Quantized Vortex Stability and Dynamics in Superfluidity and Superconductivity

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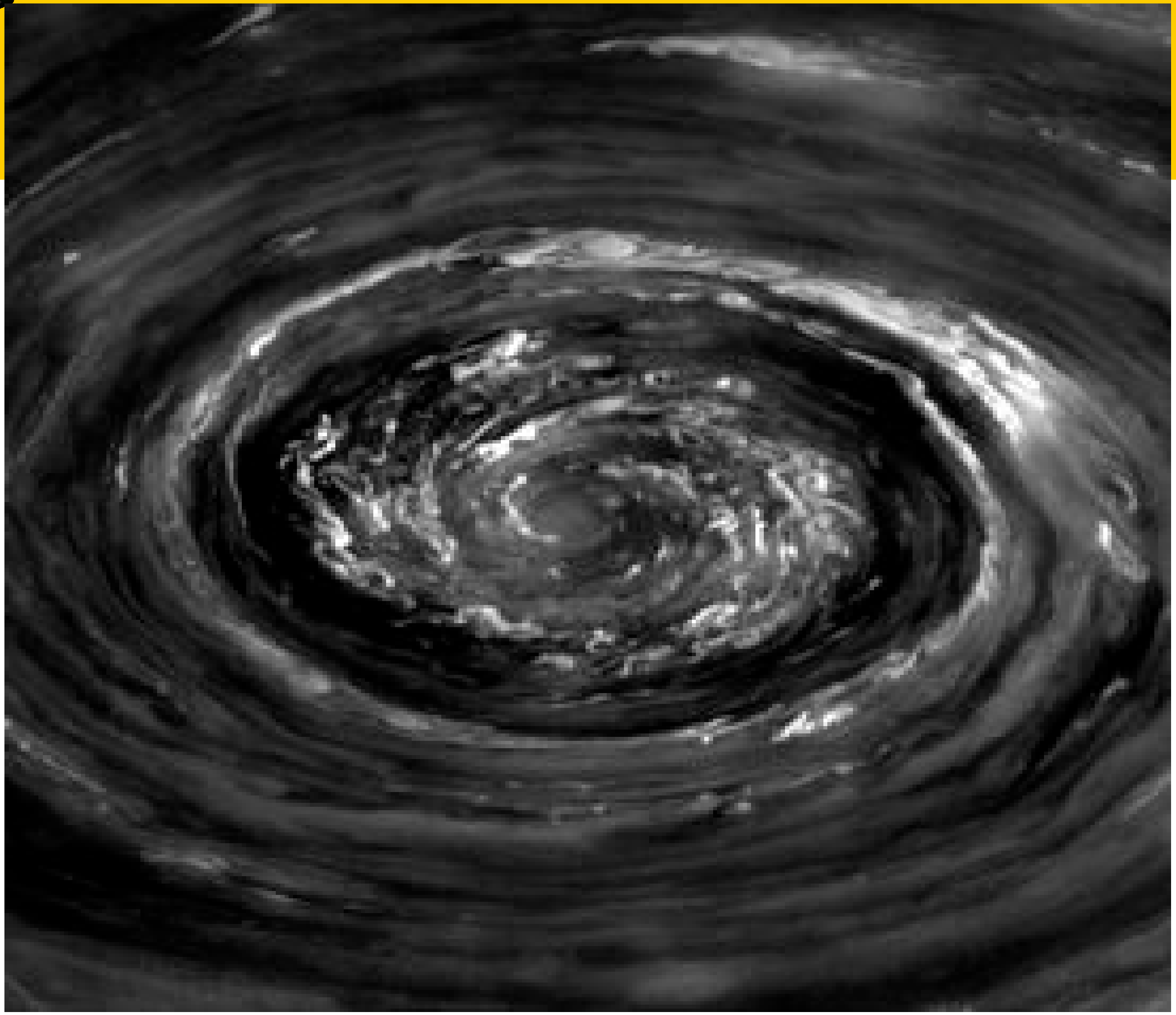
Vortex: From macroscale to microscale



Cloud vortex – Saturn hexagon at the north pole of the planet Saturn

http://en.wikipedia.org/wiki/Saturn%27s_hexagon

Vortex: From macroscale to microscale

Vortex galaxy – vortex in cosmos

Vortex: From macroscale to microscale

Tornado –Vortex in air

Vortex: From macroscale to microscale

Vortex in water – generated by a boat

Vortex: From macroscale to microscale

Vortex in water – generated by an airplane

Vortex: From macroscale to microscale

Magnetic vortex –vortex in plasma

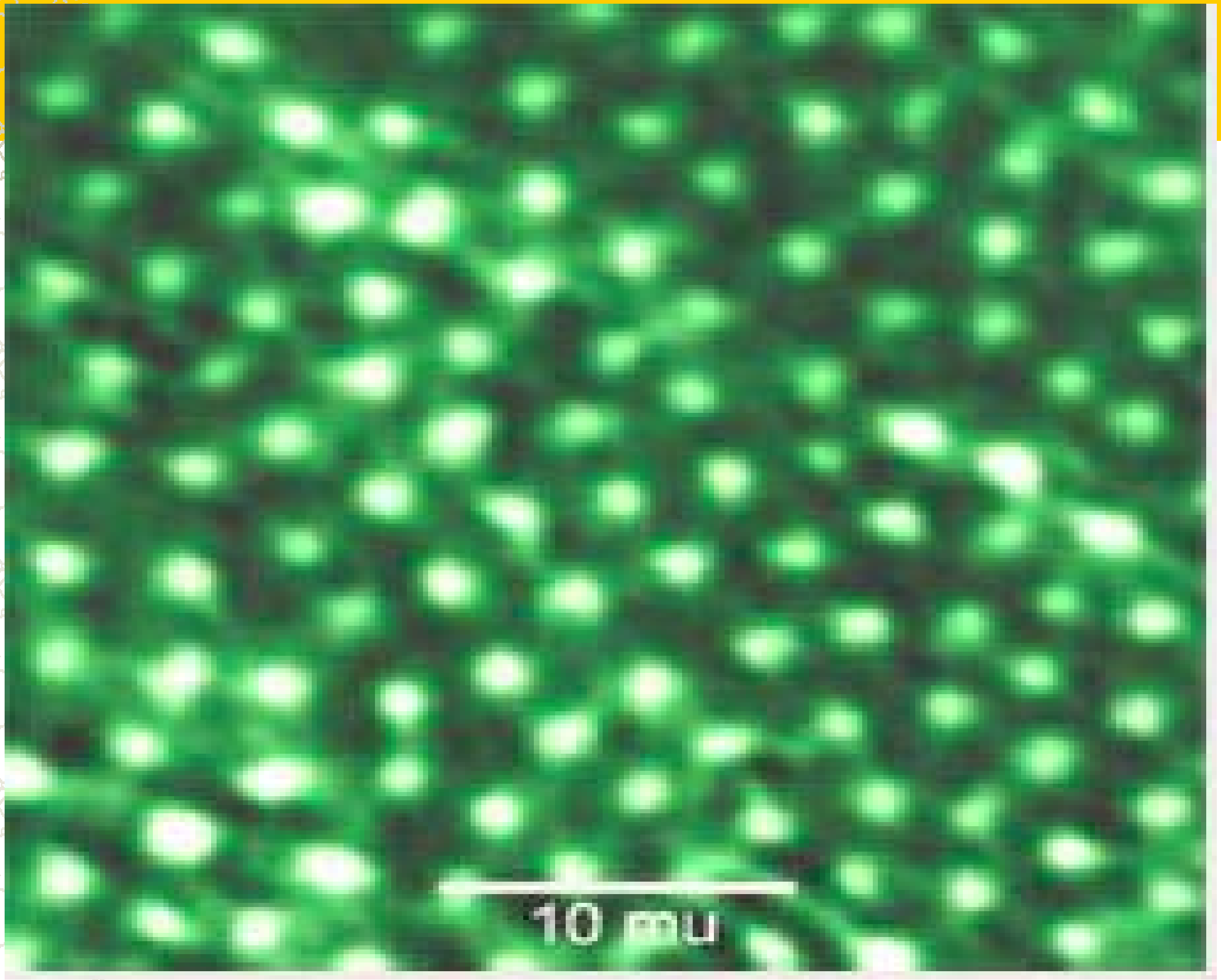
Vortex: From macroscale to microscale

Vortex in plant

Vortex: From macroscale to microscale

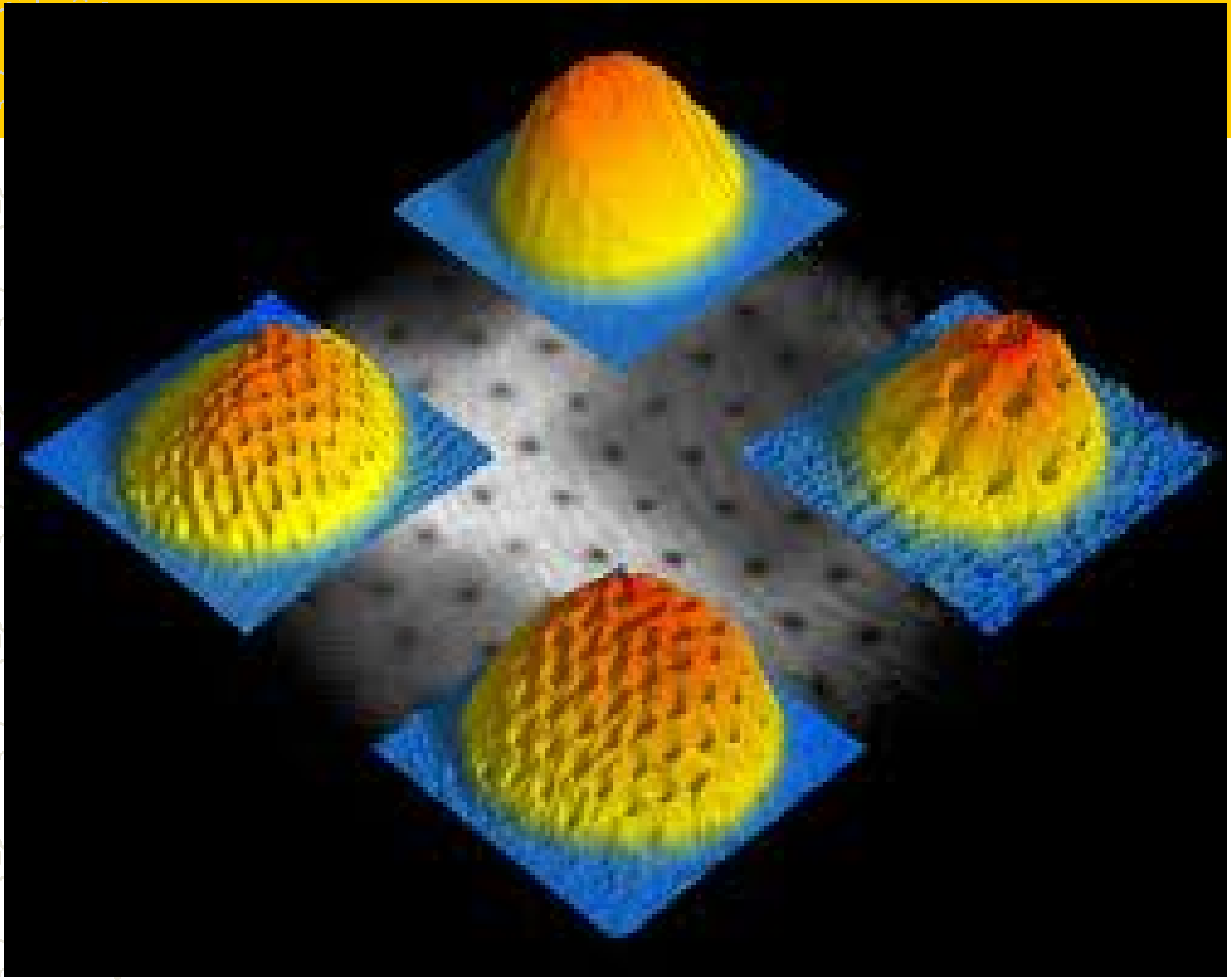
Quantized Vortex in liquid Helium 3

Vortex: From macroscale to microscale



Quantized Vortex in a type-II superconductor

Vortex: From macroscale to microscale



Quantized Vortex in Bose-Einstein condensation (BEC)

Outline

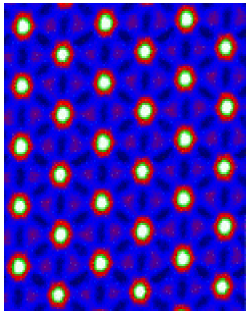
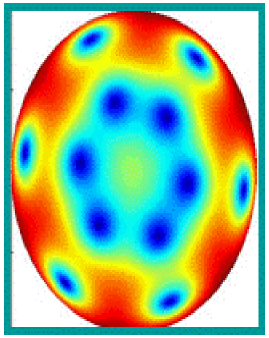
- Motivation
- Mathematical models
- Central vortex states and stability
- Vortex interaction and reduced dynamic laws
- Numerical methods
- Numerical results – whole space, bounded domain and in BEC
- Conclusions

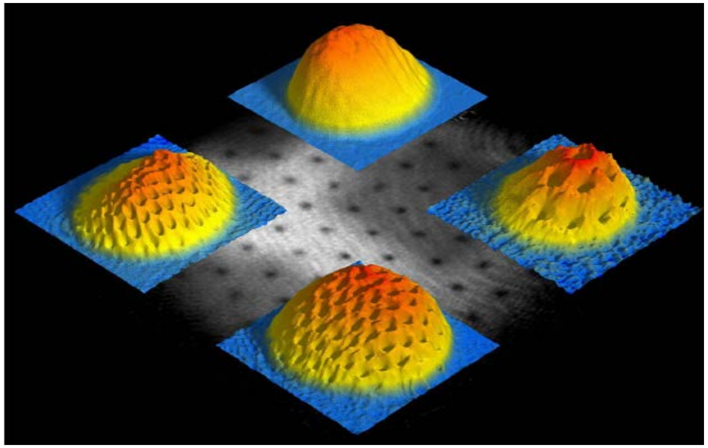
Motivation

- Quantized Vortex: Particle-like (topological) defect
  - Zero of the complex scalar field
  - localized phase singularities with integer topological charge:

