Nearly parallel vortex filaments in the Gross-Pitaevskii equation

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joint work with A Contreras and D.Smets

We consider

- the Gross-Pitasvskii equation

$$\boxed{i\partial_t u - \Delta u + \frac{1}{\varepsilon^2}(|u|^2 - 1)u = 0}$$

for $u : \Omega \times (0, T) \to \mathbb{C}$, where Ω is a cylinder in $\mathbb{R}^3$:

$$\Omega = B \times (0, L) \subset \mathbb{R}^2, \qquad B \subset \mathbb{R}^2 \text{ is a ball of radius } R;$$

- and its steady states

$$\boxed{-\Delta u + \frac{1}{\varepsilon^2}(|u|^2 - 1)u = 0}$$

for $u : \Omega \to \mathbb{C}$, with Ω as above .

Notation: points in Ω denoted (x, z), with $x \in B \subset \mathbb{R}^2$ and $z \in (0, L)$.

relevant quantities

- density: $|u|^2$
- energy density: $e_\varepsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - 1)^2$
- momentum density: $j(u) := iu \cdot \nabla u$
- vortex density:

$$Ju := \frac{1}{2}\nabla \times j(u) = \nabla u^1 \times \nabla u^2 \qquad \text{if} \quad u = u^1 + iu^2$$

conservation laws

- mass:

$$\frac{1}{2}\partial_t |u|^2 = \nabla \cdot j(u) \ .$$

- energy :

$$\partial_t e_\varepsilon(u) := \nabla \cdot (\partial_t u \cdot \nabla u) \ .$$

- momentum :

$$\partial_t j(u) := 2\nabla \cdot (\nabla u \otimes \nabla u) + \nabla[\cdots] \ .$$

- vorticity:

$$\partial_t Ju := \nabla \times \nabla \cdot (\nabla u \otimes \nabla u).$$

the eqivariant vortex solution is a solution of $(GL)_\varepsilon$ of the form

$$U_{d,\varepsilon}(x) = f_d(r/\varepsilon)e^{id\theta}, \qquad x = re^{i\theta} \in \mathbb{R}^2 \cong \mathbb{C}$$

where $d \in \mathbb{Z}$ and

$$f_d(0) = 0, \qquad f_d \text{ nondecreasing}, \qquad f_d(s) \to 1 \text{ as } s \to \infty$$

We will write U_ε for $U_{1,\varepsilon}$.

Facts

- $e_\varepsilon(U_{d,\varepsilon}) \approx \frac{d^2}{2r^2}$ for $r \geq C\varepsilon$; smooth near $r = 0$. Thus

$$\boxed{\int_{B(R)} e_\varepsilon(U_{d,\varepsilon})dx \approx \pi d^2 \log(R/\varepsilon)\ .}$$

- $JU_{d,\varepsilon}$ has the form $\pi d \eta_\varepsilon$ with $\eta_\varepsilon(x) = \frac{1}{\varepsilon^2}\eta(x/\varepsilon)$, with $\eta \geq 0$, $\int_{\mathbb{R}^2} \eta = 1$. Thus

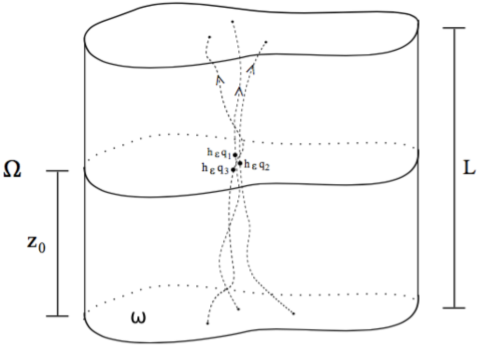
$$\boxed{\|J_x U_{d,\varepsilon} - \pi d \delta_0\|_{W^{-1,1}} \approx \varepsilon\ .}$$

Ansatz

We seek solutions of roughly the form

$$ansatz = e^{i\varphi_\varepsilon} \prod_{i=1}^{d} U_\varepsilon(x - h_\varepsilon f_i(z,t)), \qquad \mathrm{f} : (0, L) \to \mathbb{C}^n \cong (\mathbb{R}^2)^n.$$

for some intermediate length scale $\varepsilon \ll h_\varepsilon \ll 1$. The role of φ_ε is to take account of boundary conditions.



del Pino and Kowalczyk (2008) compute energy of the above ansatz.

