Holomorphic extension of the Poisson Kernel

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Outline

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Poisson kernel

Let Ω be an open subset of $\mathbb{R}^d$ with **analytic** boundary, and $y \in \partial\Omega$.

The Poisson kernel $K_y(x)$ is solution of the elliptic boundary value problem

$$\triangle K_y(x) = 0 \quad \text{in } \Omega; \quad K_y|_{\partial\Omega} = \delta_{x=y} \ . \tag{1.1}$$

When Ω is bounded, K_y is uniquely defined. When Ω is arbitrary, K_y is uniquely defined near y in Ω modulo an analytic function defined in a neighborhood of y in $\mathbb{R}^d$.

Moreover, for any $z \in \partial\Omega \setminus \{y\}$, $K_y(x)$ extends as an holomorphic function of x in a neighborhood of z in $\mathbb{R}^d$. In particular

$$\partial_n K_y \in C^\omega(\partial\Omega \setminus \{y\}) \tag{1.2}$$

Problem

Describe the "maximal" holomorphic extension in z of $K_y(z)$ in a complex neighborhood of $y \in \mathbb{C}^d$.

Outline

Balls and half spaces

In case $\Omega = \{\|x\| < 1\}$ is the unit ball in $\mathbb{R}^d$, one has

$$K_y(x) = c\,\frac{1 - \|x\|^2}{\|x - y\|^d} \tag{2.1}$$

In case $\Omega = \{x_d > 0\}$ is a half space in $\mathbb{R}^d$, one has

$$K_y(x) = c\,\frac{x_d}{\|x - y\|^d} \tag{2.2}$$

We will see later on that this two cases are the **only** cases where an "explicit" simple formula may holds...

Green function

Let g be an analytic Riemannian metric defined near $\overline{\Omega}$. For $y \in \Omega$, let $G_y(x)$ be the Green function

$$\triangle_g G_y(x) = \delta_{x=y} \quad \text{in } \Omega; \quad K_y|_{\partial\Omega} = 0 \ . \tag{2.3}$$

Then G_y is uniquely determined up to an analytic function of x defined near y. Let $d^2(x,y)$ be the square of the Riemannian distance. Then $d^2(x,y)$ is holomorphic in x in a neighborhood of y, and the following theorem is essentially due to J. Hadamard.

Theorem

G_y extends holomorphically near y on the covering of the complement of the complex cone $\{z, d^2(z,y) = 0\}$.

Outline

Let g be an analytic Riemannian metric defined near $0 \in \mathbb{R}^d$. We may assume $g = Id + O(x^2)$. We denote by $K(x)$ the associated Poisson kernel, i.e the solution, unique up to an analytic function defined near 0 in $\mathbb{R}^d$, of the elliptic boundary value problem

$$\triangle_g K(x) = 0 \quad \text{in } \{x_d > 0\}; \quad K|_{x_d=0} = \delta_{x=0} \ . \tag{3.1}$$

Let $u = (det\ g)^{1/4} K$. Then u satisfies the elliptic boundary value problem

$$Pu = 0 \quad \text{in } \{x_d > 0\}; \quad u|_{x_d=0} = \delta_{x=0} \tag{3.2}$$

Where

$$P = \partial^2_{x_d} + R(x_d, x', \partial_{x'})$$

is a second order elliptic differential operator with principal symbol

$$p(z,\zeta) = \zeta_d^2 + r(z_d, z', \zeta') \tag{3.3}$$

which is equal to the metric induced by g on the cotangent space.

Let $B = B_\varepsilon = \{z = (z_1, ..., z_d) \in \mathbb{C}^d,\ \sum |z_j|^2 < \varepsilon^2\}$
and $B_0 = B_{\epsilon,0} = B_\epsilon \cap \{z_d = 0\}$.