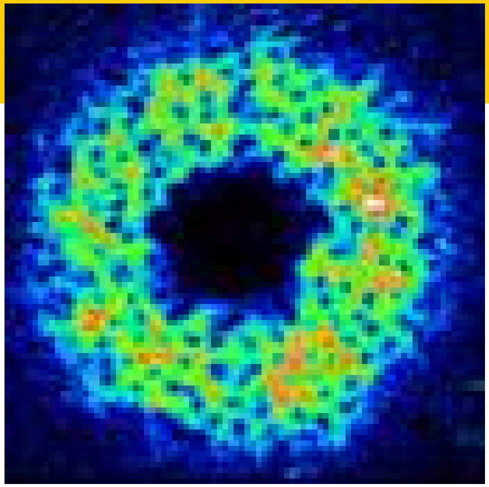
$$\psi(\vec{x}_0)=0, \quad \psi(\vec{x})=\sqrt{\rho}\, e^{i\phi}, \quad \int_\Gamma d(\arg\psi)=\int_\Gamma d\phi = 2\pi n \neq 0$$

  - Key of superfluidity: ability to support dissipationless flow

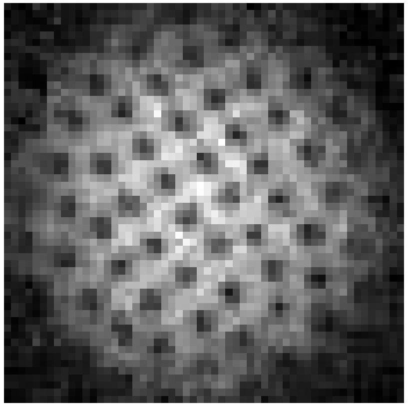





Motivation



- Existing in
  - Superconductors:
    - Ginzburg-Landau equations (GLE)
  - Liquid helium:
    - Two-fluid model
    - Gross-Pitaevskii equation (GPE)
  - Bose-Einstein condensation (BEC):
    - Nonlinear Schroedinger equation (NLSE) or GPE



  - Nonlinear optics & propagation of laser beams
    - Nonlinear Schroedinger equation (NLSE)
    - Nonlinear wave equation (NLWE)

Mathematical models

- Ginzburg-Landau equation (GLE): Superconductivity, nonlinear heat flow, etc.

$$\psi_t = \Delta \psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Gross-Pitaevskii equation (GPE): nonlinear optics, BEC, superfluidity

$$-i\,\psi_t = \Delta \psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Nonlinear wave equation (NLWE): wave motion

$$\psi_{tt} = \Delta \psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

– Here

ψ : complex-valued wave function or order parameter,

$\varepsilon > 0$: dimensionless constant,

$V(\vec{x})$: real-valued external potential

Mathematical models

- `Free Energy' or Lyapunov functional:

$$E(\psi)=\int_{\mathbb{R}^2}\left[\left|\nabla\psi\right|^2+\frac{1}{2\varepsilon^2}\left(V(\vec{x})-\left|\psi\right|^2\right)^2\right]d\vec{x},$$

- Dispersive system (GPE): $-i\frac{\partial\psi}{\partial t}=-\frac{\delta E(\psi)}{\delta\psi^*}$
  - Energy conservation: $E(\psi)\equiv E(\psi_0),\quad t\ge 0$
  - Density conservation: $\int_{\mathbb{R}^2}\left(|\psi(\vec{x},t)|^2-|\psi_0(\vec{x})|^2\right)d\vec{x}\equiv 0,\quad t\ge 0$
  - Admits particle like solutions: solitons, kinks & vortices
- Dissipative system (GLE): $\frac{\partial\psi}{\partial t}=-\frac{\delta E(\psi)}{\delta\psi^*}$
  - Energy diminishing, etc.

Two kinds of vortices

- \`Bright-tail' vortex: GLE, GPE & NLWE (\`order parameter')

$$V(\vec{x}) \to 1, \quad \text{when } |\vec{x}| \to \infty, \quad \Rightarrow |\psi(\vec{x},t)| \to 1, \qquad |\vec{x}| \to \infty, \quad (e.g.\ \psi \to e^{im\theta}),$$

  - Time independent case (*Neu, 90'*): $\varepsilon = 1, \quad V(\vec{x}) \equiv 1$
  - Vortex solutions: $\psi(\vec{x}) = \phi_n(\vec{x}) = f_n(r)\, e^{in\theta}$

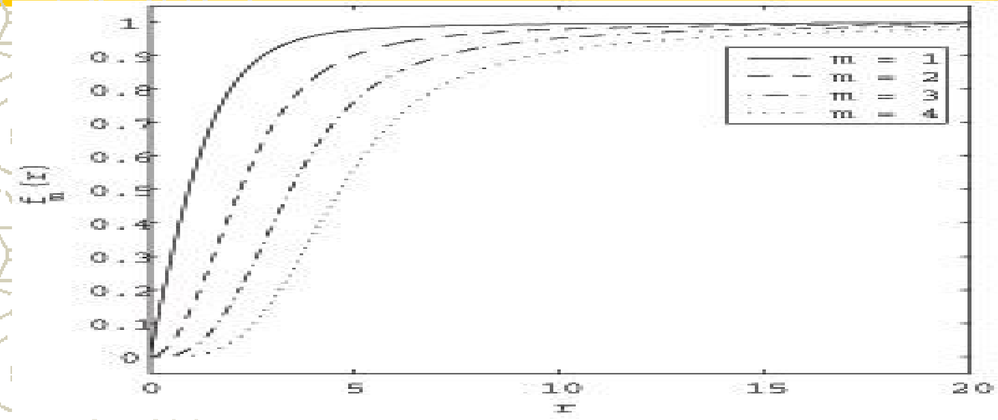
$$f_n''(r) + \frac{1}{r} f_n'(r) - \frac{n^2}{r^2} f_n(r) + (1 - f_n^2(r)) f_n(r) = 0, \quad 0 < r < \infty,$$

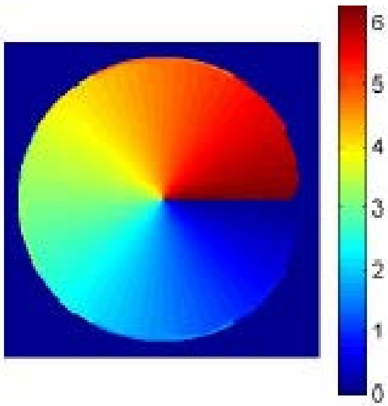
$$f_n(0) = 0, \qquad f_n(\infty) = 1.$$

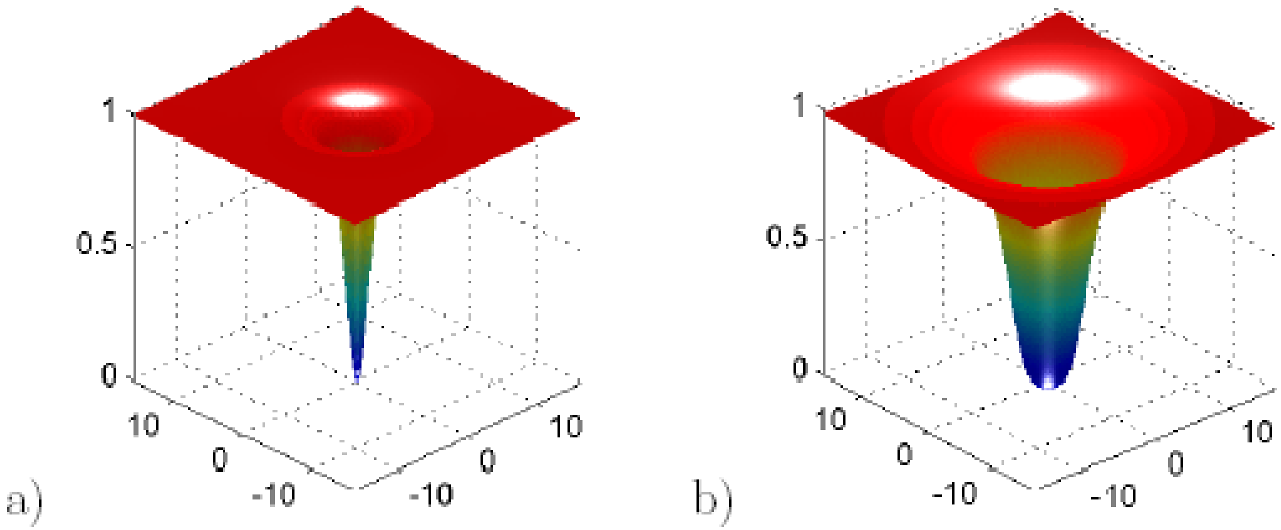
  - Asymptotic results

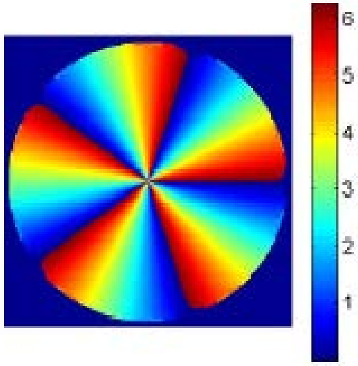
$$f_n(r) \approx \begin{cases} ar^{|n|} + O(r^{|n|+2}), & r \to 0, \\ 1 - n^2/2r^2 + O(1/r^4), & r \to \infty. \end{cases}$$

`Bright-tail' vortex









Two kinds of vortices

- `Dark-tail' vortex: BEC (wave function)

$$i\,\psi_t = -\Delta\psi + \frac{|\vec{x}|^2}{2}\psi + \beta\,|\psi|^2\,\psi, \quad \vec{x}\in R^2, \quad t>0, \quad \|\psi\|^2 = \int |\psi|^2\, d\vec{x} = 1$$

  - Center vortex states: $\psi(\vec{x},t) = e^{-i\mu_n t}\phi_n(\vec{x}) = e^{-i\mu_n t} f_n(r)\, e^{in\theta}$

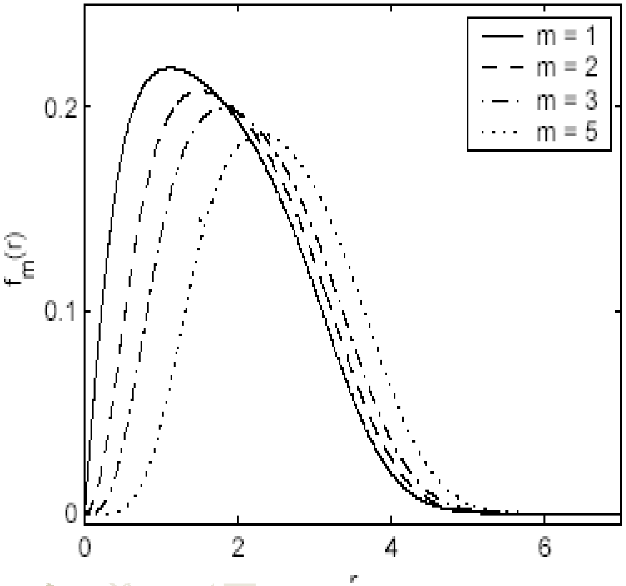
$$\mu_n\, f_n(r) = -\frac{d}{dr}\left(r\frac{df_n(r)}{dr}\right) + \left(\frac{n^2}{r^2}+\frac{r^2}{2}\right) f_n(r) + \beta f_n^3(r), \quad 0<r<\infty,$$

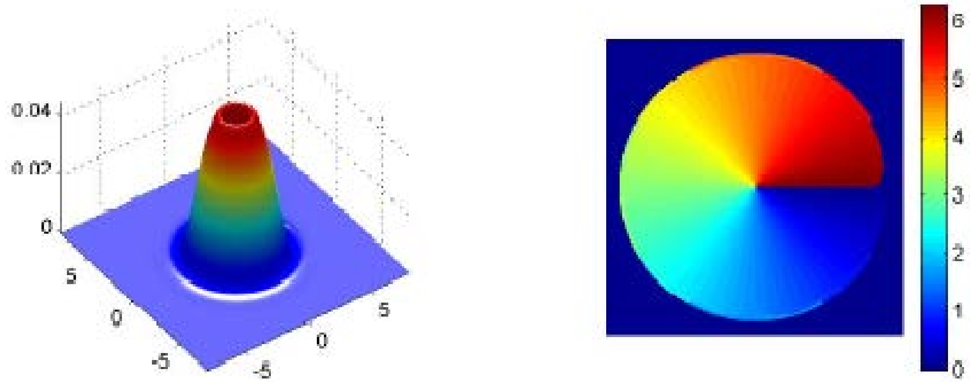
$$f_n(0)=0, \quad f_n(\infty)=0, \quad 2\pi\int_0^\infty f_n^2(r)\, r\, dr = 1$$

  - Asymptotic results

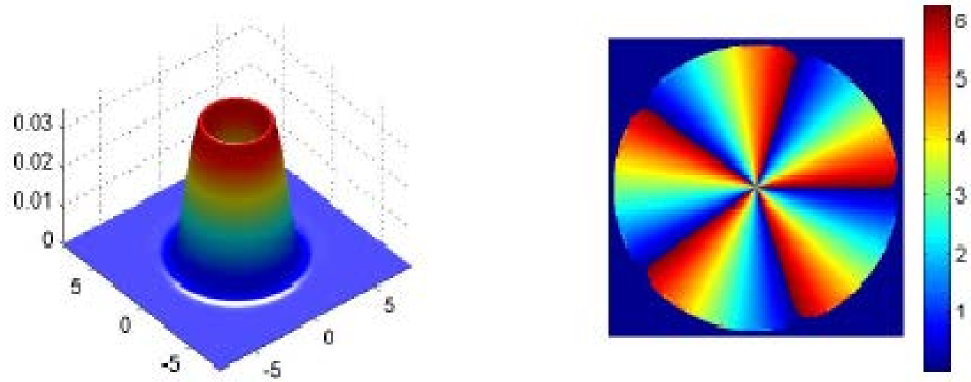
$$f_n(r) \approx \begin{cases} a\, r^{|n|} + O(r^{|n|+2}), & r\to 0, \\ b\, r^{|n|} e^{-r^2}, & r\to\infty. \end{cases}$$

