Fractional Schrödinger Equation: Stationary States and Dynamics

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Outline

Fractional Schrödinger equation

Consider the fractional nonlinear Schrödinger equation:

$$i\partial_t\psi(\mathbf{x},t) = \frac{1}{2}(-\Delta)^{\alpha/2}\psi + V(\mathbf{x})\psi + \gamma|\psi|^2\psi, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0,$$
$$\psi(\mathbf{x},0) = \psi_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d,$$

where

- $\psi(\mathbf{x},t)$: Complex-valued wave function
- $(-\Delta)^{\alpha/2}$: Fractional Laplacian
- $V(\mathbf{x})$: Real-valued external trapping potential
- $\gamma\in\mathbb{R}$: Strength of particle interactions

Fractional Laplacian

From a probabilistic point of view, it represents an infinitesimal generator of a symmetric α-stable Lévy process.

It can be defined in two different forms:

1. Pseudo-differential representation:

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) := \mathcal{F}^{-1}\left[|\xi|^{\alpha}\mathcal{F}(u)\right], \qquad \alpha > 0.$$

   where $\mathcal{F}$ represents Fourier transform, and $\mathcal{F}^{-1}$ is its inverse.

   Note:

   - This definition is usually used for problems defined on the entire domain $\mathbb{R}^d$ or a bounded domain Ω with periodic boundary conditions.
   - If $\alpha = 2$, $-(-\Delta)^{\alpha/2}$ reduces to the Laplace operator $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$.

Fractional Laplacian

From a probabilistic point of view, it represents an infinitesimal generator of a symmetric α-stable Lévy process.

It can be defined in two different forms:

1. Hypersingular integral representation:

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) = C_{d,\alpha}\,\text{P.V.}\int_{\mathbb{R}^d}\frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\,d\mathbf{y}, \qquad 0<\alpha<2,$$

where P.V. stands for principal value, and $C_{d,\alpha}$ is a normalization constant:

$$C_{d,\alpha} = \frac{2^{2\alpha}\alpha\Gamma(\alpha+d/2)}{\pi^{d/2}\Gamma(1-\alpha)}.$$

Fractional Laplacian

Remarks.

1. In the literature, the fractional Laplacian is sometimes referred to as

$$(-\Delta)_s^{\alpha/2} u(\mathbf{x}) = \sum_{k\in\mathbb{N}^d} c_k \, \lambda_k^{\alpha/2} \varphi_k(\mathbf{x}), \qquad \alpha > 0,$$

where (λ_k, φ_k) satisfies the eigenvalue problem:

$$\begin{aligned} -\Delta\varphi_k(\mathbf{x}) &= \lambda_k \, \varphi_k(\mathbf{x}), && \mathbf{x} \in \Omega, \\ \varphi_k(\mathbf{x}) &= 0, && \mathbf{x} \in \partial\Omega. \end{aligned}$$

with the normalization condition $\|\varphi_k(\mathbf{x})\|_{L^2(\Omega)} = 1$.

It $((-\Delta)_s^{\alpha/2})$ is called *the fractional power of the Laplacian operator*, or *the spectral fractional Laplacian*.

2. In this talk, we will consider the fractional Laplacian in the hypersingular integral form.

Fractional Schrödinger equation

Conservation properties:

- L_2 norm, or the total mass:

$$\begin{aligned} N(\psi) &:= \int_{\mathbb{R}^d} |\psi(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |\psi_0(\mathbf{x},t)|^2 d\mathbf{x} \\ &= N(\psi_0), \quad t \geq 0. \end{aligned}$$

- Hamiltonian, or the total energy:

$$\begin{aligned} E(\psi) &:= \int_{\mathbb{R}^d} \left[\frac{1}{2} \left| \nabla^{\alpha/2}\psi \right|^2 + V(\mathbf{x})|\psi|^2 + \frac{\gamma}{2}|\psi|^4 \right] d\mathbf{x} \\ &= E(\psi_0), \quad t \geq 0, \end{aligned}$$

where the fractional operator $\nabla^s = -(-\Delta)^{s/2}$.

Fractional Schrödinger equation

- Fractional quantum mechanics:

  The (fractional) Schrödinger equation was proposed as a fundamental model of (fractional) quantum mechanics.

