Dispersion for the wave and the Schrödinger equations outside a strictly convex obstacle and counterexamples

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The wave and Schrödinger equations

(Ω, g) = Riemannian manifold of dimension $d \geq 2$,
Δ = Laplace-Beltrami operator on (Ω, g).

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{ on } \Omega, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0. \end{cases}$$

A close relative is the Schrödinger equation (especially in the "semi-classical" setting)

$$\begin{cases} \frac{h}{i}\partial_t v - \frac{h^2}{2}\Delta v = 0, \\ v|_{t=0} = v_0, \quad v|_{\partial\Omega} = 0. \end{cases}$$

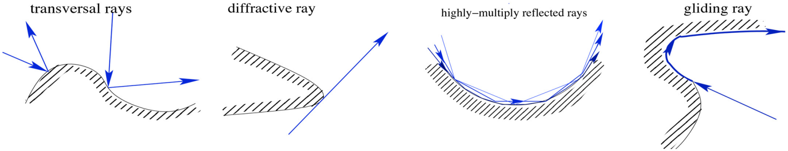
What *is* dispersion?

- The wave flow :

$$\|\chi(hD_t)e^{\pm it\sqrt{|\triangle_{\mathbb{R}^d}|}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ch^{-d}\min(1,(\frac{h}{t})^{\frac{d-1}{2}})$$

- The classical Schrödinger flow :

$$\|e^{\pm it\triangle_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ct^{-d/2}.$$


transversal rays
diffractive ray
highly−multiply reflected rays
gliding ray

Understand the dispersive effects $\Rightarrow$ study the **WAVEFRONT**

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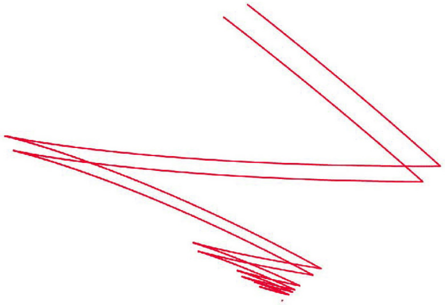
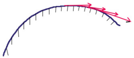


Fig: The wavefront near a point of strict convexity after 5 reflections. **Cusps** and **swallowtail** singularities.

What happens with the WF set that reaches the boundary of an obstacle?

- Tangent to a convex obstacle : can a ray carrying WF tangent to a convex obstacle stick to it and release energy in the "shadow region" ?



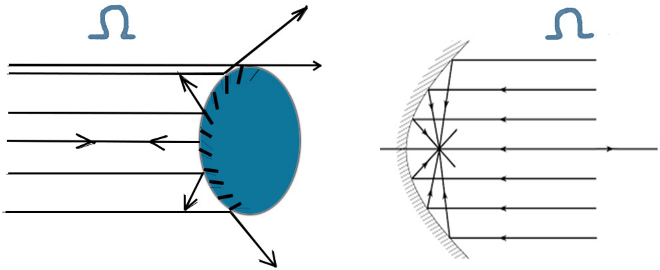
Theorem

([Melrose-Taylor] 1975) No propagation into the "shadow region". If we measure analytic singularities, this becomes false.

Theorem

([Hargé-Lebeau] 1994 - Keller's conjecture for C^∞ boundary) The decreasing rate in the shadow region is of the form $e^{-C\tau^{1/3}}$.

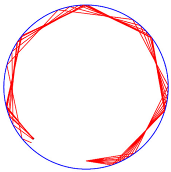
A caustic/cluster point/ singularity in the WF should yield loss in dispersion .

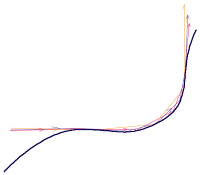


Inside a parabola : the wave shrinks in size at the focus and its L^∞ norm increases.