`Dark-tail' vortex





a)



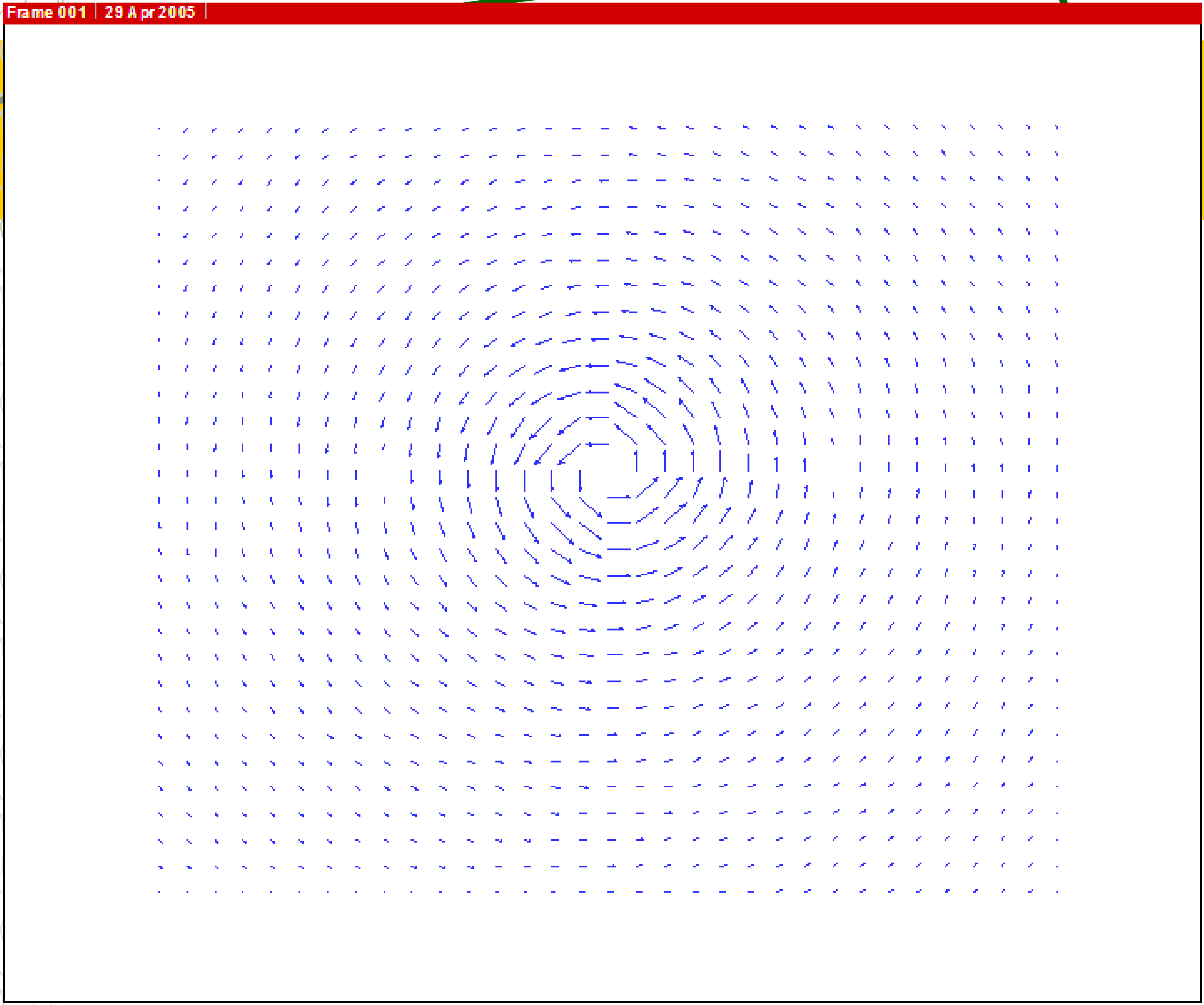
b)

Stability of vortices

- Data chosen

$$\varepsilon = 1, \qquad V(\vec{x},t) \equiv 1 \quad (+W(\vec{x},t)), \qquad \psi_0(\vec{x}) = \phi_n^h(\vec{x}) \quad (+\,\text{noise})$$

- Results (Bao,Du & Zhang, Eur. J. Appl. Math., 07')
  - For GLE or NLWE with initial data perturbed
    - n=1: velocity density
    - N=3: velocity density
  - For NLSE with external potential perturbed
    - n=1: velocity density
    - N=3: velocity density
- Results



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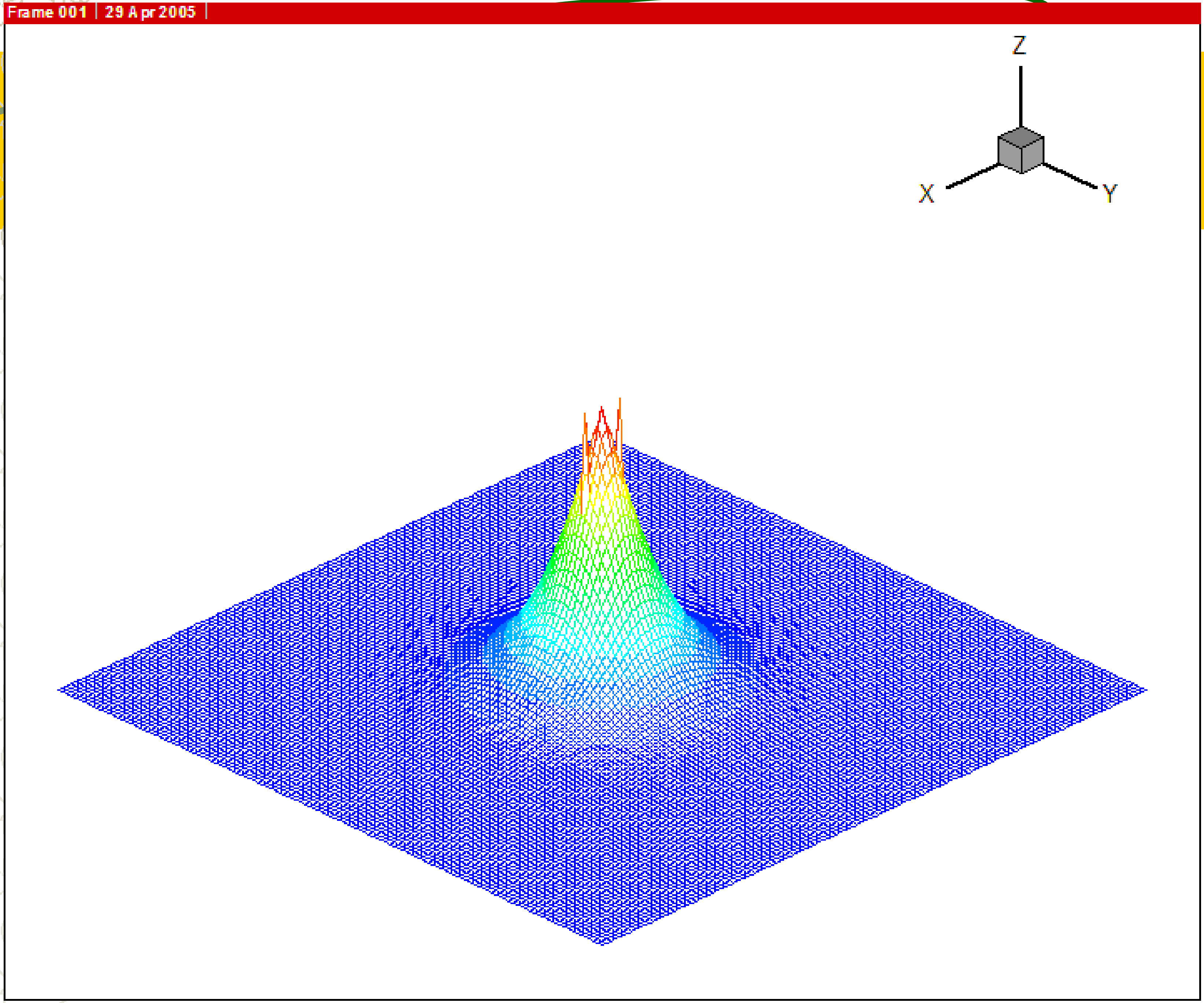

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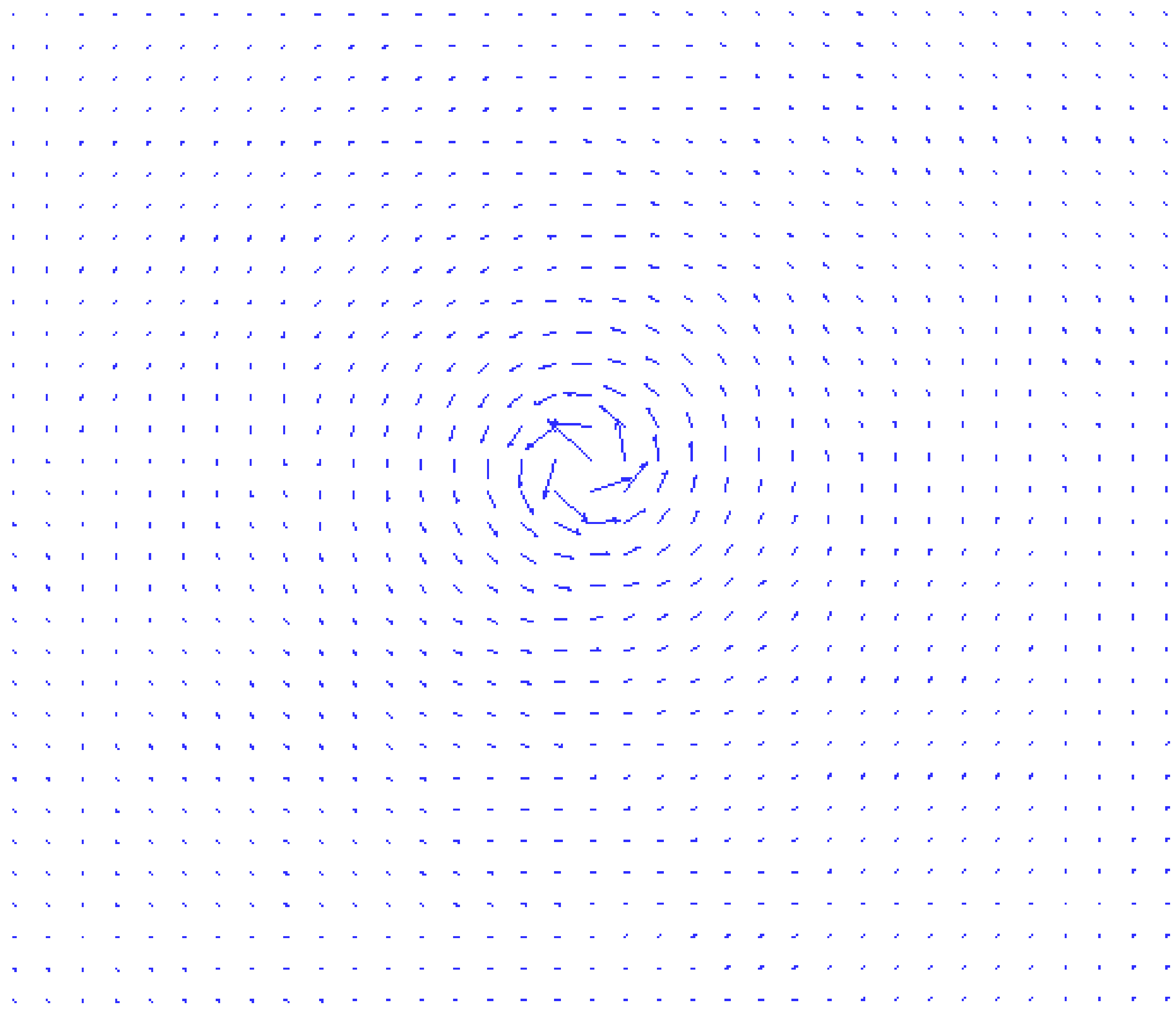

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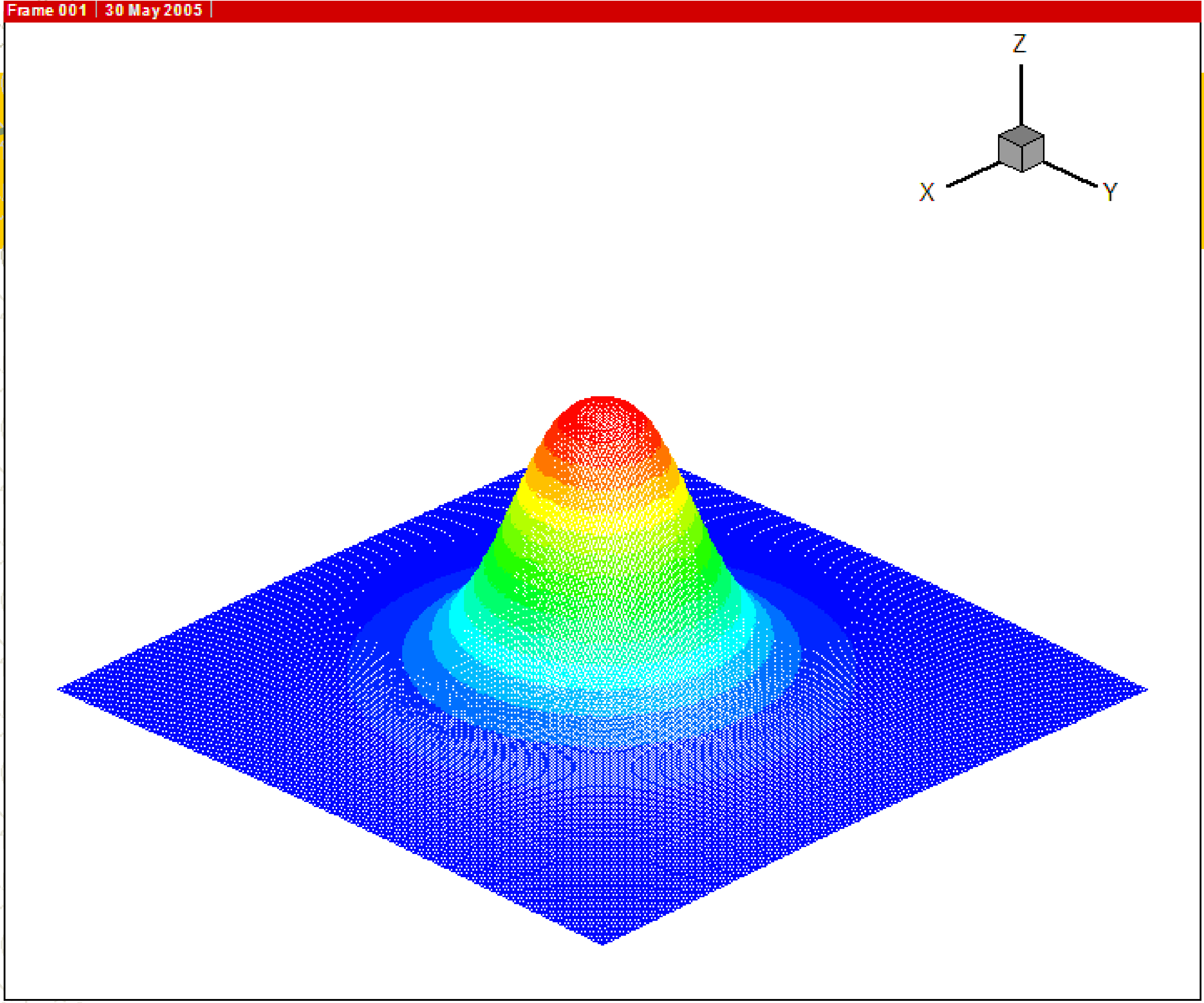