  - *Fractional quantum mechanics and Lévy path integrals*, N. Laskin, Phys. Lett. A, **268** (2000) 298–305.
  - *Fractals and quantum mechanics*, N. Laskin, Chaos, **10** (2000) 780–790.

- Experiment attempts and applications:

  - *Potential condensed-matter realization of space-fractional quantum mechanics: The one-dimensional Lévy crystal*, B. A. Stickler, Phys. Rev. E, **88** (2013) 012120.
  - *Fractional Schrödinger equation in optics*, S. Longhi, Optics Lett., **40** (2015) 1117–1120.
  - *Fractional quantum mechanics in polariton condensates with velocity dependent mass*, F. Pinsker, W. Bao, Y. Zhang, H. Ohadi, A. Dreismann, J. Baumberg, Phys. Rev. B, **92** (2015) 195310.
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Outline

1. The fractional Schrödinger equation
2. Motivation and challenges
3. Spatial discretization
4. Stationary states
5. Dynamics
6. Summary

Motivation and challenges

Motivation:

1. Understand how the fractional Laplacian affects the solutions of the Schrödinger equation.

Main challenges:

1. The fractional Laplacian is a nonlocal operator,

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) \quad = \quad C_{d,\alpha}\,\text{P.V.}\int_{\mathbb{R}^d}\frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\,d\mathbf{y}.$$

2. Accurate numerical scheme for discretizing the fractional Laplacian is still scant.

Example: Schrödinger equation in a box potential

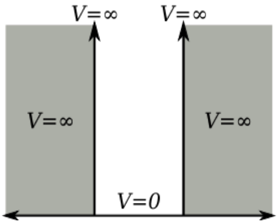
Let's consider the 1D linear Schrödinger equation:

$$i\partial_t \psi(x,t) = -\Delta\psi + V(x)\psi, \quad x \in \mathbb{R}, \quad t > 0,$$

with a box potential (or infinite well potential), i.e.,

$$V(x) = \begin{cases} 0, & \text{if } |x| < L, \\ \infty, & \text{otherwise,} \end{cases} \qquad x \in \mathbb{R}.$$

This is one important model to understand the quantum effects.



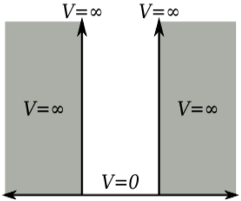
Example: Schrödinger equation in a box potential

Its stationary states can be found by solving

$$\mu\phi(x) = -\Delta\phi + V(x)\phi, \qquad x \in \mathbb{R}$$

with the normalization

$$\|\phi\|^2 = \int_{\mathbb{R}} |\phi(x)|^2 dx = 1.$$



Due to the constraint of box potential, $\phi(x) \equiv 0$ for x located outside of box.

The eigenvalue problem reduces to

$$\begin{aligned} \mu\phi(x) &= -\Delta\phi, & x &\in \Omega, \\ \phi(x) &= 0, & x &\in \partial\Omega, \\ \|\phi(\cdot)\|^2 &= 1. \end{aligned}$$

Example: Schrödinger equation in a box potential

That is, stationary states of Schrödinger equation in a box potential are equivalent to the eigenfunctions of the Dirichlet Laplacian on Ω.

The s-th eigenfunction has the form:

$$\phi_s(x) = \sqrt{\frac{1}{L}} \sin\left[\frac{s\pi}{2}\left(1 + \frac{x}{L}\right)\right], \quad x \in \Omega, \qquad s \in \mathbb{N},$$

and the corresponding eigenvalue is

$$\mu_s = \left(\frac{s\pi}{2L}\right)^2, \qquad s \in \mathbb{N}.$$

Example: Schrödinger equation in a box potential

Now, let's focus on 1D fractional linear Schrödinger equation:

$$i\partial_t \psi = (-\Delta)^{\alpha/2}\psi + V(x)\psi, \quad x \in \mathbb{R}, \quad t > 0.$$

Research questions:

- What are the eigenvalues and eigenfunctions of the fractional Schrödinger equation in a box potential?
- Are they the same as those of the standard Schrödingier equation?

Current literature: No analytical results are reported, except the estimates on the eigenvalues.

