Diversity in parametric families of number fields II

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Notations (Review)

For a separable polynomial $F(T) \in \mathbb{Z}[T]$ we denote:

- Δ_F the discriminant of F;
- $\mathcal{P}_F$ the set of p for which $F(T)$ has a root mod p, and which do not divide Δ_F.
- $\mathcal{M}_F$ the set of square-free integers composed of primes from $\mathcal{P}_F$.

By the Chebotarev Density Theorem, the set $\mathcal{P}_F$ is of positive density among all the primes denoted δ_F. Note that

$$\delta_F \geq \frac{1}{D}, \qquad (D = \deg F). \tag{1}$$

The set $\mathcal{M}_F(x)$ (Review)

Let $\varepsilon > 0$ be sufficiently small in a way to be specified later but fixed. Let x be large and put

$$y = \exp((\log x)^{1-\varepsilon}).$$

Let $P(n)$ be the largest prime factor of a positive integer n. Let

$$k := \lfloor \delta_F \varepsilon \log\log x \rfloor + 1, \qquad \kappa := \log\log x$$

and $\mathcal{M}_F(x)$ be the set of positive integers m subject to the following conditions:

(i) $m \in [x/(2\kappa), x/\kappa]$ and if $p \mid m$ is prime, then $p \in \mathcal{P}_F$;

(ii) $P(m) > x^{9/10}$;

(iii) $p \mid m$ is prime then $p \geq y$.

(iv) m is squarefree;

(v) $\omega(m) = k + 1$.

Lemma (1)

We have

$$\#\mathcal{M}_F(x) = \frac{x}{(\log x)^{1-\varepsilon\delta_F+o(1)}}$$

as $x \to \infty$.

Proof.

If $m \in \mathcal{M}_F(x)$, then $m = Pm_1$, where $P = P(m) > x^{9/10}$ and

$$m_1 = m/P < x^{1/10}.$$

Let $\mathcal{M}'_F(x)$ be the set of such m_1's. Then m_1 fulfills (iii), (iv), has $\omega(m_1) = k$ and all its prime factors are in $\mathcal{P}_F$. Further, for a fixed m_1, we have

$$P \in [x/(2\kappa m_1), x/(\kappa m_1)].$$

Since $m_1 < x^{1/10}$, it follows that

$$x/(\kappa m_1) > x^{4/5} \qquad \text{for} \qquad x > x_0.$$

Thus, for a fixed m_1, P can be chosen in

$$\begin{aligned} \pi_F(x/(\kappa m_1)) \quad &- \quad \pi_F(x/(2\kappa m_1)) \\ &= \quad (\delta_F/2 + o(1)) \frac{x}{\kappa m_1 \log(x/(\kappa m_1))} \\ &\asymp \quad \frac{x}{\kappa m_1 \log x} \end{aligned}$$

ways.

In the above, $\pi_F(T)$ counts the number of primes in $\mathcal{P}_F(T)$. Summing up over $m_1 \in \mathcal{M}'_F$, we get

$$\#\mathcal{M}_F(x) \asymp \frac{x}{\kappa \log x} \sum_{m_1 \in \mathcal{M}'_F} \frac{1}{m_1}.$$

Let's deduce bounds on $\#\mathcal{M}_F(x)$. For the upper bound:

$$\begin{aligned}
\sum_{m_1 \in \mathcal{M}'_F} \frac{1}{m_1} &\leq \frac{1}{k!} \left(\sum_{\substack{y \leq p \leq x \\ p \in \mathcal{P}_F}} \frac{1}{p} \right)^k \\
&\ll \frac{(1 + o(1))}{\sqrt{k}(k/e)^k} (\delta_F \log\log x - \delta_F \log\log y)^k \\
&\ll \frac{1}{(\log\log x)^{1/2}} \left(\frac{((e\varepsilon\delta_F + o(1)) \log\log x}{k} \right)^k \\
&\ll \frac{1}{(\log\log x)^{1/2}} (e + o(1))^{\varepsilon\delta_F \log\log x + O(1)} \\
&\ll (\log x)^{\varepsilon\delta_F + o(1)} \quad \text{as} \quad x \to \infty.
\end{aligned}$$

