Hyperbolic triangles with no positive Neumann eigenvalues

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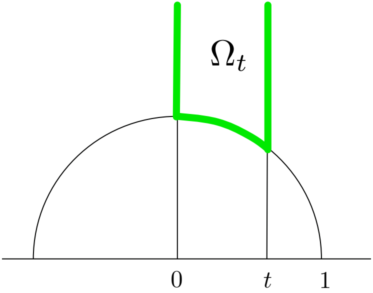
Question



- Does there exist $u \in L^2\left(\Omega_t, \frac{dx\,dy}{y^2}\right)$ and $E > 0$ with
  1. $-y^2(\partial_x^2 + \partial_y^2)\,u \;=\; E \cdot u$, and
  2. $\frac{\partial}{\partial n} u \equiv 0$ (Neumann conditions)?

- Remark: Ω_t non-compact but finite measure.

Some answers

- Selberg proved 'yes' if $t \in \left\{ \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2} \right\}$
- Phillips and Sarnak conjectured 'no' if

$$t \in \{\cos(\pi/n) : \; n \neq 3, 4, 6\}$$

Theorem (Hillairet-J.)

The answer is 'no' for all but countably many $t \in]0, 1[$.

Goal of this talk: Describe some ingredients of the proof.

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Spectral dichotomy for hyperbolic surfaces

- $\mathbb{H} = \{(x, y) \mid y > 0\}$: arclength $y^{-1} ds_{\text{Euc}}$; measure $y^{-2} dxdy$.
- $\Delta_{\mathbb{H}} = -y^2 \left(\partial_x^2 + \partial_y^2\right)$
- $\Gamma \subset \mathrm{Isom}(\mathbb{H})$ cofinite discrete subgroup with 'cusps'.
- $\Delta : L^2(\mathbb{H}/\Gamma) \to L^2(\mathbb{H}/\Gamma)$ non-negative self-adjoint operator.
- spectral decomposition (Selberg)

$$\Delta u = \sum_{k \in \mathbb{N}} E_k \langle u, u_k \rangle u_k + \int_{\mathbb{R}} (1/4 + r^2) \langle u, G_r \rangle G_r \, dr$$

- Conjectured dichotomy (Sarnak):

  Γ 'arithmetic' $\Leftrightarrow$ pure point spectrum has Weyl density 1.

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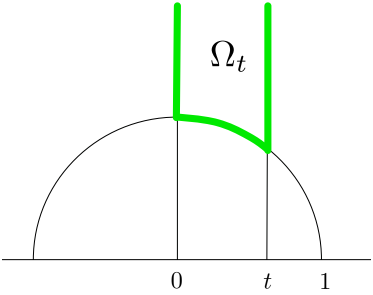
Return to Ω_t: Fourier decomposition in x



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Return to Ω_t: Fourier decomposition in x

- pure point vs. continuous spectrum explained by Fourier decomposition:
- For u smooth Neumann function on Ω_t and $y > 1$,

$$u(x,y) \;=\; \sum_{\ell=0}^{\infty} u^{\ell}(y)\cdot\cos\left(\frac{\pi\ell}{t}\cdot x\right)$$

- If u eigenfunction with eigenvalue E, then u^{ℓ} satisfies the ODE

$$\left(u^{\ell}\right)'' \;=\; \left(\left(\frac{\ell\pi}{t}\right)^2 - \;\frac{E}{y^2}\right)\cdot u^{\ell}$$

Solutions to the ODE

$$\left(u^{\ell}\right)'' = \left(\left(\frac{\ell\pi}{t}\right)^2 - \frac{E}{y^2}\right) \cdot u^{\ell}$$

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  $E = 1/4 + r^2$

- $\ell > 0$: For $y >> $ zero of $(\ell\pi/t)^2 - E \cdot y^{-2}$

$$u^{\ell} \sim c \cdot \exp\left(\pm\frac{\ell\pi}{t} \cdot y\right)$$

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Cut-off Laplacian (Lax-Phillips, Colin de Verdiere)