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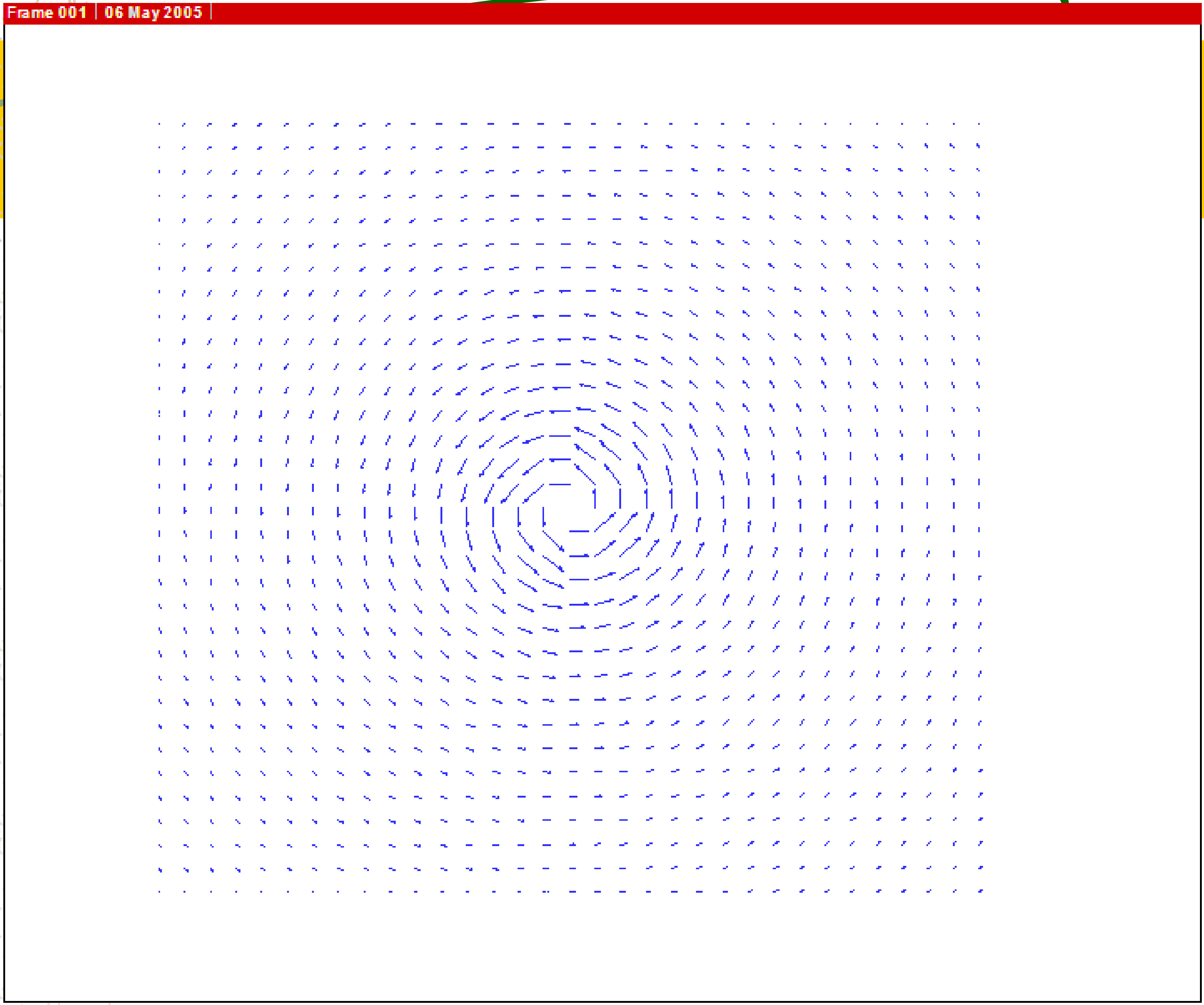

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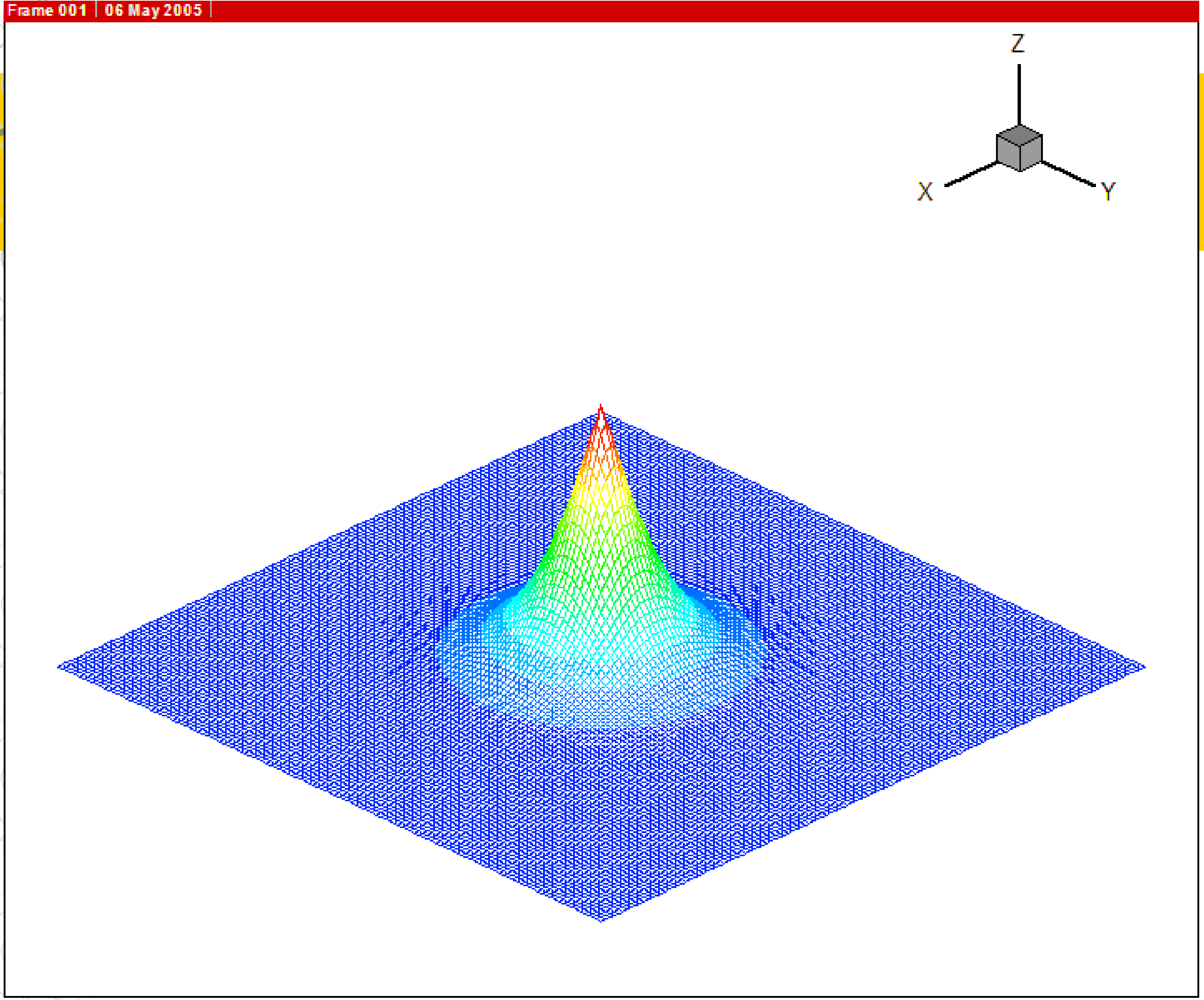

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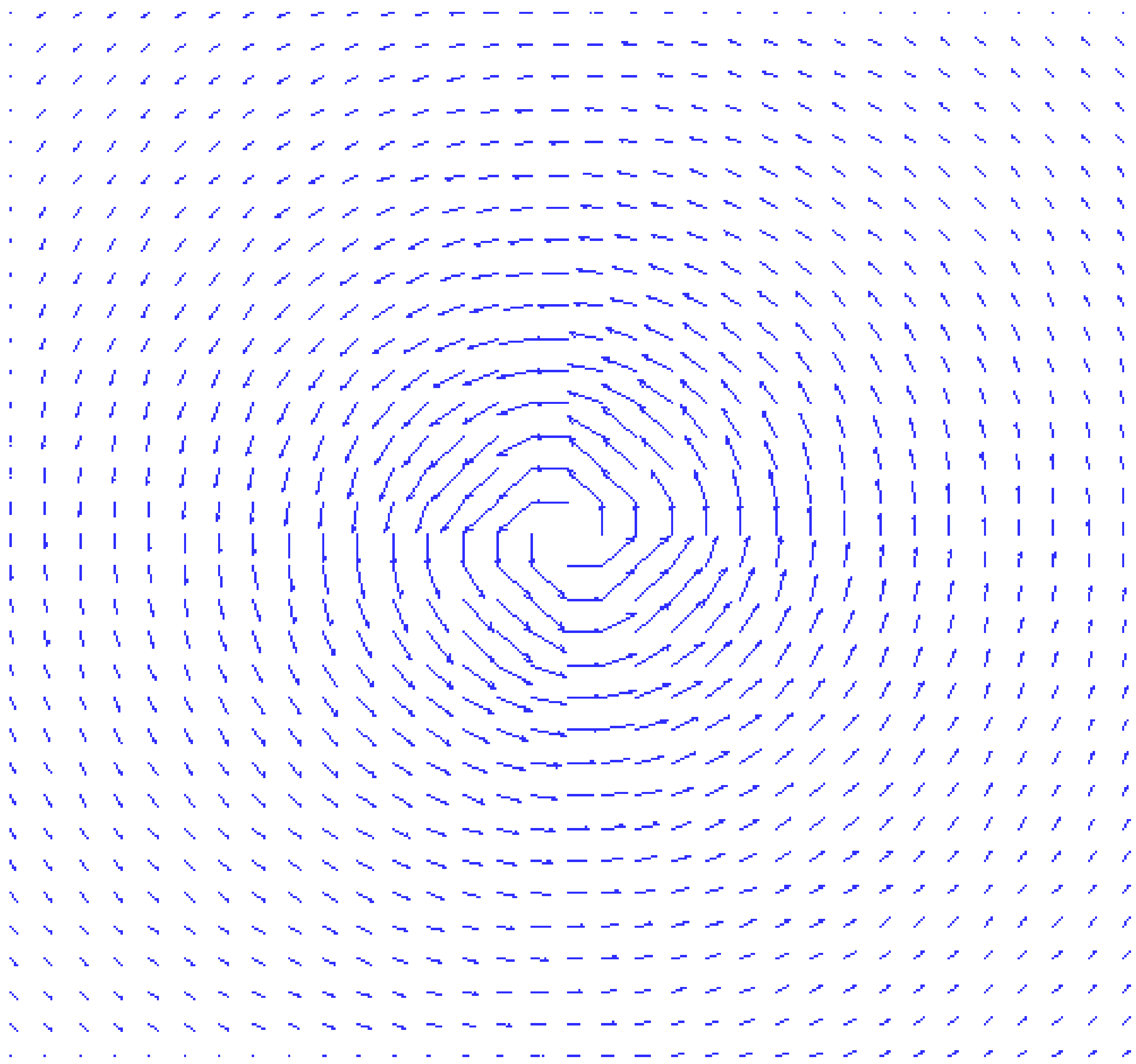