Hence,

$$\#\mathcal{M}_F(x) \ll \frac{x}{(\log x)^{1-\varepsilon\delta_F+o(1)}}$$

as $x \to \infty$. For the lower bound, consider

$$z := x^{1/11 \log\log x} \quad \text{and} \quad \mathcal{I} = [y, z],$$

and consider the set $\mathcal{M}_F''$ of squarefree numbers m_1 formed by k primes in $\mathcal{P}_F \cap \mathcal{I}$. Clearly, they satisfy (iii) and (iv) and

$$m_1 \le x^{k/(11 \log\log x)} < x^{1/11},$$

so

$$\frac{x}{2\kappa m_1} > \frac{x^{10/11}}{2\kappa} > x^{9/10}$$

for large x, so

$$[x/(2\kappa m_1), x/(\kappa m_1)] \subseteq [x^{9/10}, x/(\kappa m_1)].$$

As above, given m_1, P can be chosen in

$$\begin{aligned}\pi_F(x/(\kappa m_1)) &- \pi_F(x/(2\kappa m_1))\\ &\asymp \frac{x}{\kappa m_1 \log(x/(\kappa m_1))}\\ &\asymp \frac{x}{m_1 \log x \log\log x}.\end{aligned}$$

ways.

Hence,

$$\#\mathcal{M}_F(x) \gg \frac{x}{\log x \log\log x} \sum_{m_1\in\mathcal{M}_F''} \frac{1}{m_1}.$$

We need a lower bound for the last sum above, and we note that

$$\sum_{m_1\in\mathcal{M}_F''} \frac{1}{m_1} \ge \frac{1}{k!}\left(\sum_{p\in\mathcal{P}_F\cap[y,z]} \frac{1}{p}\right)^k - \sum_{\substack{p\mid n\Rightarrow p\in\mathcal{P}_f\cap[y,z]\\ \Omega(n)=k \text{ and } \mu^2(n)=0}} \frac{1}{n} := S_1 - S_2.$$

Every n appearing in the range of S_2 is of size at most

$$P(n)^{\Omega(n)} \le z^k < x,$$

and divisible by the square of a prime $p > y$. Hence, $n = p^2 m$ for some $m \le x$. It follows that

$$S_2 \le \left(\sum_{y<p\le z} \frac{1}{p^2} \right) \left(\sum_{m \le x} \frac{1}{m} \right) \ll \frac{\log x}{y} = o(1) \quad (x \to \infty).$$

For S_1, we use the same argument as before and get

$$\begin{aligned} S_1 &\gg \frac{1}{\sqrt{k}(k/e)^k} ((\delta_F + o(1)) \log\log z - (\delta_F + o(1)) \log\log y)^k \\ &\gg \frac{1}{\sqrt{\log\log x}} \left(\frac{(e\varepsilon\delta_F + o(1)) \log\log x}{k} \right)^k \gg (\log x)^{\varepsilon\delta_F + o(1)}. \end{aligned}$$

So, we see that $S_2 = o(S_1)$ as $x \to \infty$, therefore

$$\#\mathcal{M}_F(x) \gg \frac{x S_1}{\log x \log\log x} \gg \frac{x}{(\log x)^{1-\varepsilon\delta_F + o(1)}}.$$

Condition F' (Review)

Next, we prove the following lemma.

Lemma (2)

For large x and $m \in \mathcal{M}_F(x)$, there exists $n \in [m, 2m]$ and $j \in \{0, 1, \ldots, \kappa - 2\}$ such that $m \mid F(n + jm)$ and such that furthermore $p \| F(n + jm)$ for each prime factor p of m.

Proof. Explained by my co-author. Essentially it is the Pigeon Hole Principle.

Condition E' (Review)

Now for each $m \in \mathcal{M}_F(x)$, let n_m be the minimal positive integer $\geq m$ such that $m \mid F(n_m)$ and every prime factor p dividing m has the property that $p \| F(n_m)$.
By the Lemma 2, $n_m \leq \kappa m \leq x$. For each n let $z(n)$ be the number of $m \in \mathcal{M}_F(x)$ such that $n = n_m$. We have the following lemma.

Lemma (3)

The subset of $n \in \mathcal{N}_m(x)$ with $z(n) \geq 6D$ is of cardinality at most

$$\frac{x}{(\log x)^{2+O(\varepsilon)}}.$$

Condition G' assuming E'

Lemma 3 is an important ingredient in the proof, yet we are not done. We will prove it later. Let's see how we finish off.