- Restrict Δ_t to $u \in C_0^\infty$ with $u^0(y) = 0$ for $y \geq b > 1$
- Friedrichs extension called the *cut-off Laplacian* Δ_t^b.
- $(\Delta_t^b + Id)^{-1}$ compact, and eigendata real-analytic in t.
- $\Delta_t u = Eu \implies \Delta_t^b u = Eu$
- $\Delta_t^b u = Eu$ and $u^0(y) = 0$ for $1 < y < b \implies \Delta_t u = Eu$
- If $t \mapsto u_t$ analytic, then $t \mapsto u_t^0(y)$ is analytic.
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New ideas

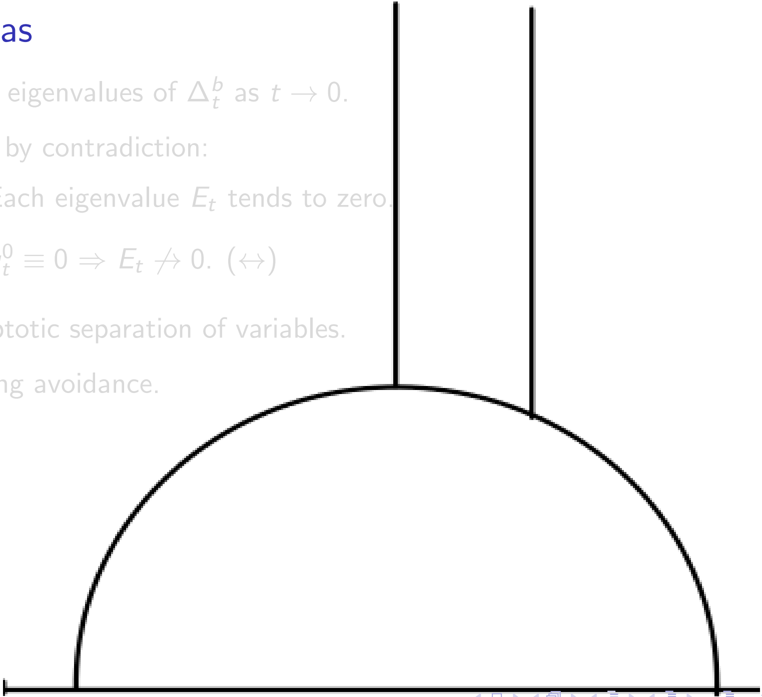
Study eigenvalues of Δ_t^b as $t \to 0$.

Proof by contradiction:

- Each eigenvalue E_t tends to zero.
- $u_t^0 \equiv 0 \Rightarrow E_t \not\to 0$. $(\leftrightarrow)$

asymptotic separation of variables.

crossing avoidance.



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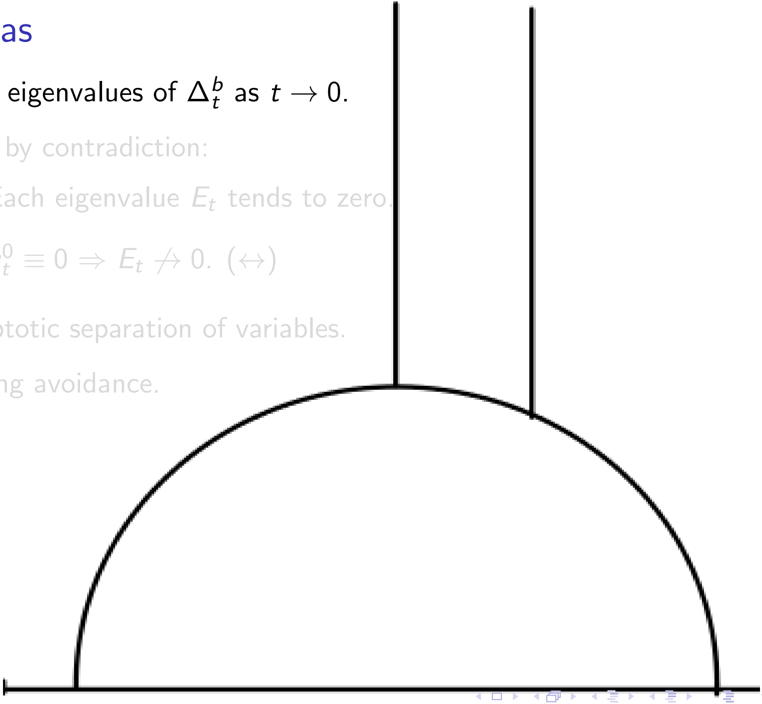

