Superfluidity & Bogoliubov Theory: Rigorous Results

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Superfluidity

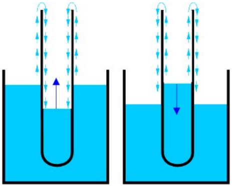
- matter behaves like a fluid with zero viscosity
- very low temperature
- discovered in 1937 for liquid Helium
- in trapped Bose-Einstein condensates, neutron stars,...

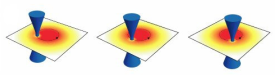
▶ A microscopic effect

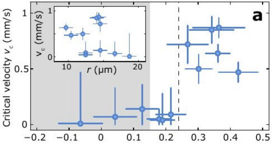
- macroscopic manifestation of quantum mechanics
- essentially for **bosons** (Helium-4),
  more subtle for **fermions** (Helium-3)

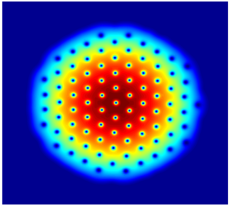
▶ Related to Bose-Einstein condensation (?)

- discovery of quantized vortex lines (50s) and rings (60s)
- Gross & Pitaevskii in 1961 for liquid Helium
- only about 10 % of the particles are condensed in superfluid Helium





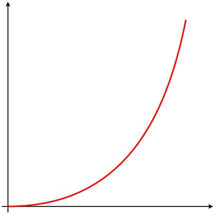




Top: lack of viscosity in superfluid Helium
Middle: The core of a trapped 2D cold Bose gas is superfluid (Dalibard et al, *Nature Physics*, 2012)
Bottom: numerical simulation of vortices in a BEC (GPE-lab, Antoine & Duboscq)

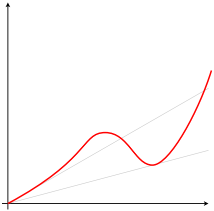
Microscopic origin of superfluidity

- Bogoliubov ('47), Feynman ('55) & Landau ('62): positive speed of sound in the gas, due to interactions between particles
- seen in **excitation spectrum**



no interaction

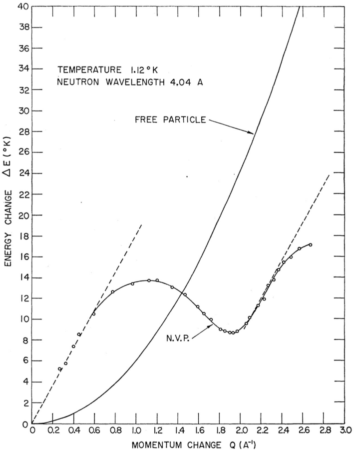
$$e(k) = |k|^2$$



with interaction w

$$e(k) \underset{\uparrow}{\simeq} \sqrt{|k|^4 + 2\widehat{w}(k)|k|^2}$$

(Bogoliubov assuming full BEC)



Henshaw & Woods, *Phys. Rev.*, 1961

Theory

1. **Bose-Einstein condensation**
   - (almost) all the particles in the gas behave the same
   - their common wavefunction u solves the **nonlinear** Gross-Pitaevskii equation

$$\left(|\nabla + iA(x)|^2 + V(x) + w * |u|^2\right)u = \begin{cases} i\partial_t u \\ \varepsilon\, u \end{cases}$$

2. **Bogoliubov**
   - fluctuations of the condensate described by a **linear** equation

$$\mathbb{H}_u \Phi = \begin{cases} i\partial_t \Phi \\ \lambda\, \Phi \end{cases}$$

$\mathbb{H}_u$ = Bogoliubov Hamiltonian
= (second) **quantization of the Hessian of the GP energy** at u
= has spectrum with the finite speed of sound when u=minimizer

Goal:

- prove this in appropriate regimes
- semi-classical theory in infinite dimension, with an effective semi-classical parameter

Regimes

- **Two typical physical systems**
  - **Confined gas:** external potential $V(x) \to \infty$
    Studied a lot since the end of the 90s
  - **Infinite gas:** $V \equiv 0$, infinitely many particles
    Very poorly understood mathematically

- **Dilute regime:** *rare collisions of order 1*
  - low density $\rho \to 0$
  - $w \rightsquigarrow 4\pi a\delta$
  - very relevant physically
  - confined gas: BEC proved by Lieb-Seiringer-Yngvason '00s,... Bogoliubov open