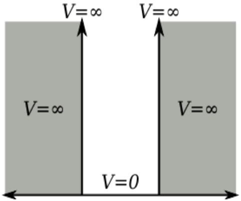
Example: Schrödinger equation in a box potential

Recall: Eigenvalue problem

$$\mu\phi(x) = (-\Delta)^{\alpha/2}\phi + V(x)\phi, \qquad x \in \mathbb{R}$$

with the normalization

$$\|\phi\|^2 = \int_{\mathbb{R}} |\phi(x)|^2 dx = 1.$$



Due to the constraint of box potential, $\phi(x) \equiv 0$ for x located outside of box.

The eigenvalue problem reduces to

$$\begin{aligned} \mu\phi(x) &= (-\Delta)^{\alpha/2}\phi, \qquad && x \in \Omega, \\ \phi(x) &= 0, && x \in \Omega^c = \mathbb{R}\backslash\Omega, \qquad x \in \partial\Omega \\ \|\phi(\cdot)\|^2 &= 1. \end{aligned}$$

Outline

Numerical methods

Goal: Discretize the fractional Laplacian

$$(-\Delta)^{\alpha/2}u(x) = C_{1,\alpha}\int_{\mathbb{R}}\frac{u(x)-u(y)}{|x-y|^{1+\alpha}}dy, \qquad x \in (-L, L)$$

with the condition

$$u(x) = 0, \qquad x \in \mathbb{R}\backslash(-L, L)$$

Numerical methods:

1. Finite element method (Duo & Zhang, 2016)
2. Finite difference method (Duo & Zhang, 2015; Duo, van Wyk & Zhang, 2016)
3. Interpolation method (Huang & Oberman, 2014)

Finite difference method

Let's first rewrite the operator

$$\begin{aligned}(-\Delta)^{\alpha/2}u(x) &= -C_{1,\alpha}\mathcal{L}_0^{\infty}u(x) \\ &= -C_{1,\alpha}\int_0^{\infty}\frac{u(x-\xi)-2u(x)+u(x+\xi)}{\xi^{1+\alpha}}\,d\xi.\end{aligned}$$

Choose a constant $A = 2L$, i.e., the length of the domain.

$$\begin{aligned}\mathcal{L}_0^{\infty}u(x) &= \int_0^{A}\frac{u(x-\xi)-2u(x)+u(x+\xi)}{\xi^{1+\alpha}}\,d\xi \\ &\quad+\int_A^{\infty}\frac{u(x-\xi)-2u(x)+u(x+\xi)}{\xi^{1+\alpha}}\,d\xi \\ &= \mathcal{L}_0^{A}u(x)+\mathcal{L}_A^{\infty}u(x).\end{aligned}$$

Finite difference method

Computation of

$$\mathcal{L}_A^{\infty} u(x) = \int_A^{\infty} \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\, d\xi.$$

Note:

- $A = 2L$;
- $u(x) = 0, \;\; \text{for} \;\; x \notin (-L, L)$.

Hence, for any $|x| < L$ and $\xi \geq A$, there is

$$|x \pm \xi| > L \Longleftrightarrow u(x \pm \xi) = 0.$$

We can *exactly* compute

$$\begin{aligned} \mathcal{L}_A^{\infty} u(x) &= \int_A^{\infty} \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\, d\xi \\ &= \int_A^{\infty} \frac{-2u(x)}{\xi^{1+\alpha}}\, d\xi = -\frac{1}{\alpha A^{\alpha}} u(x). \end{aligned}$$

Finite difference method

Discretization of

$$\mathcal{L}_0^A u(x) = \int_0^A \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\, d\xi$$

Let's rewrite the integrand

$$\begin{aligned} \mathcal{L}_0^A u(x) &= \int_0^A \underbrace{\frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\frac{\alpha}{2}}}}_{\Phi_\alpha(x,\xi)} \cdot \frac{1}{\xi^{\frac{\alpha}{2}}}\, d\xi. \\ &= \int_0^A \Phi_\alpha(x,\xi)\, \xi^{-\alpha/2} d\xi. \end{aligned}$$

Remark: As $\alpha \to 2$, we have

$$\Phi_\alpha(x,\xi) \to \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^2}.$$