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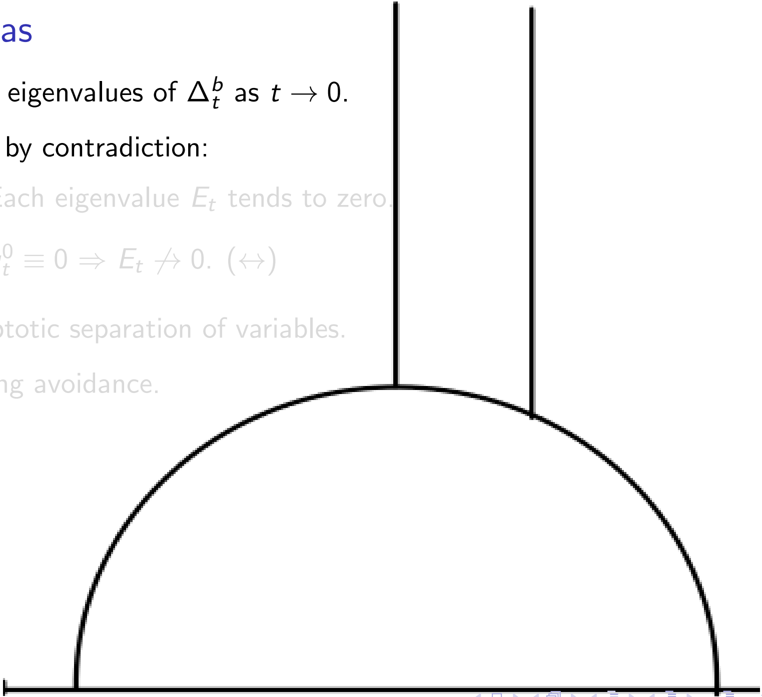

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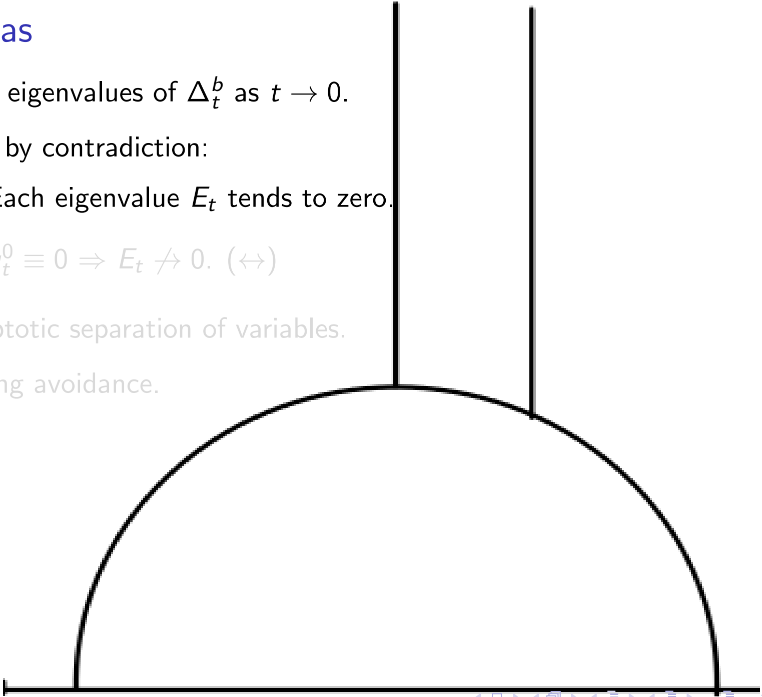

