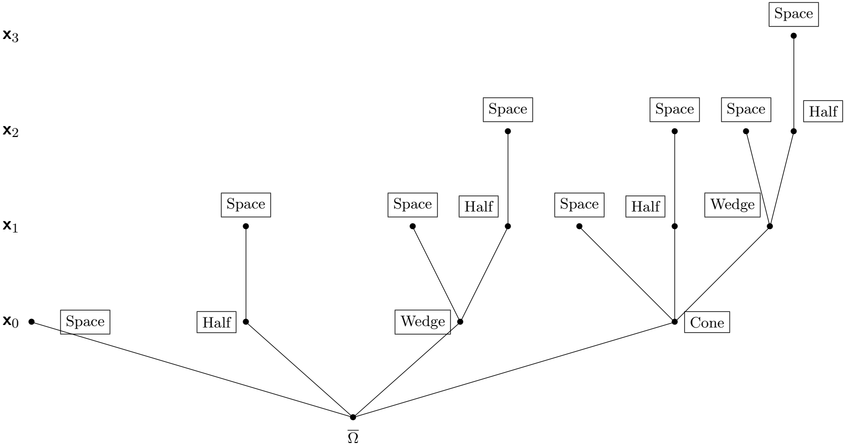
$$\zeta_{k,j}\in\mathcal{R}^1;\quad D_{\zeta_j}=\left\{\frac{1}{i}\frac{\partial}{\partial\zeta_{1,j}},\ldots,\frac{1}{i}\frac{\partial}{\partial\zeta_{d_j,j}}\right\},\quad D_{\rho_j}=\frac{1}{i}\frac{\partial}{\partial\rho_j};$$

$$Z_k=(\zeta_0,\ldots,\zeta_k),\quad R_k=(\rho_0,\ldots,\rho_k),\quad Z=Z_q,$$

$$R=R_q,\quad D_Z=\{D_{\zeta_0},\ldots,D_{\zeta_q}\},\quad D_R=\{D_{\rho_0},\ldots,D_{\rho_q}\}.$$

Будем говорить, что функция χ принадлежит классу $\mathcal{D}^l$, $l\geq 1$, если функции $R^{S(M,N)}D_Z^M D_R^N\chi$, где $|M|+|N|\leq l$ непрерывны на $\mathcal{P}$. (В подробной записи эти функции имеют вид $\rho_0^{|\mu_0|+\nu_0}\rho_1^{|\mu_0|+\nu_0+|\mu_1|+\nu_1}\cdots\rho_{q-1}^{|\mu_0|+\nu_0+\ldots+|\mu_{q-1}|+\nu_{q-1}}D_{\zeta_0}^{\mu_0}D_{\rho_0}^{\nu_0}\cdots D_{\zeta_q}^{\mu_q}D_{\rho_q}^{\nu_q}\chi$.)

Пусть ω_1, ω_2 – открытые подмножества множества $\mathcal{P}$. По



The tree of singular chains (Half is for half-space)

- With a singular chain $\mathbb{X}$ is associed a tangent structure $\Pi_{\mathbb{X}}$.

Examples: Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$, and $x_0 \in \mathfrak{A}_3(\Omega)$:

- If x_1 is a vertex of $\overline{\Pi_{x_0}} \cap \mathbb{S}^2$, then $\Pi_{(x_0,x_1)}$ is a wedge.
- If x_1 is in the regular boundary of $\overline{\Pi_{x_0}} \cap \mathbb{S}^2$, then $\Pi_{(x_0,x_1)}$ is a half-space.
- If x_1 is in $\Pi_{x_0} \cap \mathbb{S}^2$, then $\Pi_{(x_0,x_1)} = \mathbb{R}^3$.

Structure of the singular chains

- Distance between $\mathbb{X} = (x_0, \cdots)$ and $\mathbb{X}' = (x'_0, \cdots)$:

$$\mathbb{D}(\mathbb{X}, \mathbb{X}') = \|x_0 - x'_0\| + \inf_{J\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}} \|J - Id\| + \inf_{J\Pi_{\mathbb{X}'} = \Pi_{\mathbb{X}}} \|J - Id\|$$

- The infimum is taken over $J \in \mathrm{GL}_n(\mathbb{R}^3)$ and $\|J\| \leq 1$.
- Define a natural partial order on singular chains:

$$\mathbb{X} \leq \mathbb{X}' \text{ if } \mathbb{X}' = (\mathbb{X}, x'_{p+1}, \ldots)$$

Theorem [Bonnaillie-Dauge-P. 16]

Let $F : \mathfrak{C}(\Omega) \mapsto \mathbb{R}$, continuous with respect to $\mathbb{D}$ and order preserving. Then $x \mapsto F((x))$ is lower semi-continuous on $\overline{\Omega}$.