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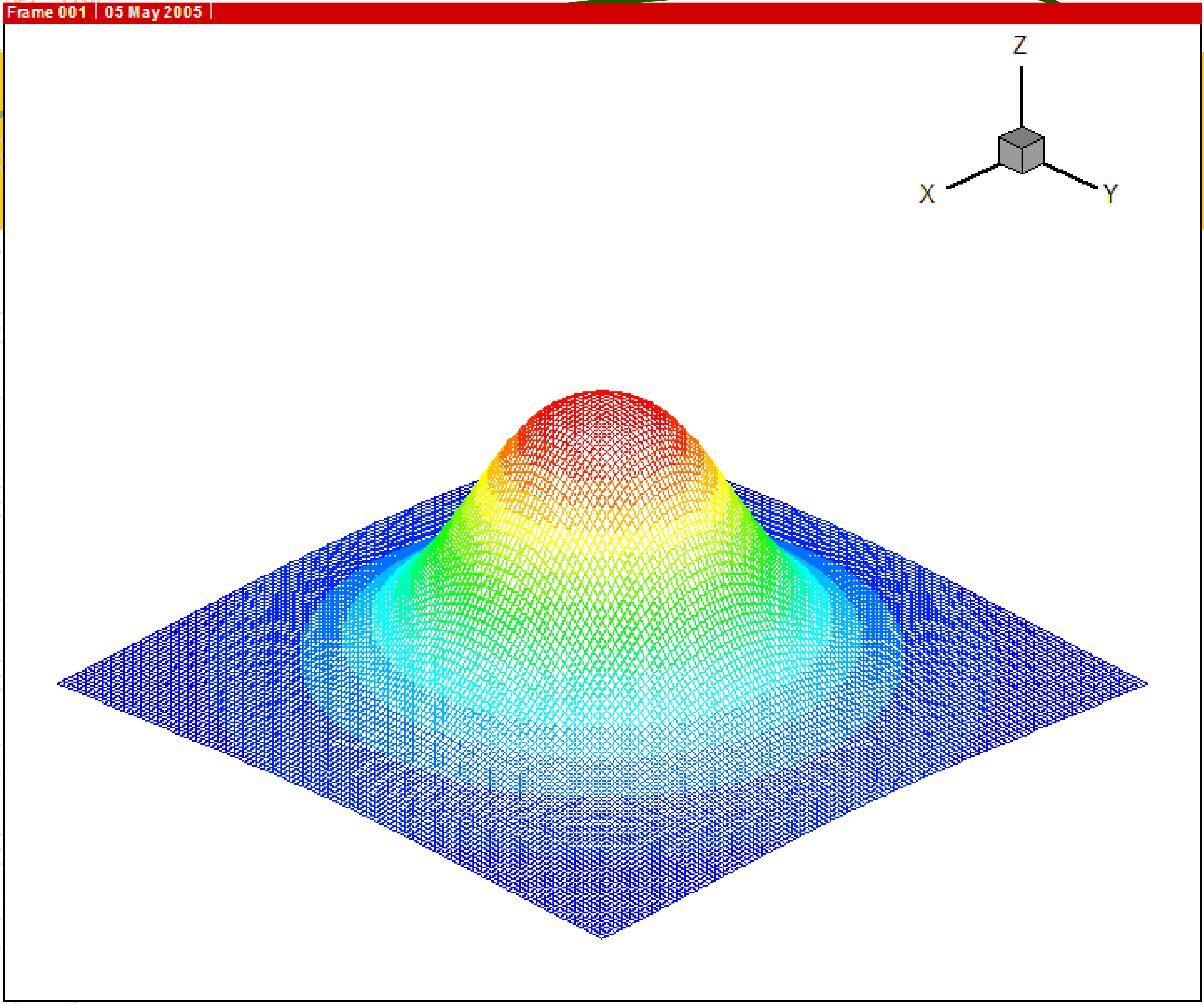

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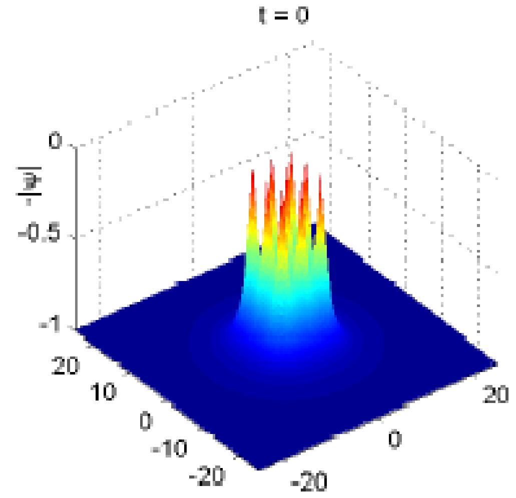
Summary of Stability

- For GLE or NLWE:
  - under perturbation either in initial data or in external potential
    - $|n|=1$: dynamically stable,
    - $|n|>1$: dynamically unstable. Splitting pattern depends on perturbation, when $t \gg 1$, it becomes n well-separated vortices with index 1 or –1.
- For NLSE or GPE:
  - under perturbation in initial data
    - dynamically stable: Angular momentum expectation is conserved!!
  - under perturbation in external potential
    - $|n|=1$: dynamically stable,
    - $|n|>1$: dynamically unstable. Vortex centers can NOT move out of core size.

Vortex interaction



- For `bright-tail' vortex:

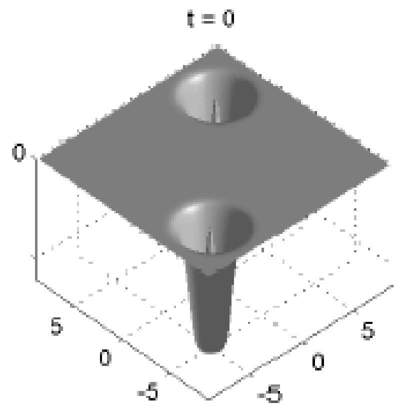
$$\psi_0(\vec{x}) = \prod_{j=1}^{N} \phi_{m_j}(\vec{x} - \vec{x}_j^0) = \prod_{j=1}^{N} \phi_{m_j}(x - x_j^0, y - y_j^0)$$

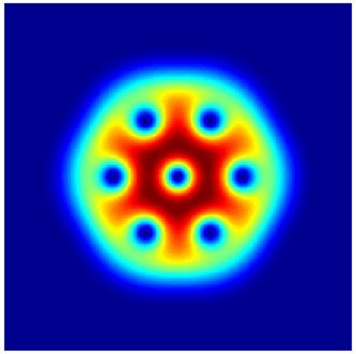
- For `dark-tail' vortex
  - Well-separate

$$\psi_0(x,y) = \sum_{j=1}^{N} \phi_{n_j}(x - x_j, y - y_j) \quad / \|\bullet\|$$

  - Overlapped

$$\psi(\mathbf{r}, t=0) = \alpha \psi_{gs}(\mathbf{r}) \prod_{j=1}^{n} p_{q_j}(\mathbf{r} - \mathbf{r}_j),$$





Reduced dynamic laws

- For `bright-tail' vortex
  - Take ansatz

$$\psi(\vec{x},t)=\prod_{j=1}^{N}\phi_{m_j}(\vec{x}-\vec{x}_j(t))+\text{ high order terms}$$

$$=\prod_{j=1}^{N}\phi_{m_j}(x-x_j(t),y-y_j(t))+\text{high order terms}$$

  - Plug into GLE or GPE or NLWE
  - Using matched asymptotic techniques
  - Prove rigorously

Reduced dynamic laws

- For `bright-tail' vortex
  - For GLE

$$\kappa \vec{v}_j(t) := \kappa \frac{d\,\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2}, \qquad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \qquad 1 \leq j \leq N.$$

  - For GPE

$$\vec{v}_j(t) := \frac{d\,\vec{x}_j(t)}{dt} = 2 \sum_{l=1, l\neq j}^{N} m_l \frac{J(\vec{x}_j(t)-\vec{x}_l(t))}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} \qquad t \geq 0,$$

$$\vec{x}_j(0) = \vec{x}_j^0 \qquad 1 \leq j \leq N; \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  - For NLWE

$$\kappa \frac{d^2\,\vec{x}_j(t)}{dt^2} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2}, \qquad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad \vec{x}_j^{'}(0) = \vec{v}_j^0, \quad 1 \leq j \leq N.$$

Existing mathematical results

- For GLE (or NLHE):
  - Pairs of vortices with like (opposite) index undergo a replusive (attractive) interaction: *Neu 90*, *Pismen & Rubinstein 91, W. E, 94,* Bethual, etc.
  - *Energy concentrated at vortices in 2D & filaments in 3D: Lin 95--*
  - *Vortices are attracted by impurities: Chapman & Richardson 97, Jian 01*
- For NLSE:
  - Vortices behave like point vortices in ideal fluid: *Neu 90*
  - Obeys classical Kirchhoff law for fluid point vortices: *Lin & Xin 98*
  - Equations for vortex dynamics & radiation: *Ovchinnikov& Sigal 98--; Jerrard, Bethual, etc.*

Conservation laws

- Mass center $\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t)$

Lemma The mass center of the N vortices for GLE is conserved

$$\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t) \equiv \overline{x}(0) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j^0$$

Lemma The mass center of the N vortices for NLWE is conserved if initial velocity is zero and it moves linearly if initial velocity is nonzero

$$\overline{\mathbf{x}}(t) := \frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j(t) = \overline{\mathbf{x}}(0) + t\,\overline{\mathbf{v}}(0) = \frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j^0 + t\left(\frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j^1\right), \quad t \geq 0,$$

Conservation laws

- Signed mass center $\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(t)$

Lemma The signed mass center of the N vortices for NLSE is conserved

$$\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(t) \equiv \tilde{x}(0) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j^{\,0}$$

Analytical Solutions

- $N\,(\geq 2)$ like vortices on a circle (Bao, Du & Zhang, SIAP, 07')

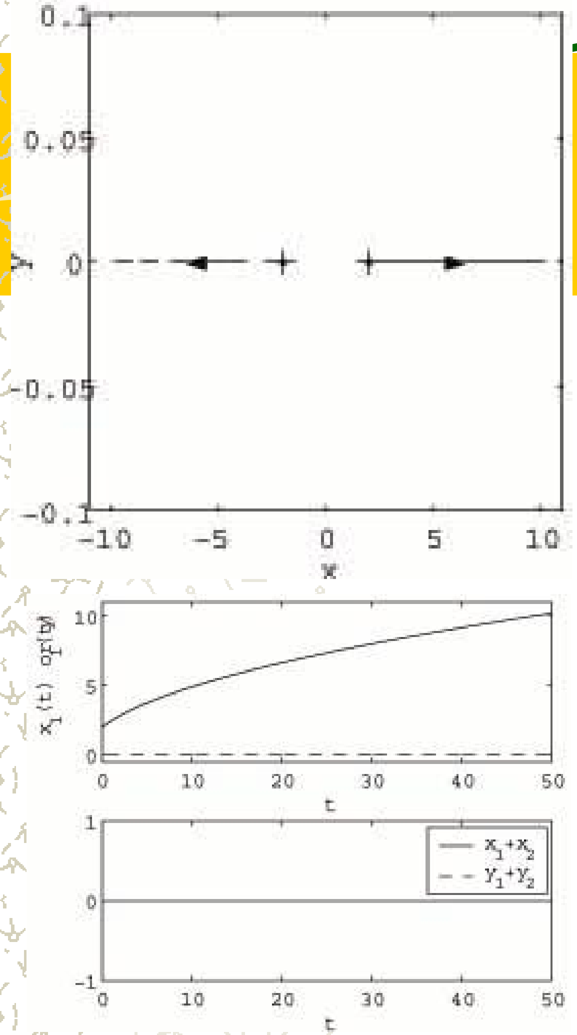
$$\vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N}\right) \;,\; \sin\left(\frac{2j\pi}{N}\right)\right), \qquad m_j = m_0 = \pm 1, \qquad 1 \leq j \leq N$$

- Analytical solutions for GLE figure

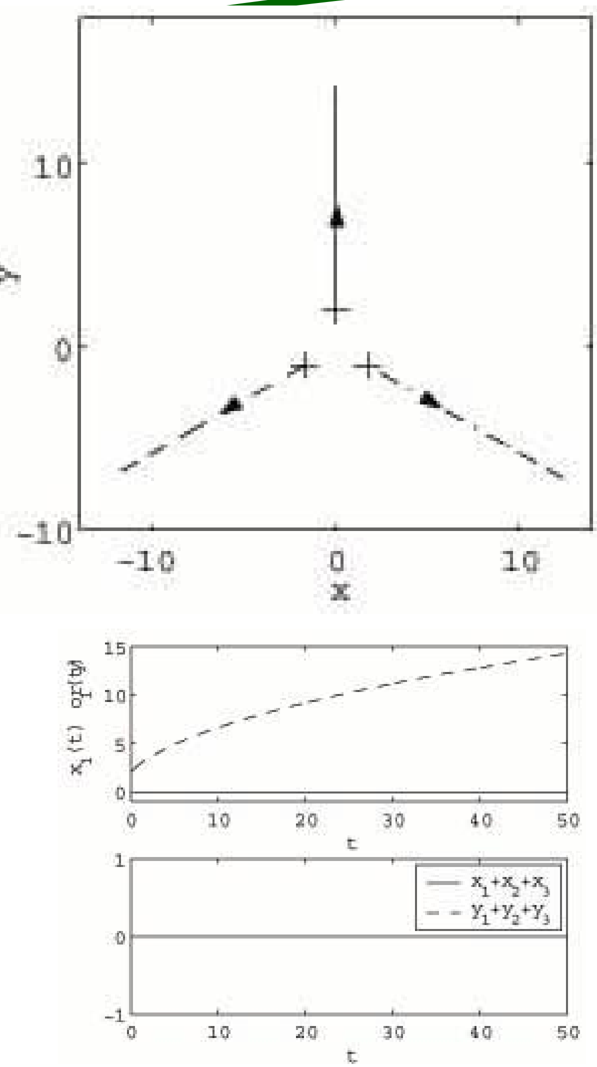
$$\vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-1)}{\kappa}t}\;\left(\cos\left(\frac{2j\pi}{N}\right) \;,\; \sin\left(\frac{2j\pi}{N}\right)\right)$$

