Primes, exponential sums, and L-functions

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Dedicated to Igor Shparlinski
in honor of his 60th year

Beatty primes

For fixed $\alpha, \beta \in \mathbb{R}$ the non-homogeneous Beatty sequence is defined by

$$\mathcal{B}_{\alpha,\beta} = \{\lfloor \alpha n + \beta \rfloor \ : \ n \geqslant 1\}$$

$\mathcal{B}_{\alpha,\beta}$ is often called a generalized arithmetic progression

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$$\mathbb{P}_{\alpha,\beta} = \{\text{primes}\} \cap \mathcal{B}_{\alpha,\beta}$$

For example, when $\alpha > 1$ is irrational, the prime counting function

$$\pi_{\alpha,\beta}(x) = \#\{p \leqslant x : p \in \mathbb{P}_{\alpha,\beta}\}$$

satisfies the expected asymptotic formula

$$\pi_{\alpha,\beta}(x) \sim \alpha^{-1}\pi(x) \qquad (x \to \infty)$$

Exponential sums with Beatty primes...

Theorem (B.–Shparlinski)

Let γ *be irrational of type* $\tau < \infty$*. For any* $\varepsilon \in (0, \frac{1}{8\tau})$ *there is a number* $\eta > 0$ *such that*

$$\Bigg| \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \Lambda(n)\, \mathrm{e}(\gamma k n) \Bigg| \leqslant x^{1-\eta}$$

holds for all $k \leqslant x^{\varepsilon}$ *and* $0 \leqslant a < q \leqslant x^{\varepsilon/4}$ *with* $\gcd(a, q) = 1$ *provided that* x *is large*

Distribution of Beatty primes in arithmetic progressions...

Theorem (B.–Shparlinski)

Let $\alpha, \beta \in \mathbb{R}$ with α positive, irrational, and of finite type. There is a constant $\kappa > 0$ such that for all integers $0 \leqslant a < q \leqslant x^\kappa$ with $\gcd(a, q) = 1$, the bound

$$\sum_{\substack{n \leqslant x \\ \lfloor \alpha n + \beta \rfloor \equiv a \bmod q}} \Lambda(\lfloor \alpha n + \beta \rfloor) = \alpha^{-1} \sum_{\substack{m \leqslant \lfloor \alpha x + \beta \rfloor \\ m \equiv a \bmod q}} \Lambda(m) + O(x^{1-\kappa})$$

holds, where the implied constant depends only on α and β

Beatty primes

Towards the k-tuple conjecture on average...

Theorem (Hao–Pan)

Fix $\beta \in \mathbb{R}$. For almost all irrational $\alpha > 0$ (in the sense of Lebesgue measure) one has

$$\limsup_{x\to\infty} \frac{\pi^2_{\alpha,\beta}(x)}{(x/\log^2 x)} \geqslant 1$$

where

$$\pi^2_{\alpha,\beta}(x) = \#\{p \leqslant x : \text{both } p \text{ and } \lfloor \alpha p + \beta \rfloor \text{ are prime}\}$$

Analogue of the Vinogradov three-prime theorem...

Theorem (B.–Güloğlu–Nevans)

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational and of finite type. Then,

(i) *Almost all even numbers can be expressed as the sum of two primes from $\mathbb{P}_{\alpha,\beta}$ if and only if $\alpha < 2$.*

(ii) *For every integer $k \geqslant 3$, any sufficiently large number with the same parity as k can be expressed as a sum of k primes from $\mathbb{P}_{\alpha,\beta}$ if and only if $\alpha < k$.*

Zeta function attached to a Beatty sequence...

Theorem (B.)

Let $\alpha, \beta \in \mathbb{R}$ *with* $\alpha > 1$. *For each* $n \geqslant 1$ *let* p_n *denote the* n*-th smallest prime. Let* $\mathbb{P}^\star_{\alpha,\beta} = \{\text{prime } p_n : n \in \mathcal{B}_{\alpha,\beta}\}$. *The function*

$$\zeta_{\alpha,\beta}(s) = \prod_{p \in \mathbb{P}^\star_{\alpha,\beta}} (1 - p^{-s})^{-\alpha} \quad (\sigma > 1)$$

extends to a meromorphic function in the region $\{\sigma > 0\}$. *There is a function* $f_{\alpha,\beta}(s)$, *analytic in* $\{\sigma > 0\}$, *such that*

$$\zeta_{\alpha,\beta}(s) = \zeta(s) \exp(f_{\alpha,\beta}(s)) \quad (\sigma > 0).$$

In particular, the Riemann hypothesis is true if and only if $\zeta_{\alpha,\beta}(s) \neq 0$ *in* $\{\sigma > \frac{1}{2}\}$

Piatetski-Shapiro sequences

Piatetski-Shapiro sequences are sequences of the form

$$\big(\lfloor n^c \rfloor\big)_{n\in\mathbb{N}} \qquad (c > 1,\ c \notin \mathbb{N}).$$

They are named in honor of Piatetski-Shapiro, who proved that for any number $c \in (1, \frac{12}{11})$ there are infinitely many primes of the form $\lfloor n^c \rfloor$.