- Application to the local ground energy

$$F : \mathbb{X} = (x_0, \cdots) \longmapsto E(\Pi_{\mathbb{X}})$$

Plan

Essential spectrum

For a m.w.c., from [Melrose GST]:

Conjecture 7.1 *For an exact b-metric*[37] *the spectrum of the Laplacian consists of possibly countable point spectrum of finite multiplicity, corresponding to L^2 eigenfunctions together with continuous spectrum consisting of an at most countable union of rays $[\tau, \infty)$ where the thresholds, τ, form the discrete set consisting of the union of the L^2 eigenvalues for the induced Laplacians on all boundary faces (of positive codimension).*[38]

Notice that Euclidean spaces are th

For an admissible cones $\Pi \in \mathfrak{P}_n$

- Note that $L[\mathbb{R}^\nu \times \Pi] = -\Delta_{|\mathbb{R}^\nu} \otimes \mathrm{Id} + \mathrm{Id} \otimes L[\Pi]$. Therefore $E(\mathbb{R}^\nu \times \Pi) = E(\Pi)$, and we focus on irreductible cones.
- Following [BN-D-P]: we parametrize the essential spectrum of $L[\Pi]$ by the singular chains of $\omega := \Pi \cap \mathbb{S}^{n-1}$:

$$\{\mathbb{X} \in \mathfrak{C}_0(\Pi), \Pi_{\mathbb{X}} \neq \Pi\} = \{\mathbb{X} = (0, x_1, \dots), x_1 \in \overline{\omega}\} \simeq \mathfrak{C}(\omega)$$

Second energy level and monotonicity

Definition: Ground energy along higher singular chains

We define

$$\mathscr{E}^*(\Pi) := \inf_{\Pi_{\mathbb{X}} \neq \Pi} E(\Pi_{\mathbb{X}})$$

where the infimum is taken over higher singular chains: $l(\mathbb{X}) \geq 2$.

Easy to see that $\mathscr{E}^*(\Pi) = \mathscr{E}(\omega)$ where ω is the section of the reduced cone.

Theorem [Bruneau P. 16]

Let $\Pi \in \mathfrak{P}_n$ be an irreductible cone and $\omega := \Pi \cap \mathbb{S}^{n-1}$ its section. Then $L(\Pi)$ is lower semi-bounded and the bottom of its essential spectrum is $\mathscr{E}^*(\Pi)$.

Therefore: $E(\Pi) \leq \mathscr{E}^*(\Pi)$, and

$$\underline{\mathbb{X} \mapsto E(\Pi_{\mathbb{X}}) \text{ is order preserving from } \mathfrak{C}(\Omega) \text{ into } \mathbb{R}}$$

Regularity of the local energy

Continuity:

- The local energy $\mathfrak{C}(\Omega) \ni \mathbb{X} \mapsto E(\Pi_{\mathbb{X}})$ is continuous for the distance $\mathbb{D}$ of the singular chains.
- In particular $x \mapsto E(\Pi_x)$ is continuous on each stratum of $\overline{\Omega}$.

The local energy is continuous and order preserving on singular chains:

Theorem [Bruneau P. 16]

The local energy $x \mapsto E(\Pi_x)$ is lower semicontinuous on $\overline{\Omega}$.Therefore

$$\mathscr{E}(\Omega) = \inf_{x \in \overline{\Omega}} E(\Pi_x) > -\infty.$$

The proof of these is done by a recursion over the dimension n.

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Theorem [Bruneau P. 16']

Let $\Omega \in \mathfrak{D}(M)$ with $n \geq 2$ the dimension of M. Then there exists $\beta_0 \in \mathbb{R}$, two constants $C^{\pm} > 0$ and two integers $0 \leq \nu \leq \nu_+ \leq n-2$ such that

$$\forall \beta \geq \beta_0, \quad -C^- \beta^{\gamma(\nu_+)} \leq \lambda(\Omega, \beta) - \beta^2 \mathscr{E}(\partial\Omega) \leq C^+ \beta^{\gamma(\nu)}, \quad \gamma(\nu) = 2 - \frac{2}{2\nu+3}.$$

The exponant:

- The exponent ν^+ is the longest chain $\mathbb{X} \in \mathfrak{C}(\Omega)$ such that $\Pi_{\mathbb{X}}$ is not polyhedral. Similar for ν, but the chains originates at a minimizer of $x \mapsto E(\Pi_x)$. Moreover $\frac{4}{3} = \gamma(0) \leq \gamma(\nu) < 2$, and $\nu^+ = 0$ iff the domain is polyhedral.