- Analytical solutions for NLSE figure next

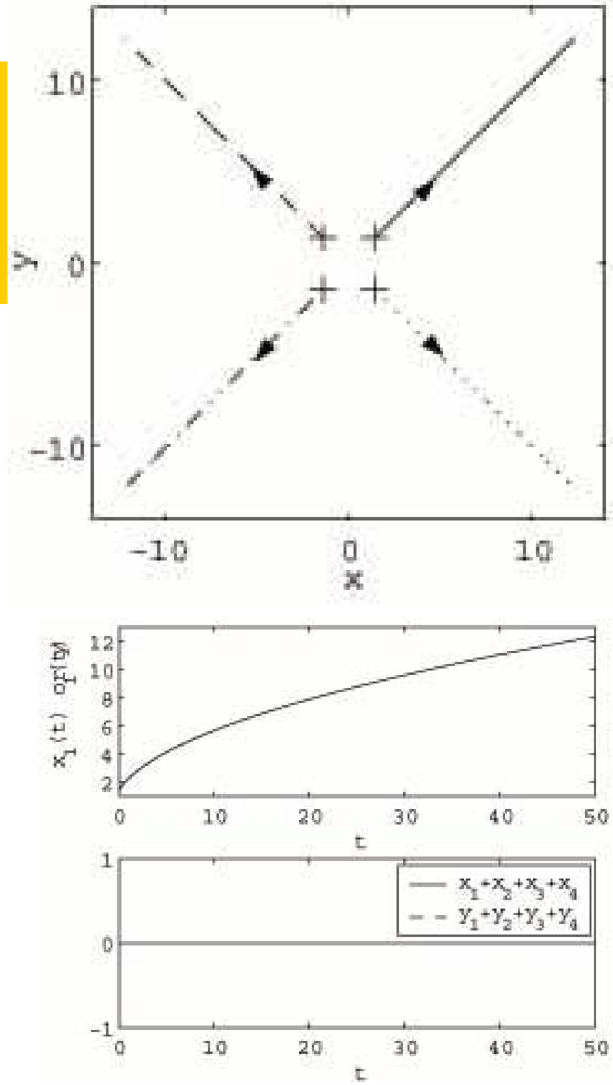
$$\vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right) \;,\; \sin\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right)\right)$$


x
y
t
x1(t) or y1(t)
x1+x2
y1+y2

N=2


x
t
x1(t) or y1(t)
x1+x2+x3
y1+y2+y3

N=3


x
y
t
x1(t) or y1(t)
x1+x2+x3+x4
y1+y2+y3+y4

N=4

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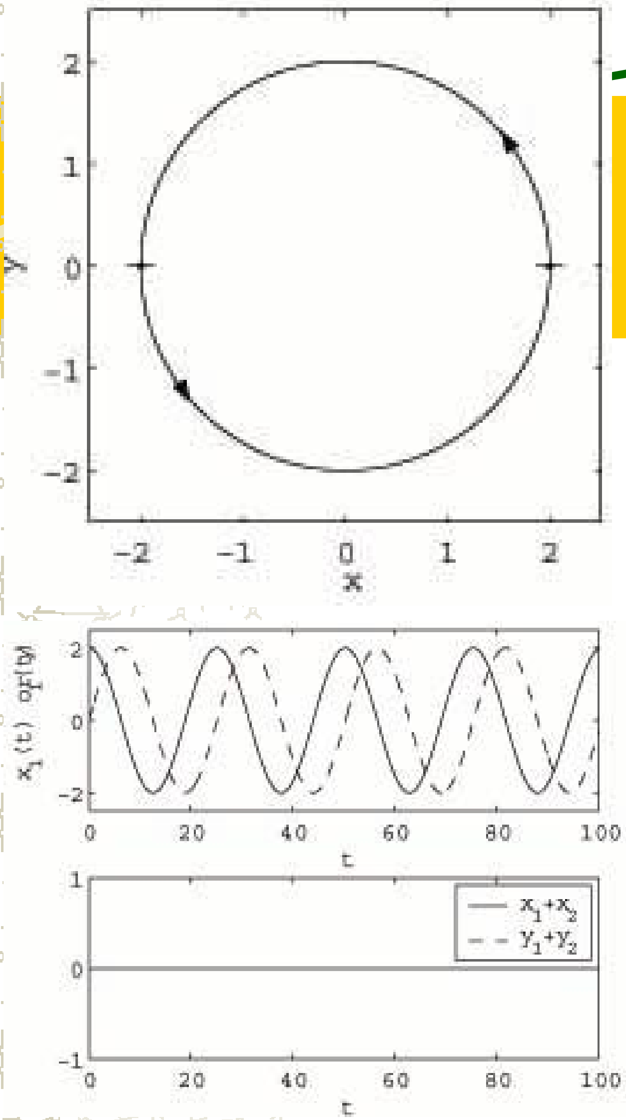
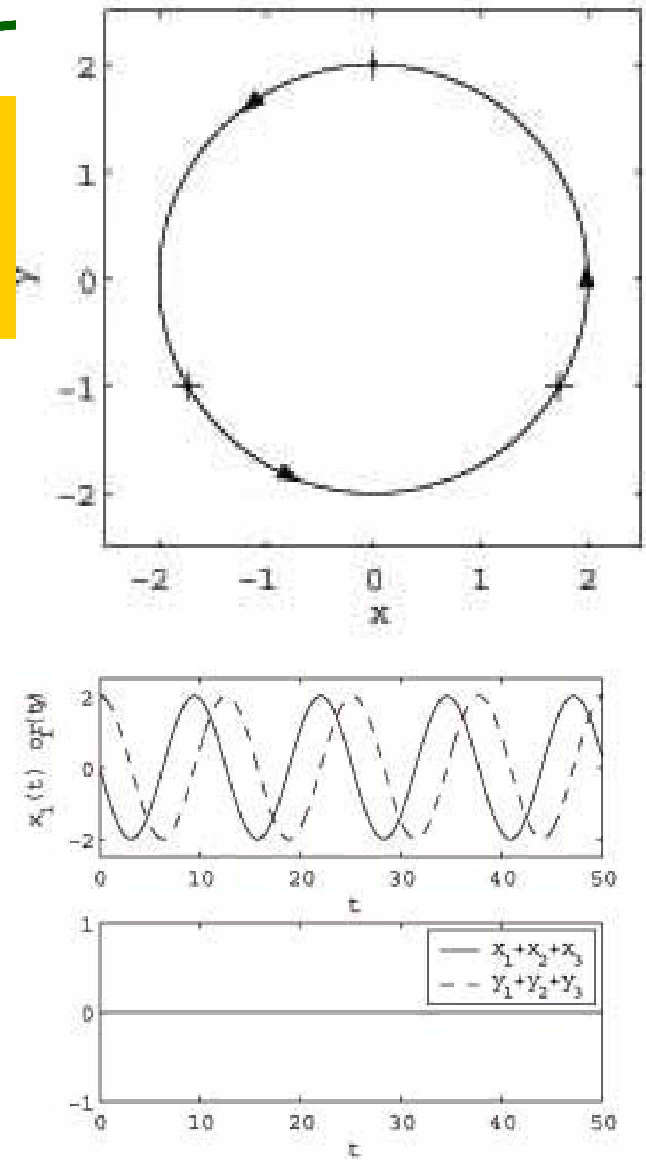


N=2



N=3

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Analytical Solutions

- $N\,(\geq 3)$ like vortices on a circle and its center (Bao,Du&Zhang, SIAP, 07')

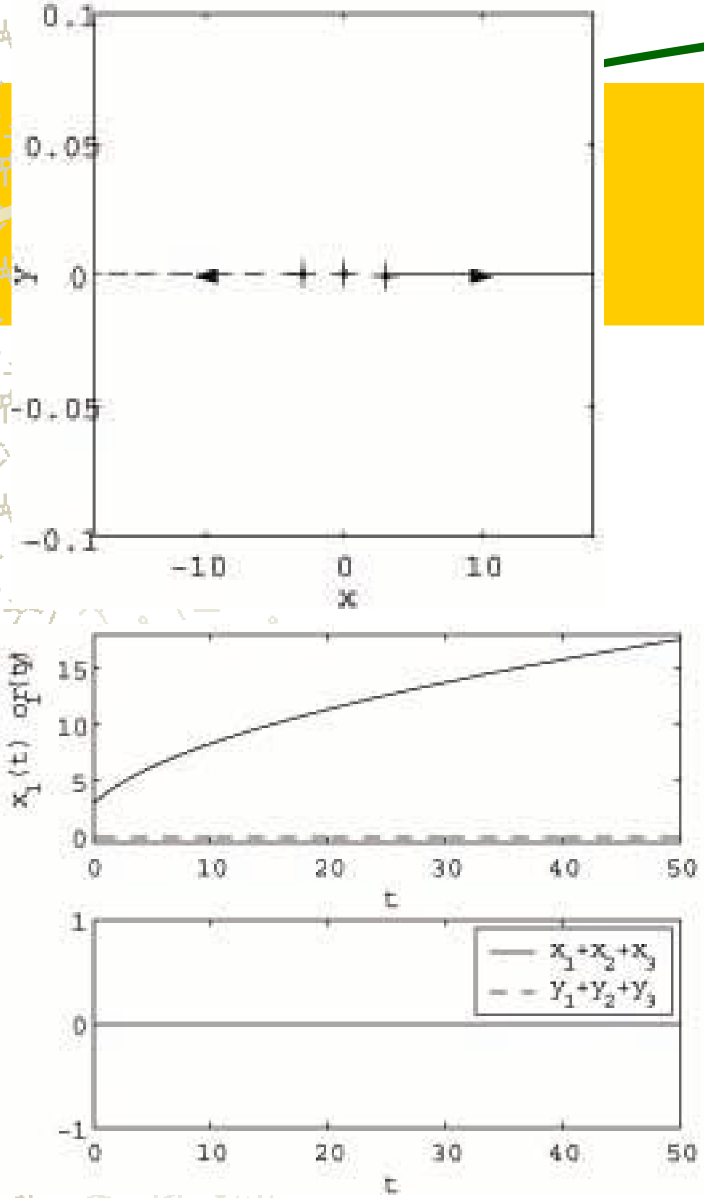
$$\vec{x}_N^0 = \vec{0}; \qquad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE figure

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + \frac{2N}{\kappa}t}\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE figure next

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right), \ \sin\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right)\right)$$



0.1
0.05
0
-0.05
-0.1
y
-10
0
10
x
15
10
5
0
x1(t) or y
0
10
20
30
40
50
t
1
0
-1
x1+x2+x3
y1+y2+y3
t

N=3

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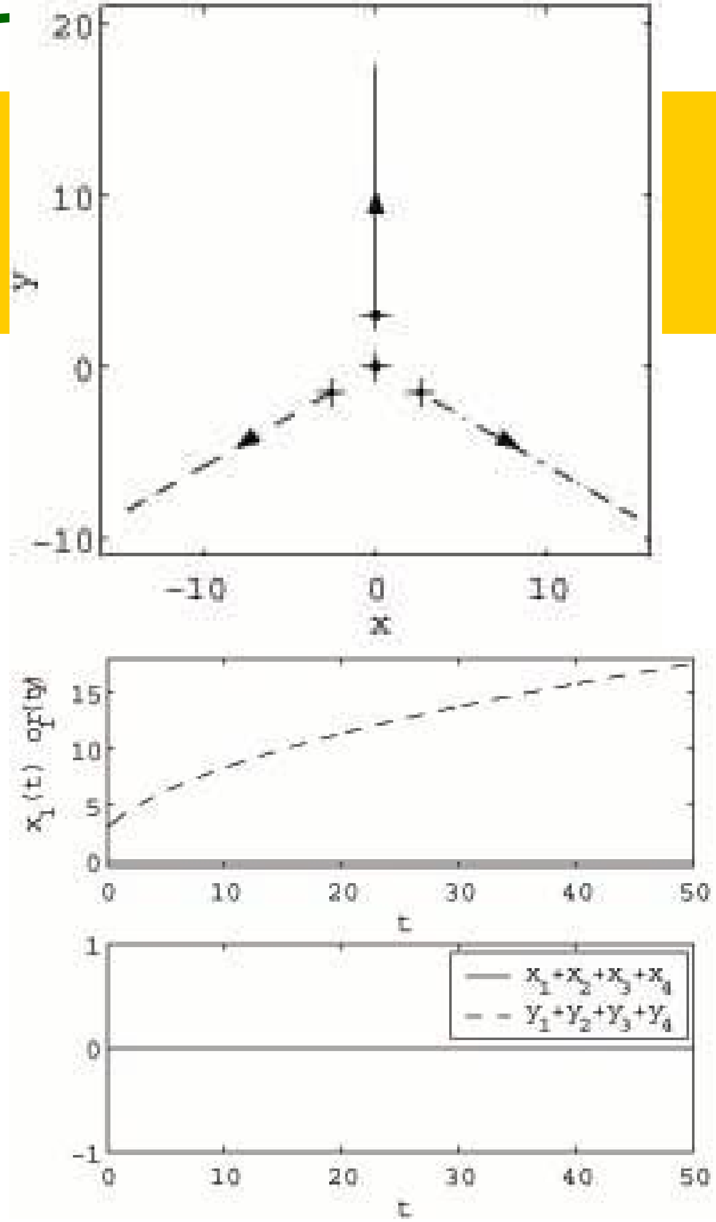
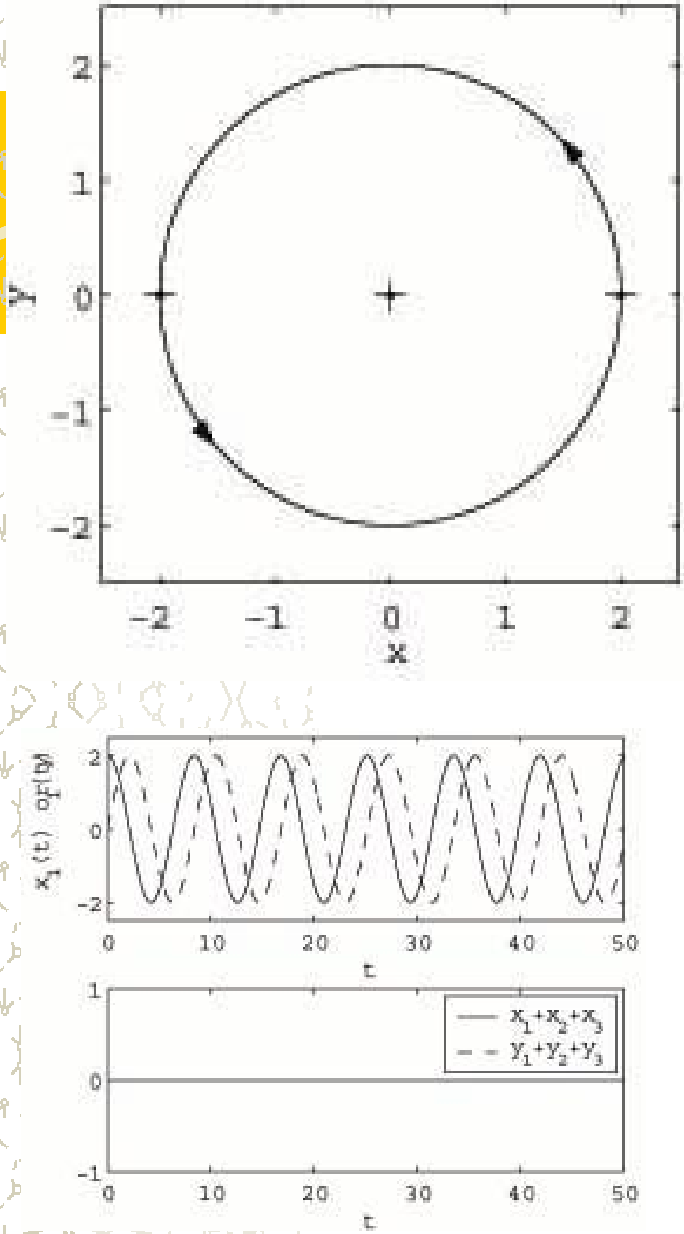