We write the list

$$\mathcal{L} := \{(n_m, m) \text{ for } m \in \mathcal{M}_F(x)\}. \tag{2}$$

The above list has, by Lemma 1,

$$\frac{x}{(\log x)^{1-\varepsilon\delta_F+o(1)}}$$

elements, all of them pairs of the form

$$(n, m),$$

where

$$n \le x, \quad m \mid F(n), \quad \mu^2(m) = 1, \quad \omega(m) = k+1,$$

and

$$p \in [y, x] \cap \mathcal{P}_F \quad \text{for all prime factors} \quad p \mid m.$$

So, let us put

$$\mathcal{J} = [y, x]$$

and try to understand the function $\omega_{\mathcal{J}}(F(n))$, where $\omega_{\mathcal{J}}(u)$ is the number of prime factors of u in the interval $\mathcal{J}$.

We split n in three sets as follows:

(i) $E(x)$ (enormous), which is the set of $n \le x$ for which

$$\omega_{\mathcal{J}}(F(n)) \ge 3D(\log\log x)^2.$$

(ii) $L(x)$ (large), which is the set of $n \le x$ for which

$$\omega_{\mathcal{J}}(F(n)) \in [KD\log\log x, 3D(\log\log x)^2],$$

where K is some constant depending on ε to be determined later.

(iii) $R(x)$ (reasonable), which is the set of $n \le x$ such that

$$\omega_{\mathcal{J}}(F(n)) \le KD\log\log x.$$

For the purpose of this argument, if $s := \omega_{\mathcal{J}}(F(n))$ we will denote by

$$p_1 < p_2 < \cdots < p_s$$

all prime factors of $F(n)$ in $\mathcal{J}$. Since

$$|F(n)| \ll n^D \ll x^D,$$

it follows that in case (i), if we put

$$U := \lfloor (\log\log x)^2 \rfloor,$$

then

$$p_1 \cdots p_U \leq x^{1/2} \quad \text{for large} \quad x.$$

Let $\rho_F : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be the multiplicative function given by

$$\rho_F(d) = \#\{0 \le n \le d-1 : F(n) \equiv 0 \pmod{d}\}.$$

Clearly,

$$\rho_F(u) \le D^{\omega(u)} \quad \text{holds for all squarefree} \quad u.$$

To count E, fix

$$p_1 < p_2 < \cdots < p_U \quad \text{all in} \quad \mathcal{J},$$

and let us count the number $n \le x$ such that $m_1 \mid F(n)$, where $m_1 = p_1 \cdots p_U$. The number of such n is

$$\frac{\rho_F(m_1)}{m_1} x + O(\rho_F(m_1)) \ll \frac{D^{\omega(m_1)}}{m_1} x. \tag{3}$$

For the above inequality, we used that

$$D^{\omega(m_1)} = D^{O(\log m_1/\log\log m_1)} = m_1^{o(1)} \le x^{o(1)},$$

so in the left–hand side of (3), the first term $D^{\omega(m_1)}x/m_1$ dominates because $m_1 \le x^{1/2}$. We sum up over the possible m_1 getting

$$\#E(x) \ll xD^U \sum_{\substack{p|m_1 \Rightarrow p\in[y,x] \\ \mu^2(m_1)=1 \\ \omega(m_1)=U}} \frac{1}{m_1}. \tag{4}$$

The last sum which we denote by S_3, is, by the multinomial coefficient trick,

$$\begin{aligned} S_3 &\le \frac{1}{U!}\left(\sum_{y\le p\le x}\frac{1}{p}\right)^U \ll \left(\frac{(e+o(1))\log\log x}{U}\right)^U \\ &\le \exp\left(-(1+o(1))(\log\log x)^2(\log\log\log x)\right). \end{aligned}$$

For each such n, since

$$|F(n)| \ll n^D \ll x^D,$$

it follows that $\omega_{\mathcal{J}}(F(n)) \leq \log x$ for large x. Thus, the number $m \mid F(n)$ can be chosen in at most

$$\binom{\lfloor \log x \rfloor}{k} \leq (\log x)^k \leq \exp((\log\log x)^2)$$

ways. So, the number of pairs

$$(n, m)$$

among $\mathcal{L}$ with n from $E(x)$ is at most

$$\frac{xD^U}{\exp((1+o(1))\log\log x)^2(\log\log\log x)} \times \exp((\log\log x)^2)$$
$$= \frac{x}{\exp((1+o(1))(\log\log x)^2(\log\log\log x))}.$$

This is $o(\#\mathcal{M}_F(x))$ as $x \to \infty$. So, we dealt with (i).