Finite difference method

We discretize it by the weighted trapezoidal method, i.e.,

$$\begin{aligned}\mathcal{L}_0^A u(x) &= \int_0^A \Phi_\alpha(x,\xi)\,\xi^{-\alpha/2} d\xi \\ &\approx \sum_{l=1}^{M} \frac{\Phi_\alpha(x,\xi_{l-1}) + \Phi_\alpha(x,\xi_l)}{2} \int_{\xi_{l-1}}^{\xi_l} \xi^{-\alpha/2} d\xi \\ &= \frac{1}{2-\alpha} \sum_{l=1}^{M} \left(\xi_l^{1-\alpha/2} - \xi_{l-1}^{1-\alpha/2}\right) \left[\Phi_\alpha(x,\xi_{l-1}) + \Phi_\alpha(x,\xi_l)\right].\end{aligned}$$

Recall

$$\Phi_\alpha(x,\xi) = \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\frac{\alpha}{2}}}.$$

Combining $\mathcal{L}_0^A$ and $\mathcal{L}_A^\infty$ gives the finite difference scheme of the fractional Laplacian.

Accuracy of spatial discretization

Example 1. Consider a function

$$u(x) = \begin{cases} -(1-x^2)^{3+\frac{\alpha}{2}}, & \text{for } x \in (-1,1), \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}.$$

The fractional Laplacian of $u(x)$ can be found exactly as

$$(-\Delta)^{\alpha/2}u(x) = \frac{2^{\alpha}\Gamma(\frac{\alpha+1}{2})\Gamma(4+\frac{\alpha}{2})}{-\sqrt{\pi}\Gamma(4)} \cdot {}_2F_1\left(\frac{\alpha+1}{2}, -3; \frac{1}{2}; x^2\right),$$

where ${}_2F_1$ denotes the Gauss' hypergeometric function.

Accuracy of spatial discretization

α	$h = \frac{1}{64}$	$h = 1/128$	$h = \frac{1}{256}$	$h = \frac{1}{512}$	$h = \frac{1}{1024}$	$h = \frac{1}{2048}$
0.2	1.2640E-5	3.1594E-6	7.8983E-7	1.9746E-7	4.9364E-8	1.2341E-8
		2.0002	2.0000	2.0000	2.0000	2.0000
0.6	5.1754E-5	1.2920E-5	3.2286E-6	8.0708E-7	2.0177E-7	5.0443E-8
	–	2.0021	2.0006	2.0001	2.0000	2.0000
1	1.3586E-4	3.3626E-5	8.3618E-6	2.0846E-6	5.2039E-7	1.3000E-7
	–	2.0145	2.0077	2.0040	2.0021	2.0011
1.5	4.9828E-4	1.1834E-4	2.8339E-5	6.8470E-6	1.6677E-6	0.4.0870E-7
	–	2.0740	2.0621	2.0492	2.0376	2.0288
1.99	3.3929E-3	8.5911E-4	2.1570E-4	5.3920E-5	1.3448E-5	3.3517E-6
	–	1.9816	1.9938	2.0001	2.0034	2.0045

Observation: It has the second-order convergence rate for $\alpha \in (0, 2)$.

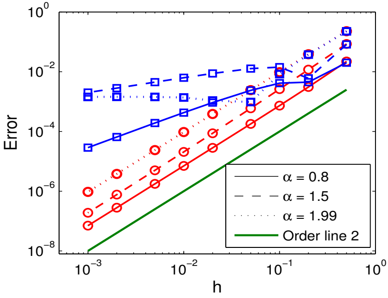
Error analysis (Duo, van Wyk & Zhang, 2016)

Comparison between methods

Example 2. Consider the function $u(x) = e^{-x^2}$. At $x = 0$, we can obtain

$$(-\Delta)^{\alpha/2}u(0) = (-\Delta)^{\alpha/2}u(x)\mid_{x=0} = \frac{2^\alpha}{\sqrt{\pi}}\,\Gamma\Big(\frac{1+\alpha}{2}\Big).$$

We compare finite difference method ('○') with interpolation method ('□') as follows:



Furthermore, the implementation of the finite difference method is straightforward.