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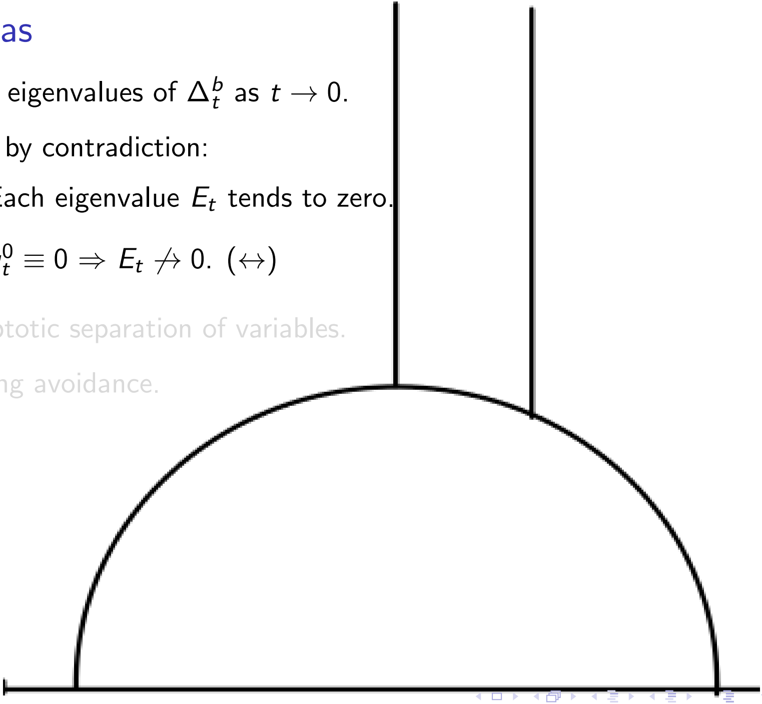

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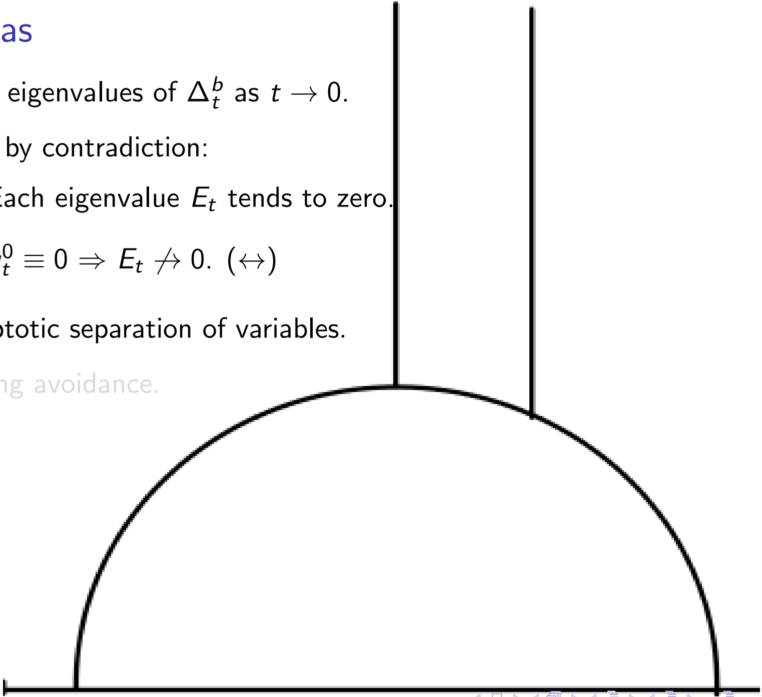