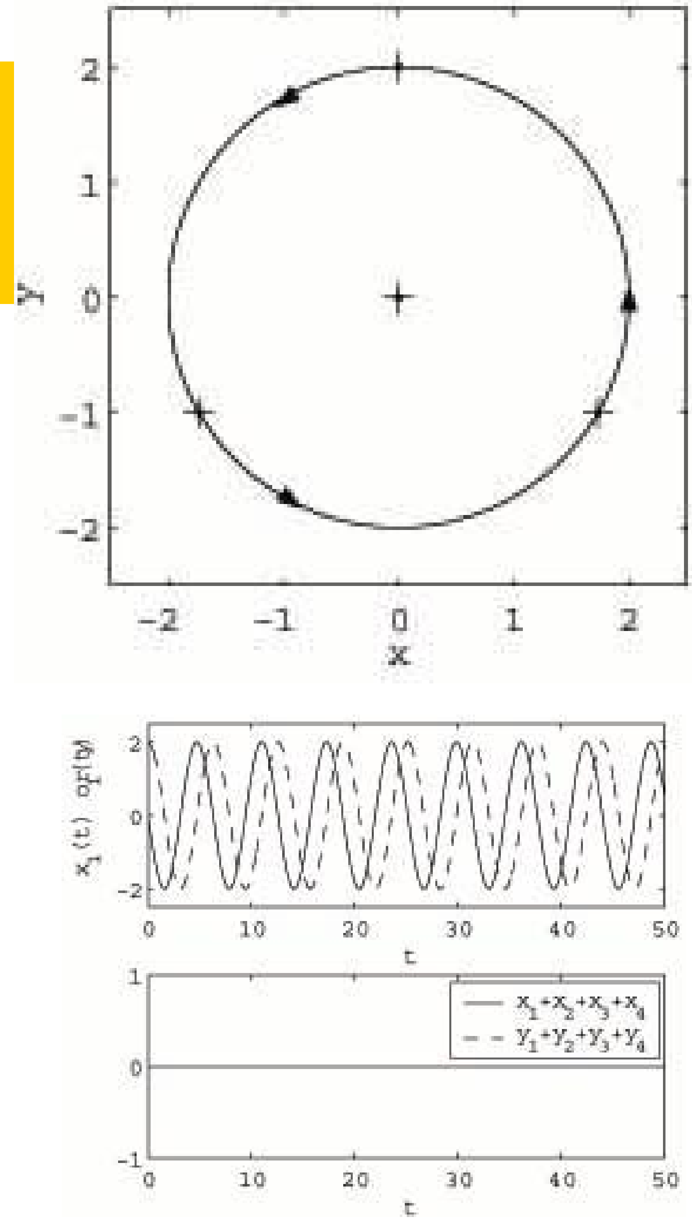
20
10
0
-10
y
-10
0
10
x
15
10
5
0
x1(t) or y
0
10
20
30
40
50
t
1
0
-1
x1+x2+x3+x4
y1+y2+y3+y4
t

N=4



N=3

back



N=4

Analytical Solutions

- Two opposite vortices (Bao, Du & Zhang, SIAP, 07')

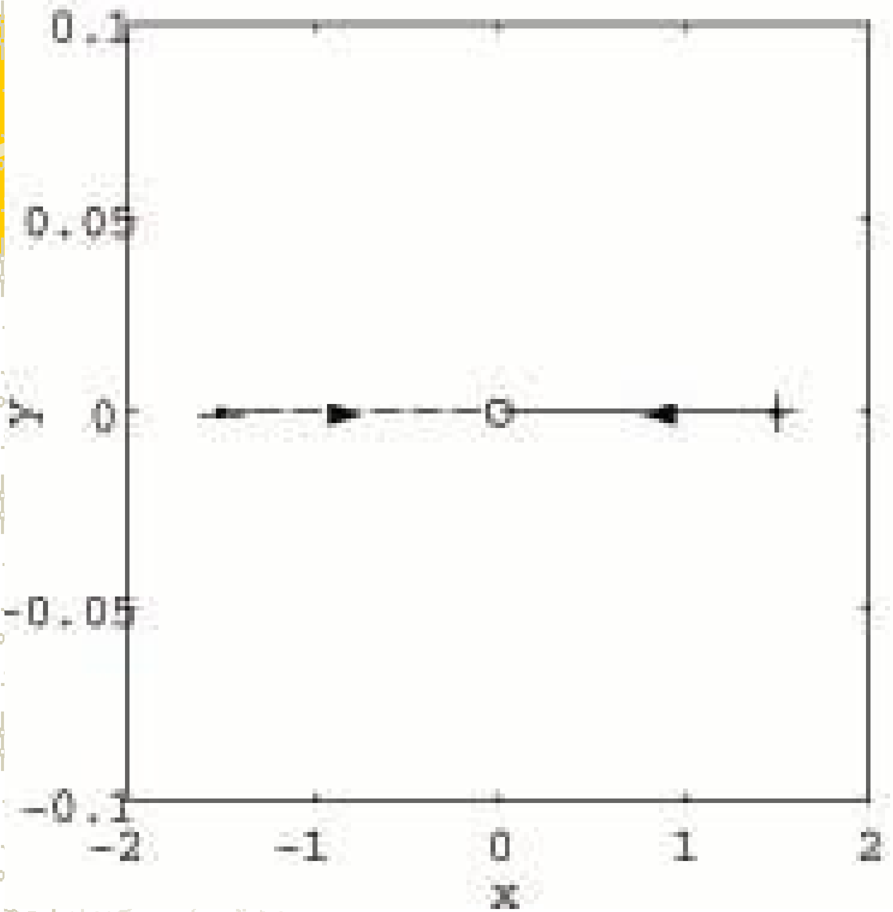
$$\vec{x}_1^0 = -\vec{x}_2^0 = a\left(\cos(\theta_0)\quad,\quad \sin(\theta_0)\right),\qquad m_1 = -m_2 = 1$$

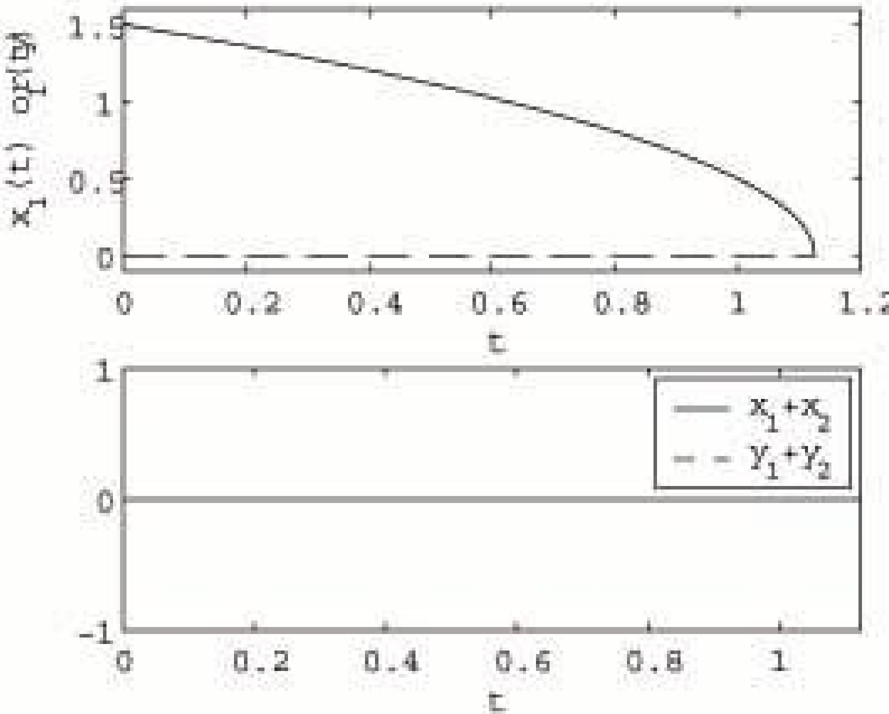
- Analytical solutions for GLE figure

$$\vec{x}_1(t) = -\vec{x}_2(t) = \sqrt{a^2 - \frac{2}{\kappa}t}\;\left(\cos(\theta_0)\quad,\quad \sin(\theta_0)\right)$$

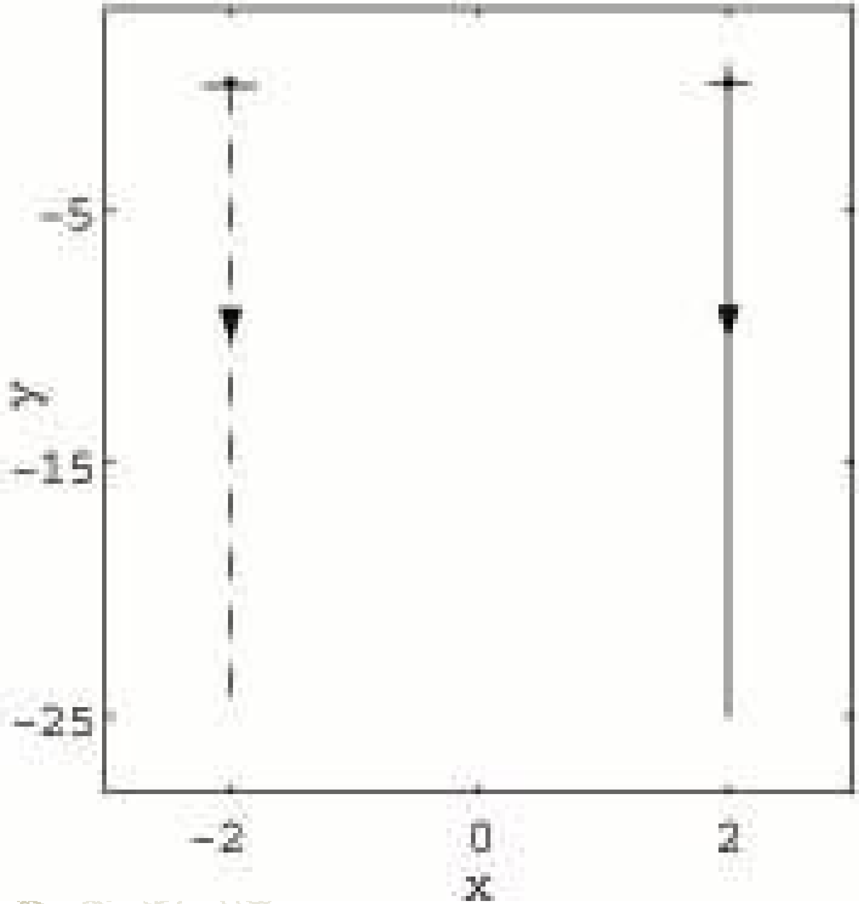
- Analytical solutions for NLSE figure next

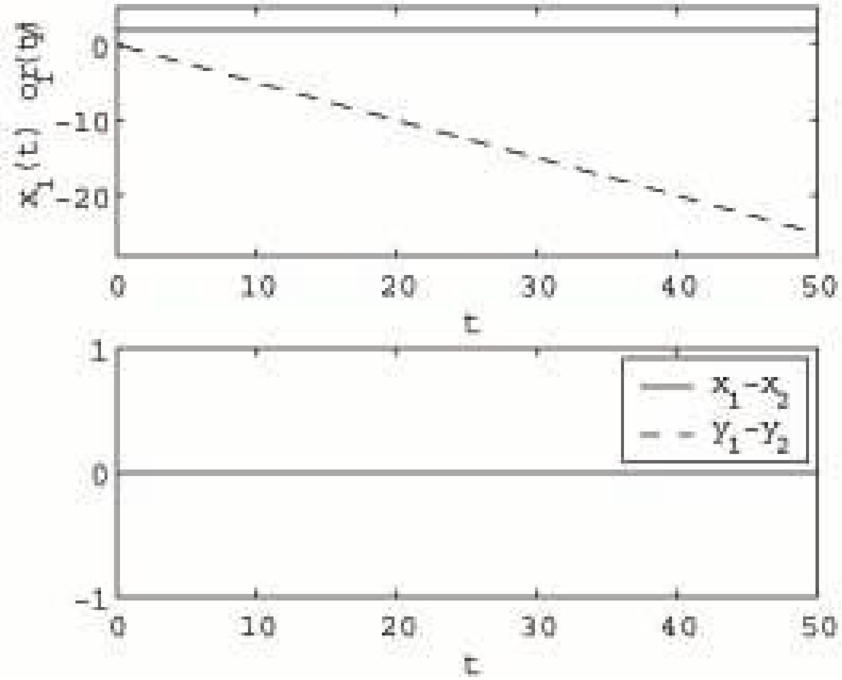
$$\vec{x}_j(t) = \vec{x}_j^0 + \frac{t}{a}\left(-\sin(\theta_0)\quad,\quad \cos(\theta_0)\right),\qquad j = 1, 2$$