They find that if $h_\varepsilon := |\log\varepsilon|^{1/2}$ (which we henceforth assume) then for fixed t,

$$G_\varepsilon(ansatz) = G_0(f),$$

where

$$G_\varepsilon(u) := \int_\Omega e_\varepsilon(u) - C(\varepsilon, n, \Omega)$$

for $C(\varepsilon, n, \Omega) = \pi L[n|\log\varepsilon| + n(n-1)|\log h_\varepsilon| + c(n, R)]$, and

$$G_0(f) := \frac{\pi}{2}\sum_{j=1}^{n}\int_0^L |\partial_z f_j|^2 - \pi\sum_{k\neq j}\int_0^L \log|f_j - f_k|\, dz\ .$$

- first term: linearization of arclength
- second term: interaction between filaments

Theorem (Contreras - J, 2016)

For suitable boundary conditions, there exist sequences $(u_\varepsilon)_{\varepsilon\in(0,1]} \subset H^1(\Omega;\mathbb{C})$ of stationary solutions of $(GP)_\varepsilon$ such that, if we rescale by settting

$$v_\varepsilon(x,z) = u_\varepsilon(h_\varepsilon x, z),$$

then

$$\det \nabla_x v_\varepsilon = J_x v_\varepsilon \to \pi \sum_{j=1}^{n} \delta_{f_j(z)} \otimes dz \qquad \text{in } W^{-1,1}_{loc}(\mathbb{R}^2 \times (0,L))$$

for some $f : (0,L) \to (\mathbb{R}^2)^n$ which is a minimizer or local minimizer of G_0 subject to appropriate boundary conditions. Moreover

$$G_\varepsilon(u_\varepsilon) \to G_0(f).$$

The proof yields much more information about the solutions u_ε.

related results:

- in 2d:
  - Bethuel - Brezs - Hélein
  - many others
- in $3d$ and higher:
  - Bethuel - Rivière
  - Lin - Rivière
  - Sandier
  - Bethuel - Brezis - Orlandi
  - Bethuel - Bourgain - Brezis - Orlandi
  - Bourgain - Brezis - Mironescu
  - J - Soner
  - Alberti - Baldo - Orandi
  - Montero - Sternberg - Ziemer
  - Bethuel - Orlandi - Smets
  - . . .

results 2: dynamics of nearly-parallel filaments

To see the vortex structure, it is convenient to rescale as above. Thus we will consider $v_\varepsilon(x, z, t)$ solving

$$i\partial_t v_\varepsilon - \Delta_x v_\varepsilon + \frac{1}{|\log\varepsilon|}\partial_{zz} v_\varepsilon + \frac{1}{\varepsilon^2|\log\varepsilon|}(|v_\varepsilon|^2 - 1)v_\varepsilon = 0$$

on $B(R|\log\varepsilon|^{1/2}) \times (0, L)$, with

$$\nu \cdot \nabla v_\varepsilon = 0 \qquad \text{for } x \in \partial B(R|\log\varepsilon|^{1/2})$$

and periodic boundary conditions with respect to z.

We seek to relate this to a solution $f(z, t) : (0, L) \times (0, T) \to \mathbb{C}^n$ of the vortex filament system

$$i\partial_t f_j - \partial_{zz} f_j - \sum_{k \neq j} \frac{f_j - f_k}{|f_j - f_k|^2} = 0, \qquad j = 1, \ldots, n$$

for periodic boundary conditions with respect to z.

Theorem (J - Smets, 2016)

Let $f = (f_1, \dots, f_n) : (0, L) \times (0, T) \to \mathbb{C}^n$ *be a smooth solution of the vortex filament system (in particular, with no collisions) with initial data* f^0*.*

Let v_ε *solve the rescaled Gross-Pitaevskii equations for initial data such that*

$$J_x v_\varepsilon \to \pi \sum_{j=1}^{n} \delta_{f_j^0(z)} \otimes dz \qquad \text{in } W^{-1,1}_{loc}(\mathbb{R}^2 \times (0, L))$$

and

$$G_\varepsilon(u_\varepsilon^0) \to G_0(f^0) \qquad \text{for } u_\varepsilon^0(x, z) := v_\varepsilon^0(x/h_\varepsilon, z).$$

Then

$$J_x v_\varepsilon(t) \to \pi \sum_{j=1}^{n} \delta_{f_j^0(z,t)} \otimes dz \qquad \text{in } W^{-1,1}_{loc}(\mathbb{R}^2 \times (0, L))$$

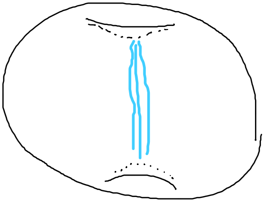
and more....

related results:

- Gross-Pitaevskii vortices in 2d:
  - Colliander - J
  - Lin - Xin
  - Spirn
  - J - Spirn
  - Bethuel - J - Smets
- related equatons (eg micromagnetics)
  - Miot
  - Kurzke - Melcher - Moser - Spirn (and subsets)
  - Serfaty - Tice
- vortex rings:
  - J - Smets
- results about the vortex filament system
  - formal derivation (in framework of classical fluids) Klein - Majda - Damodaran
  - Kenig - Ponce - Vega
  - Banica - Miot,   Banica - Faou - Miot
  - Craig - Garcia-Azpeitia, Garcia-Azpietia - Ize

1. Steady-state

- cylindrical doman $B(R) \times (0, L)$ with suitable Dirichlet data at top and bottom — minimizers (for $R > 2L$) and local minimizers
- local minimizers in domain as pictured:



2. Dynamics

- a single nearly-straight filament: centerline should follow *linear* Schrödinger equation
- $n \geq 2$ nearly parallel filament – full coupled nonlinear dynamics

Thank you!