Definition

Let $F(= F_\varepsilon)$ be the smallest closed subset of $K = \left((T^*(B_\varepsilon) \setminus B_\varepsilon) \cap \{p^{-1}(0)\}\right)/\mathbb{C}^*$, such that

$$\begin{aligned} &a)(z,\zeta) \in F \Rightarrow exp(sH_p)(z,\zeta) \in F \quad \text{for } s \in \mathbb{C} \text{ small} .\\ &b)(z',0;\zeta',\zeta_d) \in F \Rightarrow (z',0;\zeta',-\zeta_d) \in F .\\ &c)\{(0;\zeta),\ \zeta \neq 0,\ p(0;\zeta) = 0\} \subset F . \end{aligned} \tag{3.4}$$

Let π be the projection from $T^*\mathbb{C}^d$ onto $\mathbb{C}^d$.
Let $Z = \pi(F)$ and $Z_0 = Z \cap \{z_d = 0\}$.

Lemma

The open set $B \setminus Z$ is dense in B, connected, and locally connected near any point of B. The open set $B_0 \setminus Z_0$ is dense in B_0, connected, and locally connected near any point of B_0.

Conjectures

a)The Poisson kernel K (resp its normal derivative $\partial_{x_d} K|_{x_d=0}$) can be holomorphicaly extends near any path $t \geq 0 \mapsto q(t)$ with
$q(0) \in \{\|x\| < \varepsilon, x_d > 0\}$ (resp $q(0) \in \{0 < \|x'\| < \varepsilon\}$
and for $t > 0$
$q(t) \in B \setminus Z \qquad ($ resp $q(t) \in B_0 \setminus Z_0)$.

b) Let Z_{reg} (resp $Z_{0,reg}$) be the set of regular points of Z (resp Z_0), i.e the set of points $z \in Z$ (resp $z' \in Z_0$) such that near z, Z (resp Z_0) is a complex smooth hypersurface. Then Z_{reg} is dense in Z (resp $Z_{0,reg}$ is dense in Z_0) and "near" any point of Z_{reg} (resp $Z_{0,reg}$), the holomorphic extension of K (resp $\partial_{x_d} K|_{x_d=0}$) is regular holonomic.

The live of "conjectures"

.... ruinée par des erreurs géniales, la physique de Descartes a fait en tombant un bruit que l'on entend encore...

... la récurrence indéfinie de quelques questions fondamentales...

Merleau Ponty, in introduction to "La mécanique de Christian Huygens" by Christiane Vilain, Ed. Albert Blanchard, 1996.

Outline

Dimension 2

For $d = 2$, one has $p(z,\zeta) = \zeta_2^2 + r(z_1,z_2)\zeta_1^2, r(0,0) > 0$, hence

$$p^{-1}(0) = \{(z,\zeta), \zeta_2 = \pm i\sqrt{r(z_1,z_2)}\zeta_1\}$$

Thus one has

$$F = \gamma_+ \cup \gamma_-, \quad Z = Z_+ \cup Z_-, \ Z_\pm = \pi(\gamma_\pm)$$

where $\gamma_\pm$ are the integral curves of the Hamiltonians $H_{\zeta_2 \mp i\sqrt{r(z_1,z_2)}\zeta_1}$ starting from $z = 0, \zeta_1 = 1, \zeta_2 = \pm i$. The one dimensional complex curves $Z_\pm$ are transversal to the boundary $z_2 = 0$, and one has the easy lemma

Lemma

Conjectures a) and b) hold true in dimension $d = 2$.

Totally null complex geodesics boundary

Set $r_0(z',\zeta') = r(z',0,\zeta')$. Assume that the following holds true:

$$(*)\Big(r_0(0,\xi') = 0 \text{ and } (z',\zeta') = exp(sH_{r_0})(0,\xi')\Big) \Rightarrow \frac{\partial r}{\partial z_d}(z',0,\zeta') = 0$$

In that case, the sets F and Z are easy to compute:

$$F = \cup_{p(0,\zeta)=0,\zeta\neq 0} exp(sH_p)(0,\zeta), \quad Z = \{z \in \mathbb{C}^d,\ dist_g^2(z,0) = 0\}$$

Example

(*) holds true when P is of the form $P = \partial_{x_d}^2 + R_0(x',\partial_{x'}) + x_d^2 R_1(x',x_d,\partial_{x'})$, and in that case, conjectures a) and b) hold true.