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The admissible range for c in this result has been extended many times over the years and is currently known for all $c \in (1, \frac{243}{205})$ thanks to the work of Rivat and Wu.

Piatetski-Shapiro sequences

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Squarefree numbers in the P-S sequence...

Theorem (Baker–B.–Brüdern–Shparlinski–Weingartner)

For any $c \in (1, \frac{149}{87})$ *we have*

$$\#\{n \leqslant x : \lfloor n^c \rfloor \text{ is squarefree}\} = \frac{6}{\pi^2}x + O(x^{1-\varepsilon})$$

On the largest prime factor of $\lfloor n^c \rfloor$...

Theorem (Baker–B.–Brüdern–Shparlinski–Weingartner)

For any number $c \in (1, \frac{24979}{20803})$ we have

$$\#\{n \leqslant x : P(\lfloor n^c \rfloor) \leqslant n^\varepsilon\} \gg x^{1-\varepsilon}$$

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Theorem (Baker–B.–Brüdern–Shparlinski–Weingartner)

There is a positive function $\Theta(c)$ with the property that, for any non-integer $c > 1$ and any real $\varepsilon > 0$, the inequality

$$P(\lfloor n^c \rfloor) > n^{\Theta(c)-\varepsilon}$$

holds for infinitely many n

Piatetski-Shapiro sequences

For prime N, Fermat's little theorem asserts that

$$a^N \equiv a \pmod{N} \qquad \text{for all } a \in \mathbb{Z}.$$

Around 1910, Carmichael began the study of composite numbers N with this property, which are now known as Carmichael numbers

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In 1994 the existence of infinitely many Carmichael numbers was established by Alford, Granville and Pomerance

Carmichael numbers composed of P-S primes...

Theorem (Baker–B.–Brüdern–Shparlinski–Weingartner)

For every $c \in \left(1, \frac{147}{145}\right)$ *there are infinitely many Carmichael numbers composed solely of primes in the Piatetski-Shapiro sequence* $(\lfloor n^c \rfloor)_{n \in \mathbb{N}}$

For any integer $R \geqslant 1$, let P_R be the set of *R-almost primes*, i.e., the set of natural numbers having at most R prime factors, counted with multiplicity

Piatetski-Shapiro sequences

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Generating primes from almost primes...

Theorem (Baker–B.–Guo–Yeager)

For any fixed $c \in (1, \frac{77}{76})$ *we have*

$$\#\{n \leqslant x : n \in P_8 \text{ and } \lfloor n^c \rfloor \text{ is prime}\} \gg \frac{x}{(\log x)^2},$$

where the implied constant in the symbol $\gg$ *depends only on* c

Piatetski-Shapiro sequences

R	c_R	R	c_R	R	c_R
8	1.0521	12	1.1649	16	1.2073
9	1.1056	13	1.1780	17	1.2148
10	1.1308	14	1.1891	18	1.2214
11	1.1494	15	1.1988	19	1.2273

Generating almost primes from primes...

Theorem (B.–Guo–Shparlinski)

Let (R, c_R), $R = 8, \ldots, 19$, *be a pair from the table above. For any fixed* $c \in (1, c_R]$ *we have*

$$\#\{\text{prime } p \leqslant x : \lfloor p^c \rfloor \in P_R\} \gg \frac{x}{\log^2 x}$$

where the implied constant in the symbol $\gg$ *depends only on* c

Generating almost primes from primes (cont'd)...

Theorem (B.–Guo–Shparlinski)

For fixed $c \geqslant \frac{11}{5}$ there is a positive integer

$$R \leqslant \begin{cases} 16c^3 + 179c^2 & \text{if } c \in [\frac{11}{5}, 3), \\ 16c^3 + 88c^2 & \text{if } c \geqslant 3, \end{cases}$$

we have

$$\#\{\text{prime } p \leqslant x : \lfloor p^c \rfloor \in P_R\} \gg \frac{x}{\log^2 x}$$

where the implied constant in the symbol $\gg$ depends only on c

Primes that are simultaneously Beatty and P-S...