Outline

Literature review: Eigenvalues and eigenfunctions

Eigenvalues: Lower and upper bounds [1]

The lower and upper bounds of the eigenvalue μ_s are given by

$$\frac{1}{2}\left(\frac{s\pi}{2L}\right)^{\alpha} \le \mu_s \le \left(\frac{s\pi}{2L}\right)^{\alpha}, \qquad \alpha \in (0, 2],$$

for any $s \in \mathbb{N}$, where $\alpha = 2$ corresponds to the standard Laplacian.

Recently, a better estimate is found for $s = 1$,

$$\frac{(\alpha+1)(\alpha+2)(6-\alpha)}{(12+14\alpha)}\, p(\alpha) \;\le\; \mu_1 \;\le\; \frac{B(\frac{1}{2}, 1+\frac{\alpha}{2})}{B(\frac{1}{2}, 1+\alpha)}\, p(\alpha), \qquad \alpha \in (0, 2),$$

where $B(a, b)$ defines the Beta function of a and b

$$\text{with } p(\alpha) = \frac{2^{\alpha}\,\Gamma(1+\frac{\alpha}{2})\,\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})}.$$

[1] Z. -Q. Chen and R. Song, J. Funct. Anal., **226** (2005) 90–113.

Literature review: Eigenvalues and eigenfunctions

Eigenvalues: Asymptotic approximations[2]

The asymptotic approximation of μ_s in an interval $(-1,1)$ is given by:

$$\mu_s = \left[\frac{s\pi}{2} - \frac{(2-2\alpha)\pi}{8}\right]^{2\alpha} + O\left(\frac{2-2\alpha}{s\sqrt{2\alpha}}\right), \qquad \alpha \in (0,1],$$

where

$$s \geq (C/2\alpha)^{\frac{3}{4\alpha}} \qquad \text{with } C \text{ a positive constant.}$$

Eigenfunctions:

Conjecture[3]: Eigenfunctions cannot be written in terms of elementary functions.

[2] M. Kwaśnicki, J. Funct. Anal., **262** (2012) 2379–2402.
[3] Y. Luchko, J. Math. Phys., **54** (2013) 012111.

First eigenvalues

α	Lower bounds	Asymptotical results	**Our results**	Upper bounds
0.01	0.9960	0.9976	0.996636	0.9974
0.1	0.9676	0.9809	0.97261	0.9786
0.2	0.9499	0.9712	0.9575	0.9675
0.3	0.9442	0.9699	0.9528	0.9655
0.5	0.9620	0.9908	0.9702	0.9862
0.6	0.9839	1.0126	0.9913	1.0084
0.8	1.0521	1.0789	1.0576	1.0763
1.0	1.1538	1.1781	1.1578	$3\pi/8$
1.1	1.2183	1.2415	1.2222	1.2432
1.3	1.3781	1.4007	1.3837	1.4064
1.5	1.5861	1.6114	1.5976	1.6223
1.8	2.0140	2.0555	2.0488	2.0777
1.9	2.1952	2.2477	2.2441	2.2747
1.95	2.3784	2.4441	2.4437	2.4563

Note: As $\alpha \to 2$, it converges to $\pi^2/4 = 2.4674$, the first eigenvalue of $-\Delta$.

First eigenfunctions

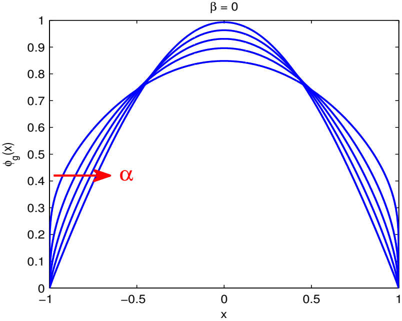


Figure: The first eigenfunction (ground state) solutions for $\alpha = 0.2, 0.7, 1.1, 1.5$, and 1.9, where the arrow indicates the change of $\phi_g(x)$ for progressively increasing α.