Remark

Let $P = \triangle$, and consider the Poisson kernel K_y on Ω. Let ω be a non void open subset of the boundary $\partial\Omega$. Then (*) holds true for all $y \in \omega$ **iff** *ω is a piece of a linear hypersurface or a piece of sphere.*

Outline

Order of contact of rays with the boundary

Let $\Omega = \{f > 0\} \subset \mathbb{R}^d$ defined near $y = 0$ by an analytic function f

$$f(0) = 0,\ df(0) \neq 0,\ f(x) = \sum_{k \geq 1} P_k(x),\ P_k(x) = \sum_{|\alpha| = k} f_\alpha x^\alpha$$

For the Laplace operator $\triangle$, the characteristic starting at 0 in a null direction $\zeta \neq 0$ is $\gamma(s\zeta) = (z = s\zeta, \zeta)$ with $\zeta^2 = 0$. One has $f(s\zeta) = \sum_{k \geq 1} s^k P_k(\zeta)$, thus $\pi(\gamma)$ has a contact of order $\geq L$ with $\partial\Omega$ iff

$$\zeta \in \mathcal{T}_L = \{\zeta \in \mathbb{C}^d \setminus \{0\}, \zeta^2 = 0, P_k(\zeta) = 0 \text{ for } 1 \leq k \leq L - 1\}$$

Therefore, we get

$$\mathcal{T}_L = \emptyset \quad \Rightarrow \quad d \leq L$$

An example with $\mathcal{T}_3 = \emptyset$ in dimension $d = 3$

$$P = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2, \quad \Omega = \{x > 0\}$$

In that case, one has $\mathcal{T}_3 = \emptyset$ and the sets F, $Z = \pi(F)$ and $Z_0 = Z \cap \{x = 0\}$ are easy to compute. In particular, one find

$$Z_0 = (\cup_{N \geq 1} Z_{N,0}) \cup Z_{\infty,0}$$

$$Z_{N,0} = \{(y,z) = 4(u + \frac{2u^3}{3N^2}, \pm iu\sqrt{1 + \frac{u^2}{N^2}})\}, \quad Z_{\infty,0} = \{z = \pm iy\}$$

Theorem

Conjectures a) and b) hold true for P.

Remark

The proof of this result uses strongly an explicit representation of the Poisson kernel in terms of an infinite sum of Airy integrals. The general case with $\mathcal{T}_3 = \emptyset$ in dimension $d = 3$ is still open, at least for me.

Outline

Cantor structure of Z when $\mathcal{T}_3 \neq \emptyset$

When $\mathcal{T}_3 \neq \emptyset$, there exists tangential "rays" with a contact of order ≥ 3 with the boundary. The simple case is $\mathcal{T}_3 \neq \emptyset, \mathcal{T}_4 = \emptyset$. Let γ_0 a "ray" with tangency of order 3 with the boundary at $y = 0$. Then a ray γ close to γ_0 will intersect the boundary at 3 distinct points generically. This means that the complex billiard dynamic is (roughly) a dynamical system of the form

$$x \mapsto (g_1(x), g_2(x)) = C(x)$$

Thus F must contain locally near $\mathcal{I}$ **the closure** of the set

$$\cup_{N\in\mathbb{N}} {C^*}^N(\mathcal{I}), \quad \mathcal{I} = \{(0,\zeta), \zeta \neq 0, \zeta^2 = 0\}$$

In particular, one find that the set of singular points of Z,

$$Z_{sing} = Z \setminus Z_{reg} \quad \text{has a "Cantor structure".}$$

...

Tu as fait de douloureux et de joyeux voyages
Avant de t'apercevoir du mensonge et de l'âge

...

J'ai vécu comme un fou et j'ai perdu mon temps

. . .

Guillaume Apollinaire, "Zone", in "Alcools", 1913