Theorem (Guo)

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational and of finite type. Let $c \in (1, \frac{14}{13})$. There are infinitely many primes that lie in both the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ and the Piatetski-Shapiro sequence $\mathcal{N}^{(c)} = (\lfloor n^c \rfloor)_{n \in \mathbb{N}}$. Moreover, the counting function

$$\pi_{\alpha,\beta}^{(c)}(x) = \#\{\text{prime } p \leqslant x : p \in \mathcal{B}_{\alpha,\beta} \cap \mathcal{N}^{(c)}\}$$

satisfies

$$\pi_{\alpha,\beta}^{(c)}(x) = \frac{x^{1/c}}{\alpha \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right),$$

where the implied constant depends only on α and c

Every hundredth prime

Let $\mathbb{P}$ denote the set of primes

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Given $\delta \in (0,1]$, $\sigma_0 \in [0,1)$ and a real function $\varepsilon(x)$ such that $\varlimsup_{x\to\infty} \varepsilon(x) \leqslant 0$, let $\mathcal{A}(\delta,\sigma_0,\varepsilon)$ denote the class consisting of sets of primes $\mathcal{P} \subseteq \mathbb{P}$ for which one has an estimate of the form

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big(x^{\sigma_0+\varepsilon(x)}\big),$$

where the implied constant may depend on $\mathcal{P}$

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Let $\mathcal{B}(\delta, \varepsilon)$ denote the class consisting of sets of primes $\mathcal{P} \subseteq \mathbb{P}$ for which the stronger estimate

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big((\log\log x)^{\varepsilon(x)}\big),$$

holds, where again the implied constant may depend on $\mathcal{P}$

More analogues of the zeta function...

Theorem (B.)

For any set $\mathcal{P} \in \mathcal{A}(\delta, \sigma_0, \varepsilon)$, the function $\zeta_{\mathcal{P}}(s)$ defined by

$$\zeta_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1/\delta} \qquad (\sigma > 1)$$

extends to a meromorphic function in the region $\{\sigma > \sigma_0\}$, and there is a function $f_{\mathcal{P}}(s)$ which is analytic in $\{\sigma > \sigma_0\}$ and has the property that

$$\zeta_{\mathcal{P}}(s) = \zeta(s) \exp(f_{\mathcal{P}}(s)) \qquad (\sigma > \sigma_0)$$

Every hundredth prime

Exact asymptotic bases...

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Every set $\mathcal{P} \in \mathcal{B}(\delta, \varepsilon)$ containing the prime 2 is an exact asymptotic additive basis for $\mathbb{N}$. In other words, there is an integer $h = h(\mathcal{P}) > 0$ such that the h-fold sumset

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From work of Sárközy it is known that any $\mathcal{P} \in \mathcal{B}(\delta, \varepsilon)$ is an asymptotic additive basis for $\mathbb{N}$, and stronger quantitative versions are known. To prove that $\mathcal{P}$ is exact, we use Shiu's theorem on strings of primes in an arithmetic progression.

Every hundredth prime

For example, an exact asymptotic additive basis for $\mathbb{N}$ is provided by the set

$$\{2, 547, 1229, 1993, 2749, 3581, 4421, 5281 \ldots\},$$

which consists of 2 and every hundredth prime thereafter

Least quadratic nonresidue

Legendre symbol

$$(n|p) := \begin{cases} +1 & \text{if } n \equiv m^2 \bmod p \text{ for some } m \not\equiv 0 \bmod p \\ -1 & \text{if } n \not\equiv m^2 \bmod p \text{ for all } m \in \mathbb{Z} \\ 0 & \text{if } p \mid n \end{cases}$$

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Unconditional bounds on $n_1(p)$

- Gauss (1801): $n_1(p) < 2\sqrt{p} + 1$ if $p \equiv 1 \bmod 8$
- Vinogradov (1918): $n_1(p) \ll p^{\kappa}$ for any $\kappa > 1/(2\sqrt{e})$
- Burgess (1957): $n_1(p) \ll p^{\kappa}$ for any $\kappa > 1/(4\sqrt{e})$

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Vinogradov's Conjecture $n_1(p) \ll p^\varepsilon$ *for any* $\varepsilon > 0$.

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Vinogradov's Conjecture *$n_1(p) \ll p^{\varepsilon}$ for any $\varepsilon > 0$.*

Conditional bounds on $n_1(p)$

- Linnik (1944): On ERH, the Vinogradov conjecture is true
- Ankeny (1952): On ERH, one has $n_1(p) \ll (\log p)^2$

Tightness of the Burgess bound leads to zeros of L-functions close to one...