Ideas of the proof

- Lower bound: Multiscale analysis on a suitable partition of the unity.
- Upper bound: Recursive quasi-mode from a singular chain minimizing $E(\Pi_{\mathbb{X}})$.

Difficulty for non polyhedral domains:

- The curvature is unbounded.
- In dimension 3, the "worst" blow-up curvature may be

$$\frac{1}{d(x_0, \mathfrak{A}_3(\Omega))}.$$

Let $\Omega \in \mathfrak{D}(M)$ and let ν_+ be the smallest integer satisfying

$$\forall \mathbb{X} \in \mathfrak{C}(\Omega), \quad I(\mathbb{X}) \geq \nu_+ \implies \Pi_{\mathbb{X}} \text{ is polyhedral.}$$

Given scales $(\delta_k)_{0\leq k\leq \nu_+}$ in $(0,+\infty)$, we construct an β-dependand finite set of points $\mathcal{C} \subset \overline{\Omega}$ such that for all $p \in \mathcal{C}$, there exists $0 \leq k \leq \nu_+$ with

- the ball $B(p, 2\beta^{-(\delta_0+\ldots+\delta_k)})$ is contained in a map-neighboorhood of p.
- the associated curvature satisfies $\kappa(p) \leq c(\Omega)\beta^{\delta_0+\ldots+\delta_{k-1}}$.
- $\overline{\Omega} \subset \cup_{p\in\mathcal{C}} B(p, \beta^{-(\delta_0+\ldots+\delta_k)})$.

Let $(\chi_p)_{p\in\mathcal{C}}$ be an associated partition of the unity:

$$\begin{aligned} Q_\beta[\Omega](u) &= \sum_{p\in\mathcal{C}} Q_\beta[\Omega](\chi_p u) - \sum_{p\in C} \|\nabla \chi_p u\|^2, \quad \forall u \in H^1(\Omega) \\ &= \sum_{p\in\mathcal{C}} Q_\beta[\Pi_p, G_p](\chi_p u) + O(\beta^{2\delta})\|u\|^2, \quad \delta = \sum_{k=0}^{\nu_+} \delta_k \\ &\geq \sum_{p\in\mathcal{C}} (1 + C\beta^{-(\delta_0+\ldots+\delta_k)}\kappa(c))\beta^2 E(\Pi_p)\|\chi_p u\|^2 + O(\beta^{2\delta})\|u\|^2 \\ &\geq \left(\beta^2\mathscr{E}(\Omega) + \sum_{k=0}^{\overline{\nu}_+} O(\beta^{2-\delta_k}) + O(\beta^{2\delta})\right)\|u\|^2 \end{aligned}$$

Plan

Reduction to the boundary for regular domains

Theorem [Pankrashkin-P.16]

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ domain with compact boundary, and H the mean curvature of the boundary. Let $-\Delta^S$ be the Laplace-Beltrami operator on $\partial\Omega$ and $\mu_j(\beta)$ the j-th eigenvalue of the operator $-\Delta^S - \beta(n-1)H$. Then for all $j \geq 1$:

$$\lambda_j(\beta) \underset{\beta\to+\infty}{=} -\beta^2 + \mu_j(\beta) + O(\log\beta).$$

Moreover, if the boundary is C^3, the remainder is improved to $O(1)$.

Comments:

- You are led to a semi-classical Schrödinger operator on the boundary.
- Asymptotic expansion when the mean curvature has wells.
- In dimension 2, precise results from Helffer-Kachmar (16').

Why a reduction on the boundary?

- The local energy can be seen as an effective potential in the harmonic approximation. For regular domains, it is piecewise constant, minimum on the boundary.
- One may think to [Helffer-Sjöstrand VI]: Cas des puits sous-variété (87').

Perspective and open questions

Reduction to boundary for corner domains?

- In dimension 2, the limit object could be a graph with model problems at vertices.
- The scattering of the model problems at vertices gives b.c. for the problem on the sides.
- What happens in higher dimension?

The method and the results

- Study the regularity of the local energy near its minimizers.
  Can you give Agmon estimates for the eigenfunctions using the distance to the minimizers?
- Find a class of operator for which the analysis remains true.