Let us deal with (ii) now.

We let i_0 and i_1 be maximal and minimal positive integers such that

$$2^{i_0} \le K \quad \text{and} \quad 2^{i_1} \ge 3(\log\log x),$$

respectively. Clearly,

$$i_1 - i_0 = O(\log\log\log x).$$

Consider

$$j \in [i_0, i_1 - 1]$$

and let us look only at those n such that

$$\omega_{\mathcal{J}}(F(n)) \in [2^j D \log\log x, 2^{j+1} D \log\log x].$$

We revisit the previous argument. We now take

$$U := \lfloor 2^{j-1} \log\log x \rfloor,$$

and let

$$m_1 = p_1 \cdots p_U.$$

Then

$$m_1^{2D} \le |F(n)| \ll x^D,$$

therefore $m_1 \ll x^{1/2}$. The argument used to prove (4) shows that

$$\#L(x) \ll xD^U \sum_{\substack{p|m_1 \Rightarrow p \in [y,x] \\ \mu^2(m_1)=1 \\ \omega(m_1)=U}} \frac{1}{m_1}. \qquad (5)$$

Let S_4 be the last sum above. Then

$$\begin{aligned} S_4 &\ll \frac{1}{U!}\left(\sum_{y \le z \le x} \frac{1}{p}\right)^U \\ &\ll \left(\frac{(e+o(1))\log\log x}{U}\right)^U \\ &\ll \frac{1}{(\log x)^{2^{j-1}\log(2^{j-1}/e+o(1))}} \end{aligned}$$

where we used that $2^{j-1}/e \ge K/(4e) > e$ for $K > 4e^2$.

Thus,

$$\begin{aligned}\#L(x) &\ll xD^U S_4 \ll x(\log x)^{2^{j-1}\log D} S_4 \\ &\ll \frac{x}{(\log x)^{2^{j-1}\log(2^{j-1}/eD+o(1))}}.\end{aligned}$$

Since $\omega_{\mathcal{J}}((F(n)) \le 4U+4$, it follows that the number of choices for m is at most

$$\begin{aligned}\binom{4U+4}{k+1} &\le \left(\frac{2^{j+2}}{\delta_F\varepsilon}+o(1)\right)^{\delta_F\varepsilon\log\log x+O(1)} \\ &\ll (\log x)^{\delta_F\varepsilon\log(2^{j+3}/\delta_F\varepsilon)}\end{aligned}$$

for large x. Thus, the number of pairs (n, m) in the list $\mathcal{L}$ coming from $n \in L(x)$ with a fixed j, is

$$\ll \frac{x}{(\log x)^{2^{j-1}\log(2^{j-2}/eD)-\varepsilon\delta_F\log(2^{j+3}/\delta_F\varepsilon)}}$$

The exponent above is

$$2^{j-1}(\log 2)j(1-\varepsilon\delta_F+O(\log(1/\varepsilon)/j))$$

where the constant implied by O depends on D. Since

$$2^j \geq 2^{i_0} \geq K/2,$$

we have

$$j \geq \log(K/2)/\log 2, \quad \text{so if} \quad K \geq (1/\varepsilon)^{O(1)}$$

is sufficiently large, then the factor

$$1-\varepsilon\delta_F+O(\log(1/\varepsilon)/j) \geq 1/2.$$

Thus, the number of such pairs for a fixed j is

$$\ll \frac{x}{(\log x)^{2^{j-2}j}}.$$

Summing over j, this sum is dominated by the first term, so if $j \geq 2$ (that is, $K \geq 8$), then the number of such pairs is

$$O\left(\frac{x}{(\log x)^2}\right).$$

It remains to deal with $n \in R(x)$. If

$$n \in R(x), \quad \text{then} \quad \omega_{\mathcal{J}}(F(n)) \leq KD \log\log x.$$

Thus, the number of m's such that (n, m) is in $\mathcal{L}$ for fixed $n \in R(x)$ is

$$\leq \binom{\lfloor KD \log\log x \rfloor}{k+1} = (\log x)^{O(\varepsilon \log(1/\varepsilon))}.$$