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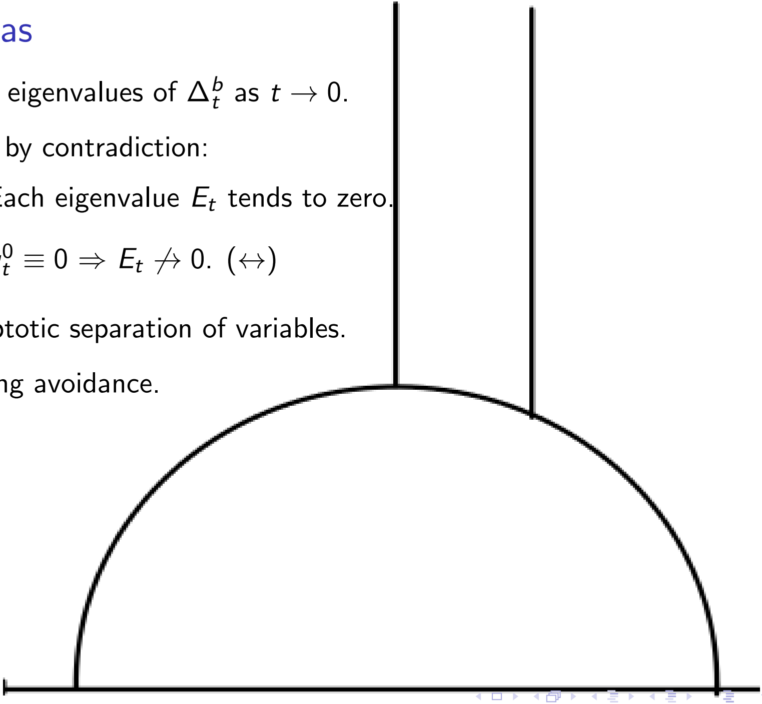
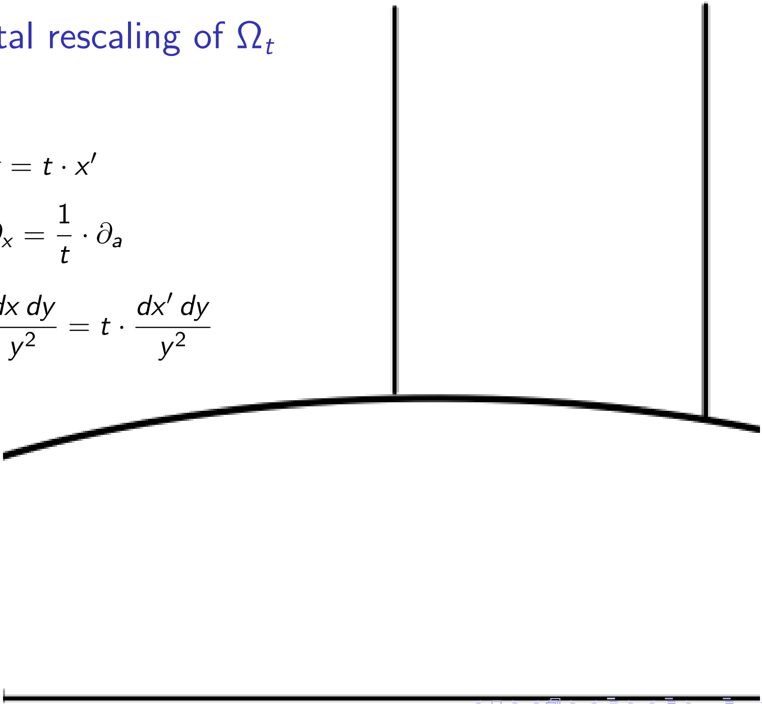


Horizontal rescaling of Ω_t

- $x = t \cdot x'$
- $\partial_x = \frac{1}{t} \cdot \partial_a$
- $\frac{dx\,dy}{y^2} = t \cdot \frac{dx'\,dy}{y^2}$



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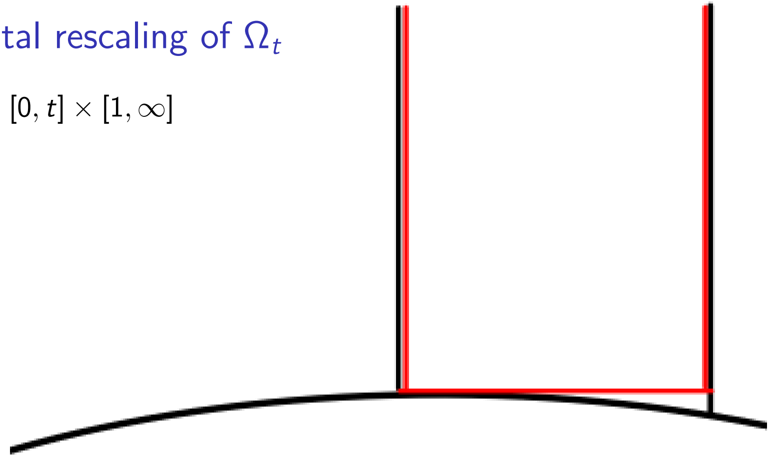

- $\langle \Delta_t u, u \rangle = \int \left(\left(\frac{u_{x'}}{t} \right)^2 + (u_y)^2 \right) t \, dx' dy$