- **Mean-field regime:** *many small collisions*
  - high density $\rho \to \infty$, small interaction $\sim 1/\rho$
  - good setting for the law of large numbers
  - a bit less relevant physically
  - confined gas: many works on BEC, Bogoliubov only understood very recently
  - **this talk**

M.L. Mean-field limit of Bose systems: rigorous results. *Proceedings of the International Congress of Mathematical Physics.* arXiv:1510.04407

Many-particle mean-field Hamiltonian

- N (spinless) bosons in $\mathbb{R}^d$ with $N \to \infty$
- A external magnetic potential or Coriolis force, V external potential (e.g. lasers)
- two-body interaction λw, with $\lambda \to 0$

$$H_N = \sum_{j=1}^{N} |\nabla_{x_j} + iA(x_j)|^2 + V(x_j) + \lambda \sum_{1 \leq j < k \leq N} w(x_j - x_k), \qquad \lambda \sim \frac{1}{N}$$

acting on $L^2_{\mathbf{s}}(\mathbb{R}^d)^N = \left\{\Psi(x_1, ..., x_N) = \Psi(x_{\sigma(1)}, ..., x_{\sigma(N)}) \in L^2, \quad \forall \sigma \in \mathfrak{S}_N\right\}$

▶ **Assumptions for this talk:**

- $h = |\nabla + iA|^2 + V$ is bounded-below and has a compact resolvent on $L^2(\mathbb{R}^d)$
- w is h–form-bounded with relative bound < 1. Can be attractive or repulsive or both

$$\lambda_\ell(H_N) := \ell\text{th eigenvalue of } H_N$$

Rmk. dilute regime corresponds to $w_N = N^{3\beta} w(N^\beta x)$ with $\beta = 1$ in $d = 3$, here $\beta = 0$.

Gross-Pitaevskii energy

- If $\lambda = 0$, then the particles are all exactly iid

$$\Psi(x_1, ..., x_N) = u(x_1) \cdots u(x_N) = u^{\otimes N}(x_1, ..., x_N)$$

where $u \in L^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u(x)|^2 \, dx = 1$

$$\begin{aligned} \frac{\left\langle u^{\otimes N}, H_N u^{\otimes N} \right\rangle}{N} &= \int_{\mathbb{R}^d} |\nabla u(x) + iA(x) u(x)|^2 dx + \int_{\mathbb{R}^d} V(x) |u(x)|^2 \, dx \\ &\quad + \frac{(N-1)\lambda}{2} \iint_{\mathbb{R}^{2d}} w(x-y) |u(x)|^2 |u(y)|^2 \, dx \, dy \\ &= \mathcal{E}_{\rm GP}(u), \qquad \text{for } \lambda = \frac{1}{N-1} \end{aligned}$$

- Minimizers:

$$e_{\rm GP} := \inf_{\int_{\mathbb{R}^d} |u|^2 = 1} \mathcal{E}_{\rm GP}(u)$$

Bose-Einstein condensation

Theorem (Derivation: ground state energy [M.L.-Nam-Rougerie '14])

For every fixed $\ell \geq 1$ *we have*

$$\lim_{N\to\infty} \frac{\lambda_\ell(H_N)}{N} = e_{\rm GP}.$$

Let Ψ_N *be such that* $\langle \Psi_N, H_N \Psi_N \rangle = N\, e_{\rm GP} + o(N)$*. Then there exists a subsequence and a probability measure* μ*, supported on*

$$\mathcal{M} = \{\text{minimizers for } e_{\rm GP}\},$$

such that

$$\left\langle \Psi_{N_j}, A_{x_1,\dots,x_k} \Psi_{N_j} \right\rangle \underset{N_j\to\infty}{\longrightarrow} \int_{\mathcal{M}} \langle u^{\otimes k}, A u^{\otimes k} \rangle \, d\mu(u),$$

for every bounded operator A *on* $L^2(\mathbb{R}^{dk})$ *and every* $k \geq 1$*.*

- Strong convergence of **(quantum) marginals** = density matrices
- Easy when $A = 0$ & $\widehat{w} \geq 0$ is smooth
- Many works since the 80s (Fannes-Spohn-Verbeure '80, Benguria-Lieb '80, Lieb-Thirring-Yau '84, Petz-Raggio-Verbeure '89, Raggio-Werner '89, Lieb-Seiringer '00s,...)
- Our method, based on quantum de Finetti thms, also works for locally confined systems
- It can be used to simplify proof in dilute case (Nam-Rougerie-Seiringer '15)

Lewin, Nam & Rougerie, Derivation of Hartree's theory for generic mean-field Bose systems, *Adv. Math.* (2014)

Describing fluctuations

Assumption

$e_{\rm GP}$ has a **unique minimizer** u_0 (up to a phase factor), which is **non-degenerate**.