Glancing

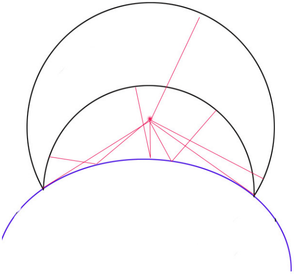
- Propagation inside a strictly convex: the "sphere" of radius t soon degenerates and develops singularities in arbitrarily small times ([I., Lebeau, Planchon] - Annals 2014, [I., Lascar, Lebeau, Planchon] general convex case 2016)
- If no convexity : even deciding what should constitute the continuation of a ray striking the boundary is difficult... .





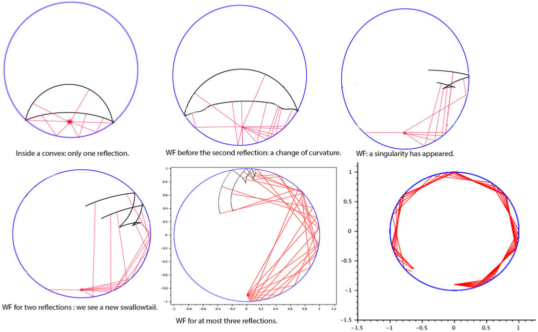
WF in the exterior/ interior of a strictly convex

Wave front outside a strictly convex obstacle



Convex obstacle

Wave front inside a convex obstacle (for at most three reflections)



Theorem

([I. - Lebeau] 2015) Θ_d = *strictly convex obstacle in* $\mathbb{R}^d$, $\Omega_d = \mathbb{R}^d \setminus \Theta_d$.

- *If* $d = 3$, *the dispersion estimates (for the wave and the Schrödinger equations inside* Ω_3*) hold true.*
- *If* $d \geq 4$, $\Theta_d = B_d(0,1) \subset \mathbb{R}^d$, *at the Poisson spot they fail.*

Recall : Strichartz do hold like in $\mathbb{R}^d$ for both wave ([Smith-Sogge] 1995) and (classical) Schrödinger ([I.] 2010) equations, $\forall d \geq 2$.

Having the full dispersion for Schrödinger in $3D$:

- we can get the endpoint Strichartz estimate.
- might also help in dealing with non-linear equation (see recent work by [Killip, Visan, Zhang] 2014 on the energy-critical NLS in $3D$).

Estimates at the Poisson spot

$Q_{\pm}(s)$ = source/observation points at (same) distance s from the ball, symmetric w.r.t. the center of the ball :

- Wave flow : $s = \gamma h^{-1/3}$, $t \simeq h^{-1/3}$

$$|(\chi(hD_t)e^{ih^{-1/3}\sqrt{|\Delta|}}(\delta_{Q_-})|(Q_+) \simeq \frac{1}{h^d}\left(\frac{h}{h^{-1/3}}\right)^{-\frac{d-1}{2}} h^{-\frac{d-3}{3}},$$

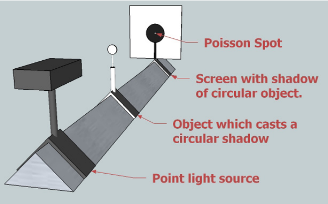
- Classical Schrödinger flow: $s \simeq \gamma h^{-1/6}$, $t \simeq h^{1/3}$

$$|(\chi(hD_t)e^{ih^{1/3}\Delta}(\delta_{Q_-})|(Q_+) \simeq (h^{1/3})^{-\frac{d}{2}} h^{-\frac{d-3}{6}},$$

- Semi-classical Schrödinger : $t \simeq h^{-2/3}$

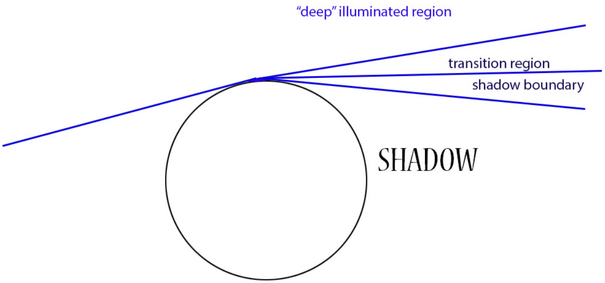
For $d \geq 4$ this contradicts the usual (flat) estimates !