Theorem (Heath-Brown)

Suppose that $n_1(p) \geqslant p^{1/(4\sqrt{e})}$ for infinitely many primes p. Then, for every root z of the function

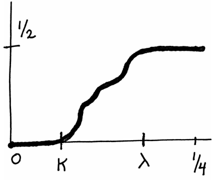
$$H(z) := \frac{2}{z}\int_{1/\sqrt{e}}^{1} \left(1 - e^{-zu}\right)\frac{du}{u},$$

there is an infinite set of primes $\mathcal{P}$ and a sequence $(s_p)_{p\in\mathcal{P}}$ such that

- $L(s_p, (\cdot|p)) = 0$ *for all* $p \in \mathcal{P}$
- $(s_p - 1)\log p \to -4z$ *as* $p \to \infty$

Generalization of Heath-Brown's theorem

- Fix κ, λ with $0 < \kappa < \lambda \leqslant 1/4$
- For any odd prime p, put $\mathcal{N}_p(X) := \{n \leqslant X : (n|p) = -1\}$
- Assume there are infinitely many primes p such that
  - $n_1(p) \geqslant p^\kappa$
  - $|\mathcal{N}_p(p^\theta)| = (\delta(\theta) + o(1))p^\theta$ as $p \to \infty$



where $\delta(\theta)$ is a function of the form

$$\delta(\theta) := \tfrac{1}{2} \int_0^\theta \underline{\mathbf{d}}(u)\, du$$

and $\underline{\mathbf{d}}(u)$ is a probability distribution, supported on $[\kappa, \lambda]$, twice-differentiable on (κ, λ), with $\underline{\mathbf{d}}(\kappa)\, \underline{\mathbf{d}}(\lambda) \neq 0$.

Under the preceding hypotheses, we have:

Theorem (B.–Makarov)

For every solution k to the equation

$$\widehat{\underline{\mathbf{d}}}(k) = 1,$$

there is an infinite set of primes $\mathcal{P}$ and a sequence $\left(s_p\right)_{p\in\mathcal{P}}$ such that

- $L(s_p, (\cdot|p)) = 0$ *for all* $p \in \mathcal{P}$
- $(s_p - 1)\log p \to -ik$ *as* $p \to \infty$

$\widehat{\underline{\mathbf{d}}}$ is the Fourier transform of $\underline{\mathbf{d}}$

Density of residues

Confirming a conjecture of Heath-Brown, in 1996 Hall proved

Theorem (Hall)

There exists an absolute constant $c > 0$ such that for all $N \geqslant 1$ and all primes p, the interval $[1, N]$ contains at least cN quadratic residues mod p.

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Theorem (Granville–Soundararajan)

One can take $c = 0.1715$ in the statement of Hall's theorem if N is large enough.

For any $N \geqslant 1$ one can find a prime p for which $[1, N]$ is free of nonresidues mod p.

Density of nonresidues

Positive density of nonresidues in the Burgess range...

Theorem (B.–Garaev–Heath-Brown–Shparlinski)

Given $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ with the following property. For every sufficiently large prime p and every integer $N \geqslant p^{1/(4\sqrt{e})+\varepsilon}$, the interval $[1, N]$ contains at least $c(\varepsilon)N$ quadratic nonresidues mod p.

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Prime nonresidues in the Burgess range...

Theorem (Pollack)

For each $\varepsilon > 0$ there are numbers $q_0 = q_0(\varepsilon)$ and $\kappa = \kappa(\varepsilon) > 0$ such that the following holds. For all $q > q_0$ and any nontrivial character χ mod q, there are more than q^κ prime χ-nonresidues not exceeding $q^{1/(4\sqrt{e})+\varepsilon}$.

Below the Burgess bound

Theorem (B.–Guo)

The bound

$$n_k(p) \ll p^{(4\sqrt{e})^{-1}} \exp\left(\sqrt{e^{-1} \log p \log\log p}\right)$$

holds for all odd primes p and all $k \geqslant 1$ such that

$$k \ll p^{(8\sqrt{e})^{-1}} \exp\left(\tfrac{1}{2}\sqrt{e^{-1} \log p \log\log p} - \tfrac{1}{2}\log\log p\right),$$

where the implied constants are absolute

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Our work relies on results of Granville and Soundararajan from

- "The spectrum of multiplicative functions"
- "Large character sums: Burgess's theorem and zeros of L-functions"

Character sums with prime-power moduli

Following Postnikov, Gallagher, Iwaniec, Chang and others, Igor and I have been studying character sums χ mod q, where the modulus q is a large power of a fixed prime p

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Our results should extend to moduli q for which the kernel $q' = \prod_{p|q} p$ is small

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Among other things, we obtain slightly stronger estimates for short character sums, a wider zero-free region for $L(s, \chi)$, and stronger bounds for $\big|L(s, \chi)\big|$ when s is close to one

Character sums with prime-power moduli

Theorem (B.–Shparlinski)

Let p be an odd prime and χ a primitive character mod $q = p^\gamma$. Then

$$L(1,\chi) \ll (\log q)^{2/3}(\log\log q)^{1/3},$$

where the implied constant depends only on p.


Happy Birthday
Igor