Any symmetric function Ψ of N variables may be uniquely written in the form

$$\Psi = \underbrace{\varphi_0}_{\in\mathbb{C}}\, u_0^{\otimes N} + \underbrace{\varphi_1}_{\in\{u_0\}^\perp} \otimes_s u_0^{\otimes N-1} + \underbrace{\varphi_2}_{\in\{u_0\}^\perp\otimes_s\{u_0\}^\perp} \otimes_s u_0^{\otimes N-2} + \cdots + \underbrace{\varphi_N}_{\in(\{u_0\}^\perp)^{\otimes_s N}}$$

with $\sum_{j=0}^{N} \|\varphi_j\|^2 = \|\Psi\|^2_{L^2}$. Here

$$f \otimes_s g(x_1, ..., x_N) = \frac{1}{\sqrt{N!}}\Big(f(x_1, ..., x_k)g(x_{k+1}, ..., x_N) + \text{permutations}\Big)$$

- Natural to express the fluctuations using $\Phi = \varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_N$.

In the limit $N \to \infty$, these live in the **Fock space**

$$\mathcal{F}_0 := \mathbb{C} \oplus \{u_0\}^\perp \oplus \bigoplus_{n\geq 2} \bigotimes_s^n \{u_0\}^\perp$$

Convergence of excitation spectrum

Theorem (Validity of Bogoliubov's theory [M.L.-Nam-Serfaty-Solovej '15])

We assume that u_0 is unique and non-degenerate, and that $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)^2|u_0(x)|^2|u_0(y)|^2\,dx\,dy<\infty$. Let $\mathbb{H}_0$ be the Bogoliubov Hamiltonian, that is, the second quantization of Hess $\mathcal{E}_{\rm GP}(u_0)/2$ *on $\mathcal{F}_0$. Then*

- $\lambda_\ell(H_N) - N\,e_{\rm GP} \to \lambda_\ell(\mathbb{H}_0)$ *for all* $\ell\geq 1$;
- *If $H_N\Psi_N=\lambda_\ell(H_N)\Psi_N$, then for a subsequence the corresponding fluctuations converge to a Bogoliubov eigenfunction: $\Phi_{N_j}\to\Phi=\varphi_0\oplus\varphi_1\oplus\cdots$ in $\mathcal{F}_0$ with $\mathbb{H}_0\Phi=\lambda_\ell(\mathbb{H}_0)\Phi$. Equivalently,*

$$\left\|\Psi_{N_j}-\sum_{n=0}^{N_j}\varphi_n\otimes_s u_0^{\otimes N_j-n}\right\|\to 0.$$

- Generalizes Seiringer '11 and Grech-Seiringer '13
- Extension to isolated local minima by Nam-Seiringer '15
- For $\ell=1$ $\varphi_{2j+1}\equiv 0$ $\forall j$, but usually $\varphi_{2j}\neq 0$ when $w\neq 0$
- Ψ_N is **not** close to $(u_0)^{\otimes N}$ since usually $\Phi\neq\varphi_0$

Lewin, Nam, Serfaty & Solovej, Bogoliubov spectrum of interacting Bose gases, *Comm. Pure Appl. Math.* (2015)