By the Lemma 3, the number of n with $z(n) \geq 6D$ is

$$O\left(\frac{x}{(\log x)^{2+O(\varepsilon)}}\right).$$

Hence, since the number of m is $(\log x)^{O(\varepsilon \log(1/\varepsilon)))}$, it follows that the number of pairs (n, m) with $n \in R(x)$ and $z(n) \geq 6D$ in $\mathcal{L}$ is

$$\ll \frac{x}{(\log x)^{2+O(\varepsilon \log(1/\varepsilon))}}.$$

Now take $\varepsilon = \varepsilon_0$ such that the above exponent of the logarithm is ≥ 1. Then the number of such pairs is

$$O\left(\frac{x}{\log x}\right).$$

Since

$$\#\mathcal{M}_F(x) = x/(\log x)^{1-\delta_F\varepsilon_0+o(1)},$$

it follows that for large x, at least half of the pairs in $\mathcal{L}$ will have

$$z(n) \leq 6D.$$

Now we are done.

It remains to prove the Lemma 3.

The proof of Lemma 3

We keep the previous notations, especially

$$y = \exp((\log x)^{1-\varepsilon})$$

and let A be the set

$$A = \{m : \mu^2(m) = 1,\ p \mid m \Rightarrow p \geq y \text{ and } p \in \mathcal{P}_F\}.$$

We study $A(t)$ for $t \in [y, x]$. We have

Lemma (4)

Uniformly for $t \in [y, x]$, we have

$$\#A(t) \leq \frac{t}{(\log x)^{1+O(\varepsilon)}}.$$

Here and in what follows, the constants implied by O depend on δ_F and D.

Proof.

Let $g(n)$ be the characteristic function of A. By a classical result

$$\#A(t) = \sum_{n \le t} g(n) \ll \frac{t}{\log t} \sum_{n \in A(t)} \frac{1}{n}. \tag{6}$$

Clearly, $\log t = (\log x)^{1+O(\varepsilon)}$ for $t \in [y, x]$. As for the sum above, we have

$$\begin{aligned} S_5 &= \sum_{n \in A(t)} \frac{1}{n} \le \prod_{\substack{y \le p \le t \\ p \in \mathcal{P}_F}} \left(1 + \frac{1}{p}\right) \le \exp\left(\sum_{\substack{y \le p \le t \\ p \in \mathcal{P}_F}} \frac{1}{p} + O\left(\sum_{p \ge y} \frac{1}{p^2}\right)\right) \\ &\le \exp((\delta_F + o(1)) \log\log t - (\delta_F + o(1)) \log\log y + O(1/y)) \\ &\le (\log x)^{O(\varepsilon)}, \end{aligned}$$

which together with (6) finishes the proof. □

Lemma (5)

Uniformly for $y \le a \le b \le x$, we have

$$\sum_{\substack{a \le n \le b \\ n \in A}} \frac{1}{n} \le \frac{\log b - \log a + 1}{(\log x)^{1+O(\varepsilon)}}.$$

Proof.

This is just Abel summation formula. Indeed,

$$\sum_{\substack{a \le n \le b \\ n \in A}} \frac{1}{n} = \left(\frac{\#A(t)}{t} \Big|_{t=a}^{t=b} \right) - \int_a^b \frac{\#A(t)}{t^2} dt.$$

In the first term we have

$$\left(\frac{\#A(t)}{t} \Big|_{t=a}^{t=b} \right) \le \frac{\#A(b)}{b} \ll \frac{1}{(\log x)^{1+O(\varepsilon)}},$$

by Lemma 4.

Also by Lemma 4,

$$\int_a^b \frac{\#A(t)}{t^2} dt \ll \frac{1}{(\log x)^{1+O(\varepsilon)}} \int_a^b \frac{dt}{t} = \frac{\log b - \log a}{(\log x)^{1+O(\varepsilon)}}.$$