Second eigenvalues

α	Lower bounds	Asymptotical results	**Our results**	Upper bounds
0.01	0.5058	1.0086	1.008719	1.0115
0.1	0.5606	1.0913	1.09221	1.1213
0.2	0.6286	1.1948	1.1966	1.2573
0.3	0.7049	1.3122	1.3148	1.4098
0.5	0.8862	1.5977	1.6016	1.7725
0.6	0.9937	1.7708	1.7753	1.9874
0.8	1.2494	2.1941	2.1995	2.4987
1.0	$\pi/2$	2.7489	2.7549	π
1.1	1.7613	3.0892	3.0954	3.5226
1.3	2.2144	3.9319	3.9380	4.4289
1.5	2.7842	5.0545	5.0600	5.5683
1.8	3.9250	7.5003	7.5033	7.8500
1.9	4.4010	8.5942	8.5959	8.8021
1.95	4.8786	9.7330	9.7332	9.7573

Note: As $\alpha \to 2$, it converges to $\pi^2 = 9.8698$, the second eigenvalue of $-\Delta$.

Second eigenfunctions

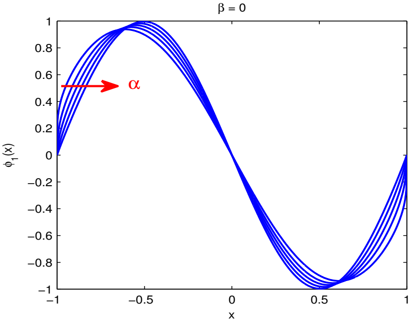
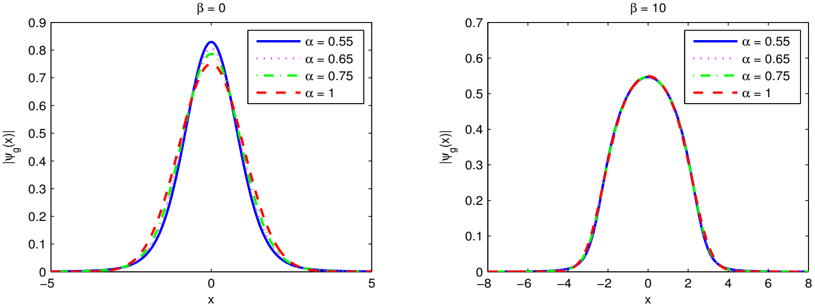


Figure: The second eigenfunction (the first excited state) solutions for$\alpha = 0.2, 0.7, 1.1, 1.5,$ and 1.9, where the arrow indicates the change of $\phi_1(x)$ for progressively increasing α.

Extensions

1. Stationary states of fractional NLS: Imaginary time method (Duo & Zhang, 2015)
2. Stationary states in other potentials (Kirkpatrick & Zhang, 2016)



Ground states of fractional Schrödinger equation with a harmonic potential. (*Legend of the plots corresponding to* $(-\Delta)^{\alpha}$).

Outline

Fractional Schrödinger equation

Consider the 1D fractional Schrödinger equation with harmonic potential

$$i\partial_t \psi(x,t) = \frac{1}{2}(-\Delta)^{\alpha/2}\psi + \frac{x^2}{2}\psi + \gamma|\psi|^2\psi, \qquad x \in \mathbb{R}.$$

Numerical methods for temporal discretization:

- Splitting step method
- Crank-Nicolson method
- Besse Relaxation method

Equations of motion

- Center of mass:

$$\langle X \rangle := \langle \psi, X\psi \rangle = \int_{\mathbb{R}^d} \mathbf{x} |\psi(\mathbf{x}, t)|^2 d\mathbf{x}.$$

- Expected fractional momentum:

$$\langle P_\alpha \rangle := \langle \psi, P_\alpha \psi \rangle = -i\frac{\alpha}{2} \int_{\mathbb{R}^d} \psi^* \nabla^{\alpha-1} \psi \, d\mathbf{x}.$$

where we define the fractional momentum operator

$$P_\alpha := -i\frac{\alpha}{2} \nabla^{\alpha-1} = \frac{\alpha}{2} |P^2|^{\alpha/2-1} P,$$

with $P = -i\nabla$ the standard momentum operator.