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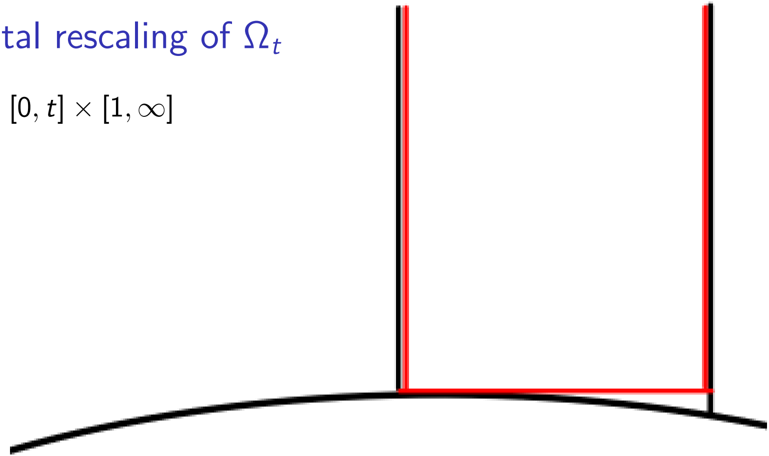

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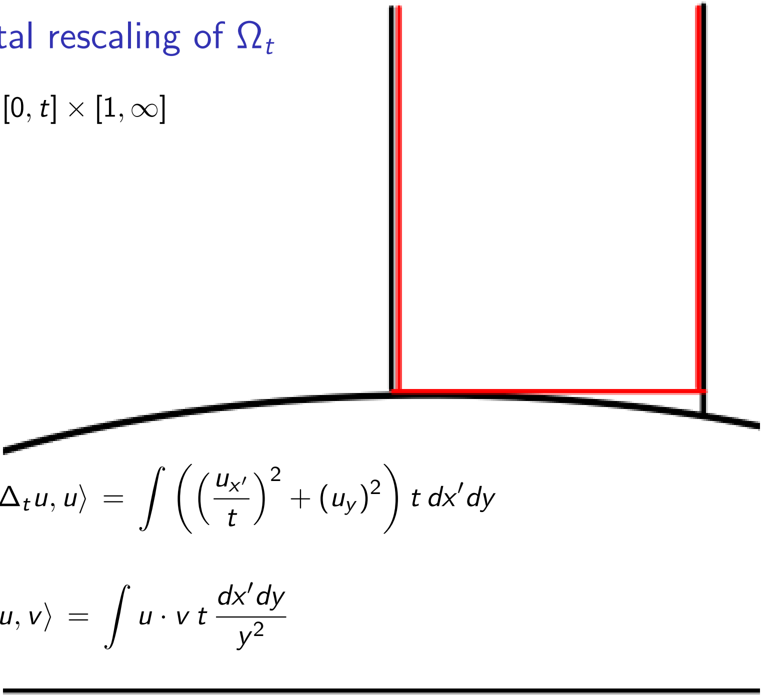

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A tale of two quadratic forms

- Horizontally stretched domain Ω_t
- Rectangle $R = [0,1] \times [1,\infty]$
- Map R onto Ω_t via diffeo 'supported' on $y < b$.
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- Expansion at $t = 0$

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$$q_t(u) \;=\; \int_R \left(\frac{(u_{x'})^2}{t^2} \;+\; (u_y)^2 \right) t \, dx \, dy \;+\; O(1)$$

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A tale of two quadratic forms

- Horizontally stretched domain Ω_t
- Rectangle $R = [0,1] \times [1,\infty]$
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Some generalities: 'Asymptotic at first order'

- a_t and q_t nonnegative quadratic forms.
- $|q_t(u,v) \ - \ a_t(u,v)| \ \leq \ C \cdot t \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$
- $|\dot{q}_t(u) \ - \ \dot{a}_t(u)| \ \leq \ C \cdot a_t(u)$ (dot signifies t-derivative)
- $\dot{a}(u) \ = \ 2t \cdot \int \ u_y^2 \, dx \, dy$
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A first consequence: convergence of eigenvalue branches

Proposition

If a_t and q_t asymptotic at first order and $\dot{a}_t \geq 0$, then each real-analytic eigenvalue branch $t \mapsto E_t$ converges as $t \to 0$.