Same results for the δ-interaction with large coupling constant:

$$-\Delta - \beta\delta_H \text{ on } M, \quad H \subset M \text{ an hypersurface with corners.}$$

The quadratic form is $u \mapsto \int_M |\nabla u|^2 \mathrm{d}x - \beta \int_H |u|^2 \mathrm{d}S$

The magnetic Laplacian

Semiclassical Magnetic Laplacian with Neumann magnetic b.c. :

$$H_h[\Omega, \mathbf{A}] := (-ih\nabla - \mathbf{A})^2 \quad \text{on } \Omega \quad \text{with} \quad h > 0$$

For Ω simply connected, the eigenvalues depends on $\mathbf{A}$ through the magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$. Tangent operator ar $x \in \overline{\Omega}$:

$$H_1[\Pi_x, \mathbf{A}_x], \quad \text{with } \mathbf{A}_x \text{ the linear part of } \mathbf{A} \text{ at } x.$$

Denote by $E(\Pi_x, \mathbf{B}_x)$ the corresponding local energy ($\mathbf{B}_x$ is constant) and $\mathscr{E}(\Omega, \mathbf{B})$ their infimum.

Theorem [Bonnaillie-Dauge-P. 16]

Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{B}$ a regular magnetic field. The local energy $x \mapsto E(\Pi_x, \mathbf{B}_x)$ is lower semicontinuous and therefore $\mathscr{E}(\Omega, \mathbf{B}) > 0$. Moreover there exists $h_0 > 0$

$$\forall h \in (0, h_0), \quad |\lambda_h(\Omega, \mathbf{B}) - h\mathscr{E}(\Omega, \mathbf{B})| \leq C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{\kappa},$$

- Ω polyhedral: $\kappa = 5/4$,
- Ω general: $\kappa = 11/10$.

Specificities of the magnetic Laplacian

Comment on the dimension

- We prove a lower bound valid in any dimension
- No easy decomposition for tensor products. Possible partial Fourier transform, depending on the dimension.

The exemple of a wedge $W_\beta := S_\beta \times \mathbb{R}$:

- $\mathbf{B} = (b_1, b_2, b_3)$ and $\mathbf{A}(x) = (0, b_3 x_1, b_1 x_2 - b_2 x_1)$.
- Model operator: $H(W_\beta, \mathbf{A}) = \mathrm{D}_1^2 + (\mathrm{D}_2 + b_3 x_1)^2 + (\mathrm{D}_3 + b_1 x_2 - b_2 x_1)^2$.
- $\mathcal{F}_3$: Partial Fourier transform w.r.t. x_3,

$$\mathcal{F}_3 H(W_\beta, \mathbf{A}) \mathcal{F}_3^* = \int_{k\in\mathbb{R}}^{\oplus} \widehat{H}_k(S_\beta, \mathbf{A}) dk\,,$$

$$\widehat{H}_k(S_\beta, \mathbf{A}) = \mathrm{D}_1^2 + (\mathrm{D}_2 + b_3 x_1)^2 + (k + b_1 x_2 - b_2 x_1)^2 \text{ sur } S_\beta.$$

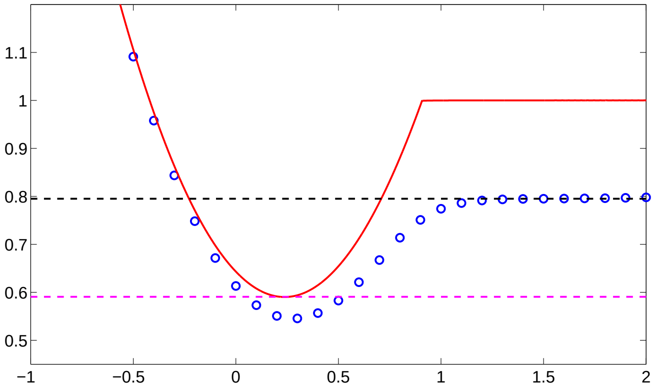
Minimization of a band function

Let $s(k)$ be the ground energy $\widehat{H}_k(S_\beta, \mathbf{A})$. Then we have

$$E(W_\beta, \mathbf{B}) = \inf_{k\in\mathbb{R}} s(k)\,.$$

Numerical simulations

Band function
for a magnetic field tangent to a face and close to the edge



$k \mapsto s(k)$, $k \mapsto s_{\text{ess}}(k)$ and $\mathscr{E}^*(W_\beta, \mathbf{B})$.

Associated eigenfunctions