Poisson - Arago spot



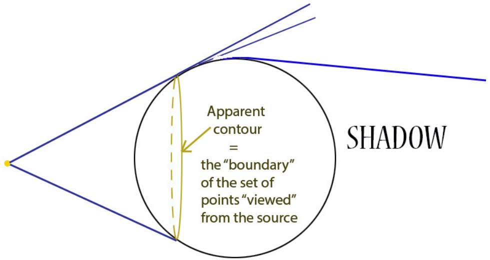
When light shines on a circular obstacle, Huygens's principle says that every point of the obstacle acts as a new point source of light. The light coming from points on the circumference of the obstacle and going to the centre of the shadow travel exactly the same distance and give rise to a bright spot at the shadow's center.

Diffraction $\Omega = \mathbb{R}^d \setminus \Theta$, Θ strictly convex



- Coordinates : $(x, y) \in \mathbb{R}_+ \times \partial\Omega$ such that $x \to (x, y)$ is the ray orthogonal to $\partial\Omega$ at y. Then $\forall Q \in \Omega$, $Q = y + xn_y$, where n_y is the outward unit normal to $\partial\Omega$ pointing towards Ω.

Glancing : $h^{1/3}$ neighbourhood of the apparent contour



- "Glancing" : a $h^{1/3}$ neighbourhood of the apparent contour.

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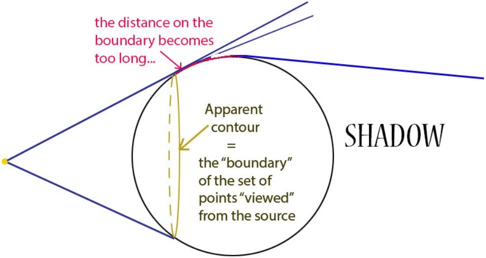
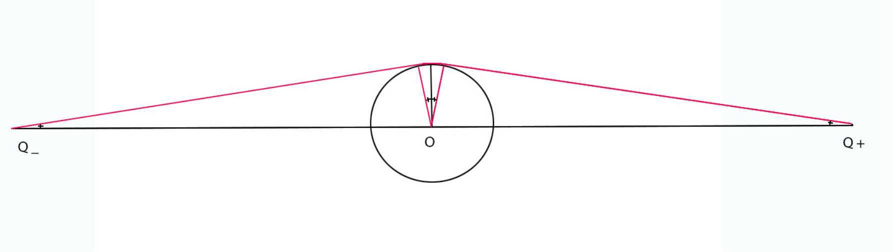


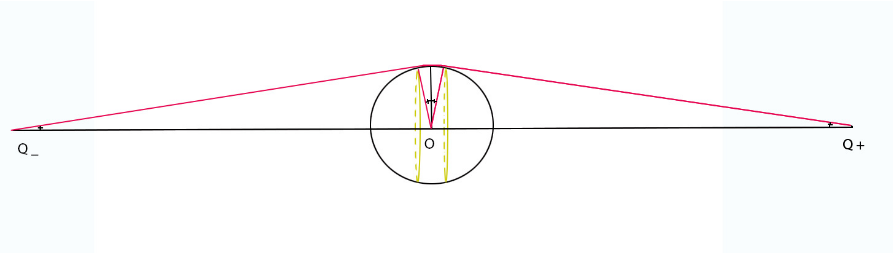
- "Glancing" : a $h^{1/3}$ neighbourhood of the apparent contour.
- If the source is not sufficiently far from Θ, in order to re-focalise, the wave would need to spend a longer time on the boundary ($\gg h^{1/3}$), but then we use [Hargé-Lebeau].

Glancing : $h^{1/3}$- neighbourhood of the apparent contour



- To obtain re-focalisation we need to put the source at distance $h^{-1/3}$.

Glancing : $h^{1/3}$- neighbourhood of the apparent contour



- The distance between the apparent contour of Q_- and the apparent contour of Q_+ is $\simeq h^{1/3}$ (*allowed* distance, not exponential decay like for the shadow region !).