Bogoliubov Hamiltonian

- Hessian of GP energy

$$\frac{1}{2}\text{Hess}\,\mathcal{E}_{\rm GP}(u_0)(v,v) = \Big\langle v, \big(\underbrace{|\nabla+iA|^2+V+|u_0|^2*w-\varepsilon_0}_{h_0}\big)v\Big\rangle$$
$$+\frac{1}{2}\iint_{\mathbb{R}^{2d}} w(x-y)\Big(\overline{u_0(x)}u_0(y)\overline{v(x)}v(y)+\overline{u_0(x)u_0(y)}v(x)v(y)+c.c.\Big)\,dx\,dy$$
$$=\frac{1}{2}\left\langle\begin{pmatrix}v\\ \overline{v}\end{pmatrix},\begin{pmatrix}h_0+K_1 & K_2^*\\ K_2 & \overline{h_0+K_1}\end{pmatrix}\begin{pmatrix}v\\ \overline{v}\end{pmatrix}\right\rangle$$

where $K_1(x,y)=w(x-y)u_0(x)\overline{u_0(y)}$ and $K_2(x,y)=w(x-y)u_0(x)u_0(y)$

- Bogoliubov $\mathbb{H}_0=\mathbb{H}_{\rm d}+\mathbb{H}_{\rm p}+(\mathbb{H}_{\rm p})^*$ where $\mathbb{H}_{\rm d}$ is diagonal and $\mathbb{H}_{\rm p}$ creates pair using the projection of K_2 on $(\{u_0\}^\perp)^{\otimes_s 2}$.

$\mathbb{H}_0\Phi=\lambda\Phi$ is an infinite system of linear equations:

$$(\mathbb{H}_{\rm d})_n\varphi_n+(\mathbb{H}_{\rm p})_{n-2,n}\,\varphi_{n-2}+(\mathbb{H}_{\rm p})_{n+2,n}\,\varphi_{n+2}=\lambda\,\varphi_n$$

$$\lambda\in\sigma(\mathbb{H}_0)\iff\begin{pmatrix}h_0+K_1 & K_2^*\\ K_2 & \overline{h_0+K_1}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\lambda\begin{pmatrix}u\\ -v\end{pmatrix}\ \text{(Bogoliubov-de Gennes)}$$

Solovej, Lecture notes on *Many-body quantum mechanics*, 2007

A word on the dynamics

Theorem (Time-dependent Bogoliubov [M.L.-Nam-Schlein '15])

Let u_0 *with* $\int_{\mathbb{R}^d} |u_0|^2 = 1$ *and* $\langle u_0, h u_0\rangle < \infty$. *Let* $\Phi = (\varphi_n)_{n\geq 0} \in \mathcal{F}_{u_0}$ *with* $\sum_{n\geq 0} \|\varphi_n\|^2 = 1$ *and* $\sum_{n\geq 0} \left\langle \varphi_n, \sum_{j=1}^n h_j \varphi_n \right\rangle < \infty$.
Then the solution of

$$\begin{cases} i\,\dot\Psi_N = H_N \Psi_N \\ \Psi_N(0) = \displaystyle\sum_{n=0}^N \varphi_n \otimes_s u_0^{\otimes N-n} \end{cases}$$

has converging fluctuations $\Phi_N(t) \to \Phi(t) = \bigoplus_{n\geq 0} \varphi_n(t)$ *for every* t, *or equivalently,*

$$\left\| \Psi_N(t) - \sum_{n=0}^N \varphi_n(t) \otimes_s u(t)^{\otimes N-n} \right\| \to 0,$$

where $\begin{cases} i\,\dot u = \left(|\nabla + iA|^2 + V + w * |u|^2 - \varepsilon(t)\right) u \\ u(0) = u_0 \end{cases}$ *and* $\begin{cases} i\,\dot\Phi = \mathbb{H}(t)\,\Phi \\ \Phi(0) = \Phi_0 \end{cases}$

with $\mathbb{H}(t)$ *the Bogoliubov Hamiltonian describing the excitations around* $u(t)$.

Hepp '74, Ginibre-Velo '79, Spohn '80, Grillakis-Machedon-Margetis '00s, Chen '12, Deckert-Fröhlich-Pickl-Pizzo '14, Benedikter-de Oliveira-Schlein '14, ...

Lewin, Nam & Schlein, Fluctuations around Hartree states in the mean-field regime, *Amer. J. Math.* (2015)

Conclusion

- Cold Bose gases pose many interesting questions to mathematicians, which are also physically important
- All are still open for the infinite gas, even in the dilute and mean-field regimes
- Trapped gases are better understood since the beginning of the 00s

▶ **Mean-field microscopic model**

- simpler theory with weak interactions and high density
- physically justified in some special cases (stars, tunable interactions mediated through a cavity)
- full justification of Bose-Einstein Condensation & Bogoliubov excitation spectrum achieved only recently