Equations of motion (Standard NLS)

Theorem: For a solution $\psi = \psi(\mathbf{x}, t)$ of the standard NLS with harmonic potential, we have the following equations of motion for $t > 0$:

$$\frac{d}{dt}\langle X\rangle = \langle P\rangle,$$
$$\frac{d}{dt}\langle P\rangle = -\Lambda\langle X\rangle,$$

where the matrix Λ in the case $d = 1$ is $\Lambda = \gamma_x^2$, and

$$\Lambda = \begin{pmatrix} \gamma_x^2 & 0 \\ 0 & \gamma_y^2 \end{pmatrix} \text{ if } d = 2, \qquad \Lambda = \begin{pmatrix} \gamma_x^2 & 0 & 0 \\ 0 & \gamma_y^2 & 0 \\ 0 & 0 & \gamma_z^2 \end{pmatrix} \text{ if } d = 3.$$

Remarks:

- It is a closed system with periodic solution.
- Its dynamics is independent the initial condition and the nonlinearity.

Equations of motion (Fractional NLS)

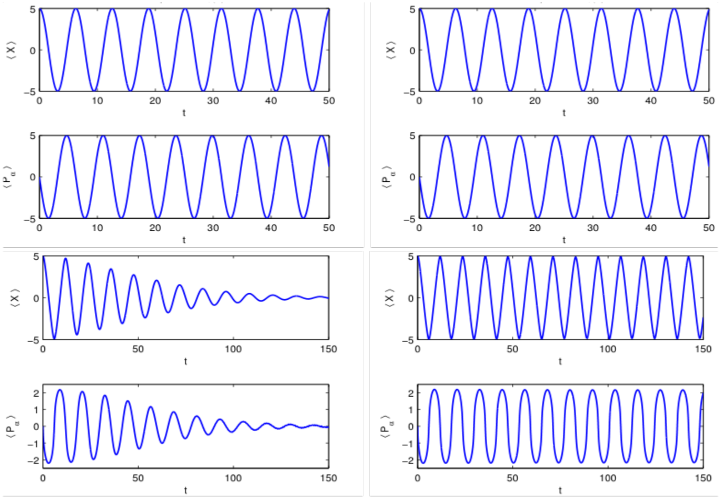
Theorem: For a solution $\psi = \psi(\mathbf{x}, t)$ of the fractional NLS with harmonic potential, we have the following equations of motion for $t > 0$:

$$\begin{aligned} \frac{d}{dt}\langle X\rangle &= P_\alpha, \\ \frac{d}{dt}\langle P_\alpha\rangle &= W_\alpha, \end{aligned}$$

where the quantity W_α is the expectation of an operator and can be defined by:

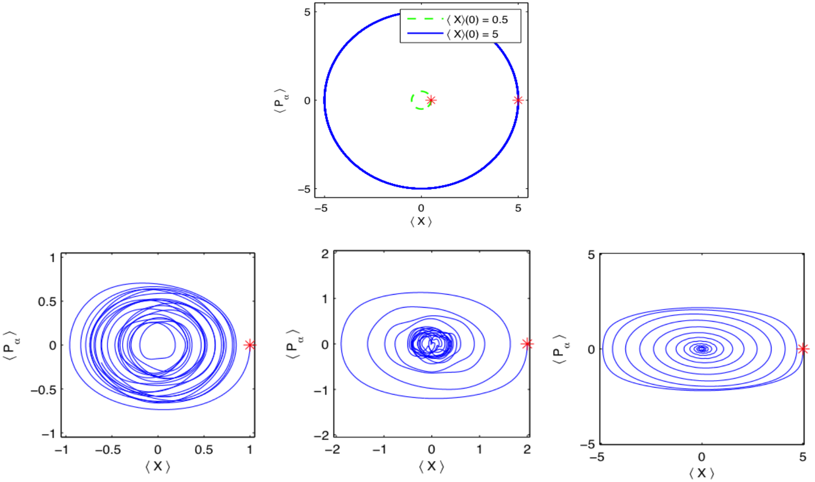
$$\begin{aligned} W_\alpha \quad := \quad & \frac{\alpha}{2}(\alpha-1)(-\nabla V)|P^2|^{\alpha/2-1} - \frac{\alpha}{2}\big(\frac{\alpha}{2}-1\big)(\alpha-1)\left(\nabla^2 V\right)\nabla^{\alpha-3} \\ & -\frac{\alpha}{2}\gamma\sum_{j\geq 1}\binom{\alpha-1}{j}\ \psi, \left(\nabla^{\alpha-1-j}\psi\right)\left(\nabla^j(|\psi|^2)\right) \end{aligned}$$

Comparison 1: Equations of motion



Top: Standard case; Bottom: Fractional case; Left: Linear; Right: Nonlinear.