How do we construct the parametrix ?

Let $\Theta = B_d(0,1) \subset \mathbb{R}^d$:

- if polar coordinates : problems when the source point $\to \infty$;
- anyway not useful in the general case, so forget it ....
- [Melrose-Taylor] parametrix : OK, but it is defined only near $\partial\Omega$
- To handle this, use the fundamental solution and the Neumann operator to reduce to estimations for (in $d = 3$):

$$\int \int_{P \in \partial\Omega} (\partial_x u_{free}|_{\partial\Omega} - N(u_{free}|_{\partial\Omega}))(s, P) \frac{\delta(t - s - |Q - P|)}{4\pi |P - Q|} d\sigma(P) ds$$

- use NOW [Melrose-Taylor] (in the glancing region) : it gives the form of the left factor in terms of Airy functions.

Form of the parametrix

- For positive results ($d = 3$) forget the time :
  $t \simeq |Q_- - P| + |P - Q_+|$ is uniformly bounded w.r.t. $d(Q_\pm, \partial\Omega)$
  ([Melrose] 1979 : the outgoing solution decays exponentially if time is larger than the escape time);
- [Melrose-Taylor]: $\exists \theta, \zeta$ phase functions near glancing, such that for v solution to $(\tau^2 + \Delta)v \in O_{C^\infty}(\tau^{-\infty})$ there exists an unique F such that $v = T_\tau(F)$, where

$$T_\tau(F)(x,y) = \left(\frac{\tau}{2\pi}\right)^{d-1} \int e^{i\tau\theta(x,y,\alpha)} \Big(aA + b\tau^{-1/3}A'\Big)(\tau^{2/3}\zeta)\hat{F}(\tau\alpha)d\alpha.$$

Moreover $\theta_0 = \theta|_{\partial\Omega}$ defines a canonical relation ($\theta(y,\eta) = y\eta$ outside the ball), $\zeta_0 = \zeta|_{\partial\Omega}$ doesn't depend on the tangential variable s.t. the left factor is

$$\tau^{d-1+\frac{2}{3}} \int e^{i\tau\theta_0(P,\eta)} \frac{\hat{F}_\tau(\tau\eta, Q_0)}{A_+(\tau^{2/3}\zeta_0(\eta))} d\eta,$$

$$\hat{F}_\tau(\tau\eta, Q_0) = \Big(\frac{\tau}{|Q_- P_0|}\Big)^{\frac{d-1}{2}} \tau^{-1/3} e^{i\tau(\theta_0(P_0,\eta)-|Q_0-P_0|)},$$

where $P_0 \in$ the apparent contour $\mathcal{C}_{Q_0}$ of Q_0.

Details... for the exterior of the ball

- [Melrose-Taylor] in the glancing region :

$$u_{free}|_{\partial\Omega}(\tau, P, Q_0) = \left(\frac{\tau}{|P-Q_0|}\right)^{\frac{d-1}{2}} e^{-i\tau|P-Q_0|}\gamma(\tau, P, Q_0)$$
$$= Op(e)(\mathcal{F}^{-1}(A(\tau^{2/3}\zeta_0)\mathcal{F}(F(., Q_0)),$$

  e elliptic symbol.
- To get F : degenerate critical points of order 2 on $\mathcal{C}_{Q_0}$.
- In the formula to estimate : a new, similar phase, hence a second degenerate stationary phase with critical points on $\mathcal{C}_Q$, at distance at most $\gamma h^{1/3}$ from $\mathcal{C}_{Q_0}$.
- If $Q_0 = Q_-(\gamma h^{-1/3})$, $Q = Q_+(\gamma h^{-1/3})$ and $t \simeq h^{-1/3}$ then the announced estimates.

Open questions

- What about the generic case?
- A "twisted sphere" still yields a counterexample...
- classify the obstacles for which dispersion holds in every d?
- what about $L^{p'} \to L^p$ $(2 < p < \infty)$? Do they still hold, even outside a sphere?