Lemma 5 now follows. □

The proof of Lemma 3

Suppose that

$$z(n) \geq 6D.$$

Thus, there are $6D$ different m's such that $n = n_m$. Each of them has

$$P = P(m) \geq x^{9/10}.$$

Let s be the number of such P's. Then

$$x^{9s/10} \leq |F(n)| \ll x^D,$$

so

$$s \leq 10D/9 + o(1) \quad \text{as} \quad (x \to \infty).$$

In particular

$$s < 2D$$

for large x. Since $z(n) \geq 6D$, it follows that there exists P and $m_1,\ m_2,\ m_3$ with these last three numbers distinct in $\mathcal{M}'_F(x)$ such that

$$n = n_m \quad \text{for each of} \quad m \in \{Pm_1, Pm_2, Pm_3\}.$$

Let's forget about P and just keep the condition that

$$m_i \mid F(n) \quad \text{for} \quad i = 1, 2, 3.$$

Since $n \le x$, this shows that the number of such n is at most

$$\frac{\rho_F([m_1, m_2, m_3])}{[m_1, m_2, m_3]} x + O(\rho_F([m_1, m_2, m_3])).$$

For us,

$$m_i \le x^{1/10} \quad \text{for} \quad i = 1, 2, 3, \quad \text{so} \quad [m_1 m_2, m_3] \le x^{3/10} \le x^{1/2}.$$

Further,

$$\omega([m_1, m_2, m_3]) \le 3k = O(\varepsilon \log\log x),$$

therefore

$$\rho_F([m_1, m_2, m_3]) \le D^{\omega([m_1, m_2 m_3]} = (\log x)^{O(\varepsilon)}.$$

Hence, the number of our n is

$$\ll x(\log x)^{O(\varepsilon)} \frac{1}{[m_1, m_2, m_3]}.$$

It remains to study the sum

$$S_6 := \sum_{m_1, m_2, m_3} \frac{1}{[m_1, m_2, m_3]}.$$

This is what remains. We shall ignore multiplicative factors of size

$$(\log x)^{O(\varepsilon)}$$

from now on. Since m_1, m_2, m_3 are squarefree with the same number of prime factors, it follows that

$$m_i < [m_i, m_j] \qquad \text{for all} \qquad i < j \in \{1, 2, 3\}.$$

We distinguish two cases.

Case 1. *There is some relabeling of the indices such that*

$$[m_1, m_2] \neq [m_1, m_2, m_3].$$

Fix m_1, m_2. Then

$$m_3 \nmid [m_1, m_2].$$

Fix u such that

$$u = \gcd(m_3, [m_1, m_2]).$$

With m_1, m_2 being fixed, u is fixed in only

$$(\log x)^{O(\varepsilon)}$$

ways as a divisor of $[m_1, m_2]$. Since

$$Pm_1, Pm_2, Pm_3 \quad \text{are all in} \quad [x/(2\kappa), x/\kappa],$$

it follows that

$$m_j/2 \leq m_i \leq 2m_j \quad \text{holds for all} \quad i, j.$$

Now write

$$m_3 = ud.$$

Since u is fixed, we get

$$m_1/(2u) \le d \le 2(m_1/u).$$

Since u is a proper divisor of m_1, we have $d \ge y/2$. Then

$$[m_1, m_2, m_3] = [m_1, m_2]d,$$

so the sum while keeping m_1, m_2, u fixed and summing up over all possible numbers d, we get

$$\begin{aligned} \sum_{m_1, m_2, u \text{ fixed}} \frac{1}{[m_1, m_2, m_3]} &\le \frac{1}{[m_1, m_2]} \sum_{u \mid [m_1, m_2]} \sum_{\substack{m_1/(2u) \le d \le 2(m_1/u) \\ d \in A}} \frac{1}{d} \\ &\ll \frac{1}{[m_1, m_2](\log x)^{1+O(\varepsilon)}} \sum_{u \mid [m_1, m_2]} 1 \\ &\ll \frac{1}{(\log x)^{1+O(\varepsilon)}[m_1, m_2]}. \end{aligned}$$

In the above, we applied Lemma 5 with the choices

$$b = 2m_1/u, \; a = m_1/(2u)$$

in the inner sums. We now fix m_1 and vary m_2. To this end, we fix

$$v = \gcd(m_1, m_2),$$

and let

$$m_2 = vd'.$$

Then $d' > 1$ otherwise $m_1 = m_2$ because m_1 and m_2 are squarefree and they have the same number of prime factors k. Thus, since

$$m_1/2 \le m_2 \le 2m_1, \quad \text{we get} \quad m_1/(2v) \le d' \le 2m_1/v.$$

As before, $d' \in A$ and $d' \geq y/2$.