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If u_t is the associated eigenfunction branch, then

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Thus,

$$-2C \;\leq\; \frac{\dot{E}_t}{E_t}$$

□

Quasimodes from spectral projections

- u eigenfunction of q_t with respect to $L^2(dx\,dy/y^2)$ norm:

$$q_t(u,v) \;=\; E \cdot \langle u, v \rangle \;\; \text{for all} \;\; v$$

- I = interval in $\mathbb{R}$.
- $w :=$ projection of u onto a_t eigenspaces with $\lambda \in I$

Proposition

w is an a_t quasimode at order t and energy E: For each v

$$|a_t(w,v) \;-\; E \cdot \langle w, v \rangle| \;\leq\; C \cdot t \cdot \|w_t\| \cdot \|v\|$$

for all v. Moreover,

$$a_t(u-w) \;+\; \|w-u\|^2 \;\leq\; C \cdot t \cdot \|u\|^2.$$

Consequences for real-analytic eigenbranches of q_t

- $t \mapsto u_t$ eigenfunction branch of q_t with eigenvalue E_t.
- w_t corresponding spectral projection associated to $I \subset \mathbb{R}$

Proposition

The function $t \mapsto \dfrac{\dot{a}(w_t)}{\|w_t\|^2}$ *is integrable over* $]0, t_0[$

and $\left|\dot{a}(w_t) - \dot{E}_t \cdot \|u_t\|^2\right| \;\leq\; C \cdot \|u_t\|$

Variational formula in Ω_t context

To obtain finer information we use our specific context:

$$a_t(u) \; = \; \int (u_x)^2 + t^2 \cdot (u_y)^2$$

$$a_t(u) - E \cdot \|u\|^2 \; = \; \int (u_x)^2 + t^2 \cdot (u_y)^2 - E \int \frac{u^2}{y^2}$$

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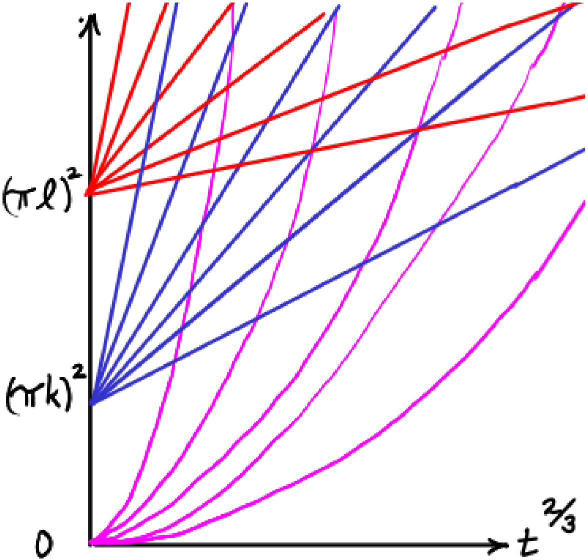
Spectrum of a_t via separation of variables

- $u(x,y) = \sum_\ell u^\ell(y) \cdot \cos(\pi \cdot \ell \cdot x)$
- $a_t(u) = \sum a_t^\ell(u^\ell)$ where

$$a_t^\ell(v) \;=\; \int_1^\infty \left(t^2 \cdot (v')^2 \;+\; (\pi\ell)^2 \cdot v^2\right)\, dy$$

- spectrum of $a_t \;=\; \bigcup_\ell$ spectra of a_t^ℓ
- $\ell > 0 \Rightarrow$ eigenvalue $\lambda_t = (\pi\ell)^2 \;+\; c \cdot t^{\frac{2}{3}} + O(t)$ (Airy)

Eigenvalue branches of a_t



Variation and non-concentration

- $\dot{a}$ is ‘diagonal’ with respect to a_t, and so $\dot{a}_t = \sum_{\ell} \dot{a}_t^{\ell}$ where

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Proposition (Nonconcentration at the turning point)

If w is an a_t^{ℓ} quasimode of order t at energy $E >> (\ell\pi)^2$, then there exists $\kappa > 0$ so that for all small t

$$\int \left(\frac{E}{y^2} - (\ell\pi)^2\right) w^2 \; \geq \; \kappa \cdot \|w\|^2.$$