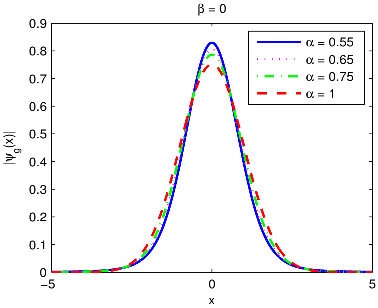
Comparison 1: Equations of motion



Linear Schrödinger equation. Top: Standard case; Bottom: Fractional case.

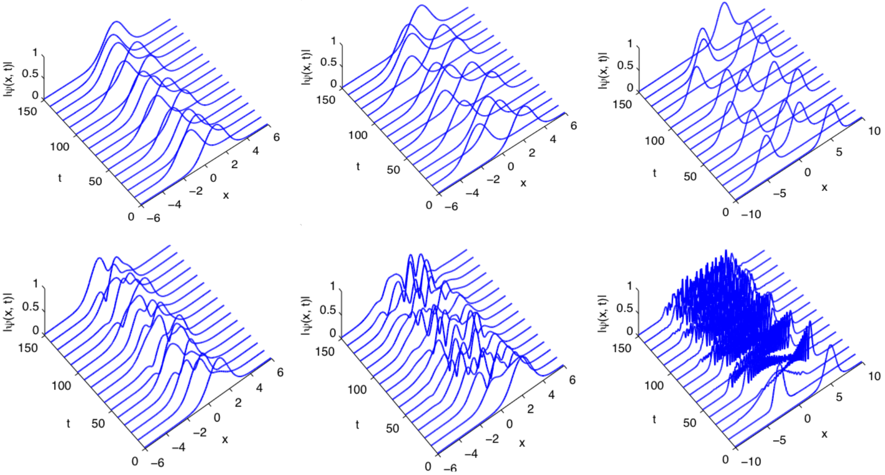
Comparison 2: Solution dynamics

Initial condition: Shift the center of the ground state from $x = 0$ to $x = \langle X \rangle(0)$.



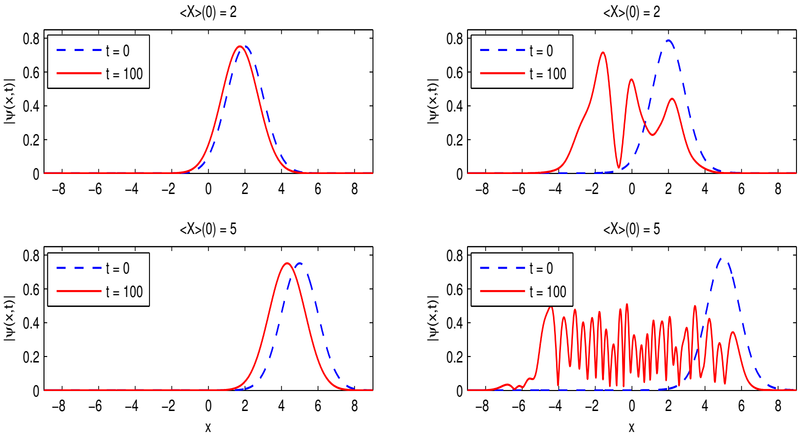
Ground states of fractional Schrödinger equation with a harmonic potential. (*Legend of the plots corresponding to* $(-\Delta)^{\alpha}$)

Comparison 2: Solution dynamics



Linear Schrödinger equation. Top: Standard case; Bottom: Fractional case. From left to right: $\langle X\rangle(0) = 1, 2, 5$.

Comparison 2: Solution dynamics



Linear Schrödinger equation. Left: Standard case; Right: Fractional case.

Summary

Motivation:

Understand nonlocal effects of $(-\Delta)^{\alpha/2}$ on the solutions of the Schrödinger equation

Challenges:

Accurate numerical methods for discretizing the hypersingular integral

Numerical methods:

Weighted trapezoidal method, FEM, ...

Solution properties of fractional Schrödinger equation

- Stationary states in box or harmonic potential
- Equation of motions, solution dynamics

Merci!