Keeping m_1 fixed, we get

$$\sum \frac{1}{[m_1, m_2]} = \frac{1}{m_1} \sum_{\substack{v|m_1 \\ v<m_1}} \sum_{\substack{m_1/(2v) \le d' \le 2m_1/v \\ d' \in A}} \frac{1}{d'}$$

$$\ll \frac{1}{m_1 (\log x)^{1+O(\varepsilon)}} \sum_{v|m_1} 1$$

$$\ll \frac{1}{m_1 (\log x)^{1+O(\varepsilon)}}.$$

Putting everything together, we get that the set of n that fall into such a case is

$$\ll \frac{x}{(\log x)^{2+O(\varepsilon)}} \sum_{m_1 \in \mathcal{M}'_F} \frac{1}{m_1}.$$

The proof of Lemma 1 tells us that the last sum is $(\log x)^{O(\varepsilon)}$. So, the set of n in this category is of cardinality

$$\frac{x}{(\log x)^{2+O(\varepsilon)}},$$

Case 2. $[m_1, m_2] = [m_1, m_3] = [m_2, m_3] = [m_1, m_2, m_3]$.

Write

$$m_1 = du, \; m_2 = dv, \quad \text{where} \quad d = \gcd(m_1, m_2).$$

Then

$$u > 1, \; v > 1 \quad \text{and} \quad \gcd(u, v) = 1.$$

Hence,

$$[m_1, m_2] = duv.$$

Since

$$m_3 \mid duv \quad \text{and} \quad m_2 \mid [m_1, m_3],$$

we get that $v \mid m_3$. Similarly,

$$u \mid m_3 \quad \text{so} \quad m_3 = d' uv, \quad \text{where} \quad d' \mid d.$$

Let

$$d = d' d''.$$

Since

$$m_1 \asymp m_2, \quad \text{we get that} \quad u \asymp v.$$

Since

$$d'd''u = m_1 \asymp m_3 = d'uv, \quad \text{we get} \quad d'' \asymp v.$$

Further, given

$$[m_1, m_2, m_3] = d'd''uv,$$

the number of ways of choosing $m_1,\ m_2$ and m_3 is

$$(\log x)^{O(\varepsilon)}.$$

Hence,

$$\sum_{m_1,m_2,m_3} \frac{1}{[m_1, m_2, m_3]} = (\log x)^{O(\varepsilon)} \sum_{\substack{d',d'',u,v \in A \\ d'' \asymp u \asymp v}} \frac{1}{d'd''uv}.$$

Keeping $d', \ d''$ fixed and summing over $u \asymp d''$ and $v \asymp d''$, we get

$$\frac{(\log x)^{O(\varepsilon)}}{d'd''}\left(\sum_{u \asymp d''} \frac{1}{u}\right)^2 \ll \frac{1}{d'd''(\log x)^{2+O(\varepsilon)}},$$

by Lemma 5. Now $d'd'' = d \in A$ is in $[y, x]$. Further, given d there are $(\log x)^{O(\varepsilon)}$ possibilities for d' and d''. Hence,

$$\frac{1}{(\log x)^{2+O(\varepsilon)}} \sum_{\substack{d',d'' \in A \\ (d',d'')=1}} \frac{1}{d'd''} = \frac{1}{(\log x)^{2+O(\varepsilon)}} \sum_{\substack{d \in A \\ d \le x}} \frac{1}{d} \ll \frac{1}{(\log x)^{2+O(\varepsilon)}},$$

where we used again Lemma 5 with

$$b = x, \quad a = y$$

to deduce that the last inner sum in the middle above is $(\log x)^{O(\varepsilon)}$. Lemma 3 is proved and we are done.

Not quite the same but of a similar flavor

Let $A_0 = 1$ and $A_n = \lfloor en! \rfloor$ for $n \geq 1$. For $m \geq 2$, let

$$S_m(N) = \#\{\mathbb{Q}(A_n^{1/m}) : 1 \leq n \leq N\}.$$

In **2007**, L., Shparlinski proved the following result.

Theorem

We have:

(i)

$$\#S_2(N) \geq (\log N)^{1/3+o(1)} \quad \text{as} \quad N \to \infty.$$

(ii)

$$\#S_m(N) \geq N^{1/2m+o(1)} \quad \text{as} \quad N \to \infty$$

uniformly in $3 \leq m \leq \log N / \log\log N$.

Happy birthday Igor!