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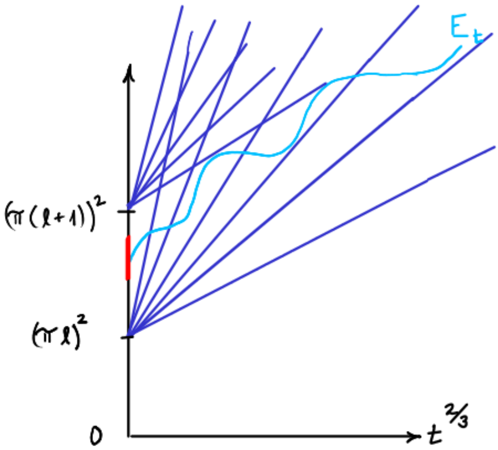

Integrality condition

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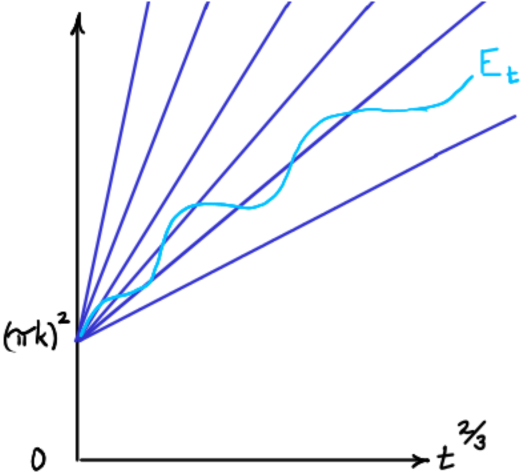

▸ Integrality condition

Relatively small k-mode implies large variation

$w_t^\ell =$ projection of w_t onto a_t eigenspaces with $\lambda_t \to (\pi\ell)^2$.

Lemma

Let $\rho < 1$. There exists $\delta > 0$ such that for sufficiently small t

$$\frac{\|w_t^k\|}{\|w_t\|} < \rho \implies \dot{E}_t \geq \frac{\delta}{t}.$$

Crossings imply relatively small k-mode

Lemma

Let $\rho < 1$. If for sufficiently small η and t

$$\mathrm{dist}(E_t, \mathrm{spec}(a_t^0)) \; < \; \eta \cdot t^{\frac{5}{3}}$$

then

$$\|w_t^k\| / \|w_t\| \; \leq \; \rho.$$

Idea of Proof:

$(E_t - \lambda_t^0) \cdot \langle u_t, \psi_t \rangle = (q_t - a_t)(u_t, \psi_t^0) = t \cdot b_t(u_t, \psi_t^0) + O(t^2) \cdot \|u_t\| \cdot \|\psi_t^0\|.$

ODE (Airy) approximation $\Rightarrow$ there exists $\delta > 0$ so that

$$|b_t(u_t, \psi_t^0)| \; \geq \; \delta \cdot \left(\|w_t^k\| \cdot t^{\frac{2}{3}} \; - \; O(t^\delta) \|w_t\| \right) \cdot \|\psi_t^0\|$$

Here

$$b_t(u) \; = \; 2t \int_{y \leq b} p(y) \cdot u_x \cdot u_y \; dx \, dy$$

with p is smooth and $p(1) = 1$

Zeroth order approximation of proof of main theorem

- $(\rightarrow \leftarrow)$ Suppose there exists u_t with $u_t \equiv 0$ and $E_t \not\to 0$.
- $E_0 > 0 \implies$ infinitely many eigenvalue branches of a_t^0 cross E_t.
- previous two lemmas imply that at each crossing $\dot{E}_t > \frac{\delta}{t}$
- Summing over crossings leads to $E_t \to 0$.

Additional considerations for summing

- Frequency of crossings: $\exists\ t_n \to 0$ with $E_{t_n} \in \mathrm{spec}(a_{t_n})$ and

$$\lim_{n\to\infty} n \cdot t_n = \ k \cdot \ln(b).$$

- Width of crossings: $\lambda_t \in \mathrm{spec}(a_t^0) \Rightarrow$

$$|s-t| < t^{\frac{8}{3}} \implies \mathrm{dist}(\lambda_s, \mathrm{spec}(a_s^0))$$

- 'Tracking': There exists unique eigenvalue branch λ_t^* of a_t^k so that $|E_t - \lambda_t^*| < Ct$ for t small.

- In truth, we use crossings to show that

$$\int_0^t \left(\dot{E}_s - \dot{\lambda}_s^*\right) ds \ > \ O(t^{\frac{2}{3}}).$$

a contradiction.