0.1
0.05
0
-0.05
-0.1
-2
-1
0
1
2
x
y


x1(t) or y1(t)
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
t
x1+x2
y1+y2
-1
t

back





back

Analytical Solutions

- $N\,(\geq 3)$ opposite vortices on a circle and its center

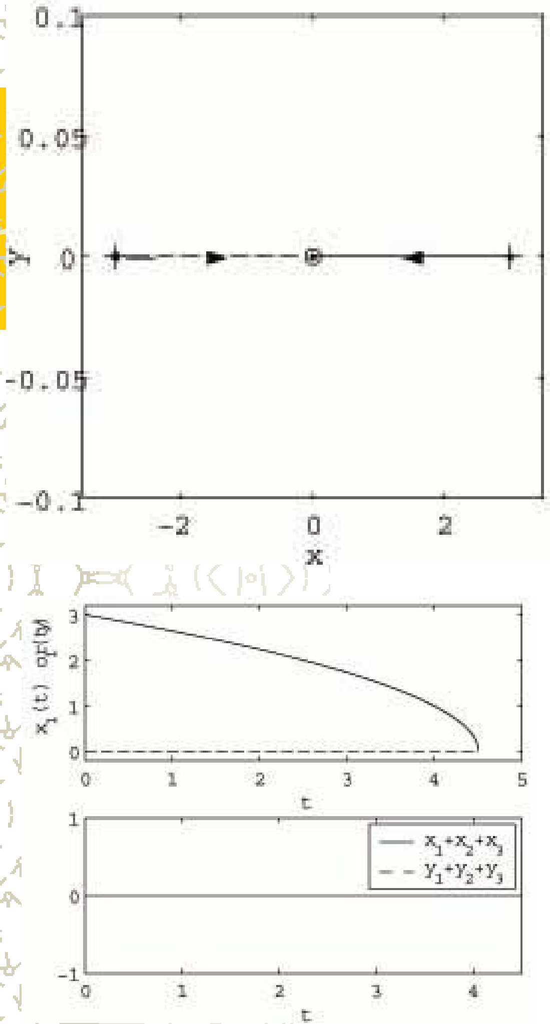
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right) \quad (+), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE figure

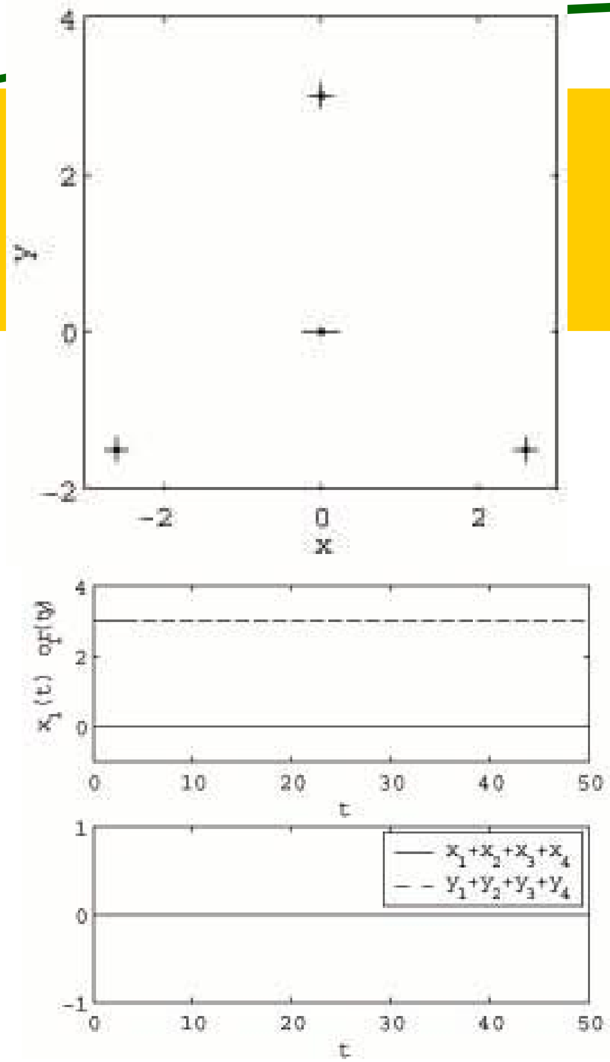
$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-4)}{\kappa}t}\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE figure next

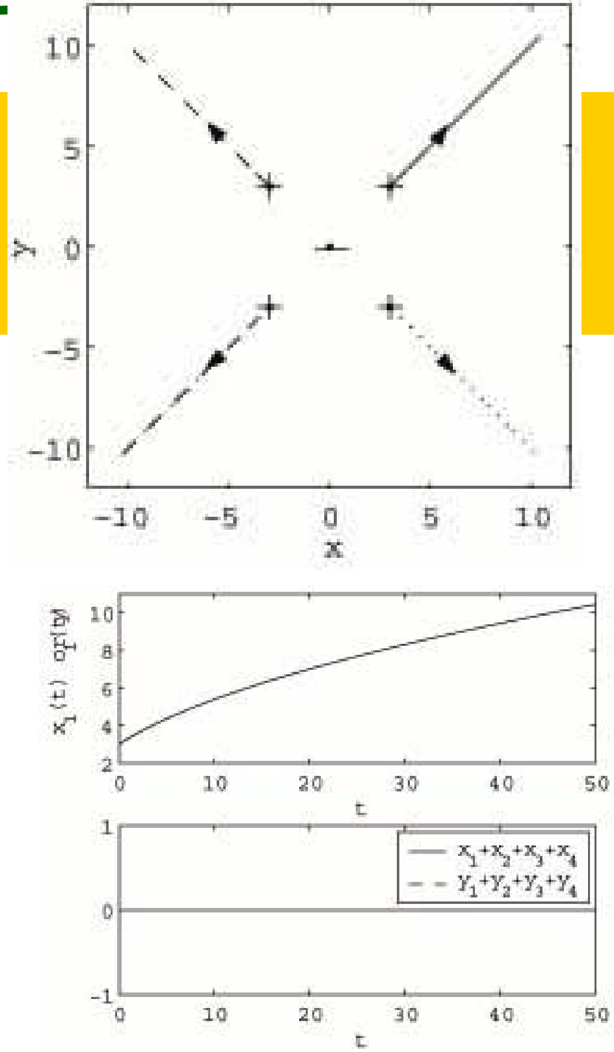
$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right), \ \sin\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right)\right)$$


x
y
t
x1(t) or(y)
x1+x2+x3
y1+y2+y3

N=3

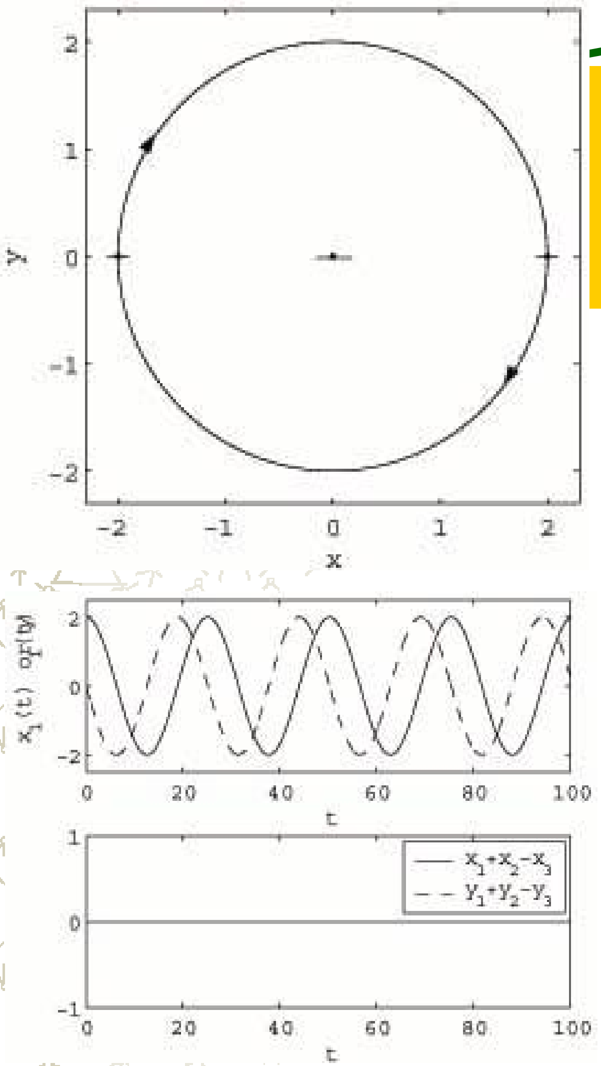

x
y
t
x1(t) or(y)
x1+x2+x3+x4
y1+y2+y3+y4

N=4


x
y
t
x1(t) or(y)
x1+x2+x3+x4
y1+y2+y3+y4

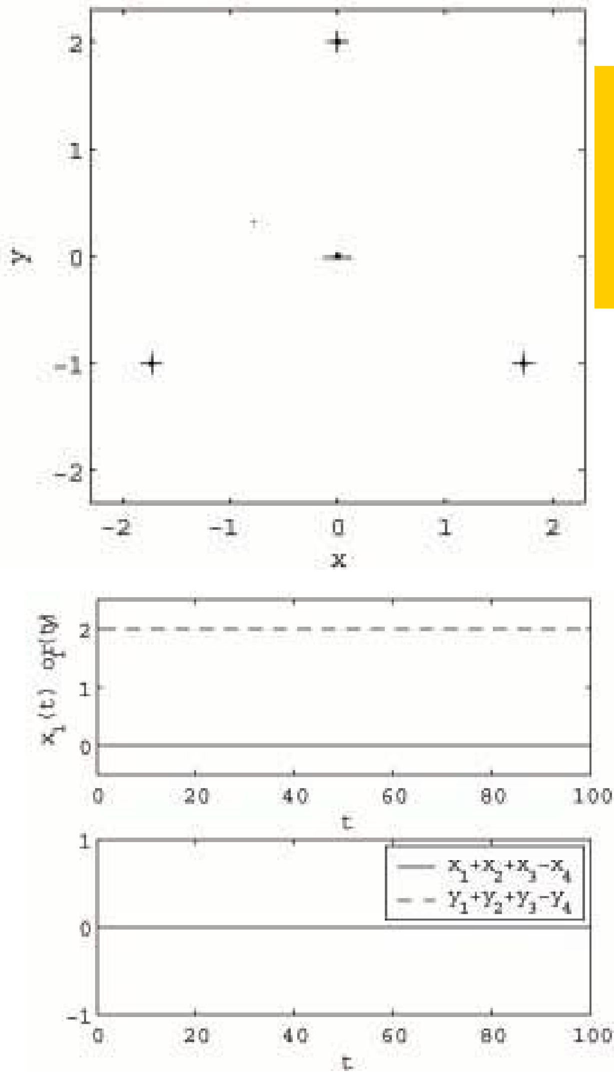
N=5

back



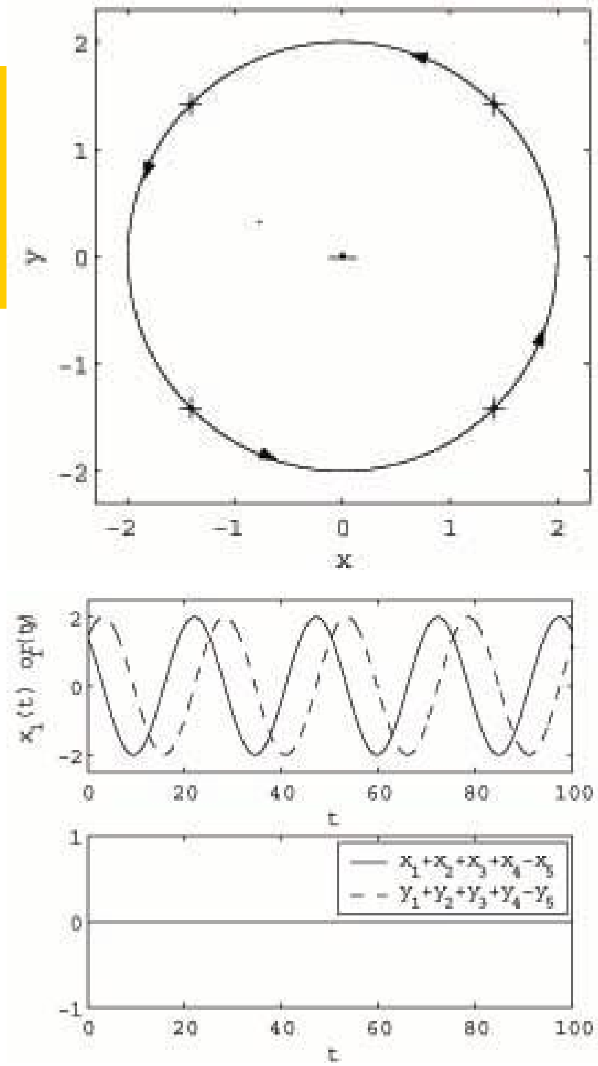
y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2-x_3$
$y_1+y_2-y_3$

N=3



y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2+x_3-x_4$
$y_1+y_2+y_3-y_4$

N=4



y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2+x_3+x_4-x_5$
$y_1+y_2+y_3+y_4-y_5$

N=5

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Numerical difficulties

- Numerical difficulties:
  - Highly oscillatory nature in the transverse direction: resolution
  - Quadratic decay rate in radial direction: large domain
  - For NLSE, more difficulties:
    - Time reversible
    - Time transverse invariant
    - Dispersive equation
    - Keep conservation laws in discretized level
    - Radiation

Numerical methods

- For `bright-tail' vortex (Bao,Du & Zhang, Eur. J. Appl. Math., 07')
  - Apply a time-splitting technique: decouple nonlinearity
  - Adapt polar coordinate: resolve solution in transverse better
  - Use Fourier pesudo-spectral discretization in transverse
  - Use 2nd or 4th order FEM or FDM in radial direction
- For `dark-tail' vortex (Bao & Zhang, M3AS, 05')
  - Apply a time-splitting technique: decouple nonlinearity
  - Use Fourier pseudo-spectral discretization
  - Use generalized Laguerre-Hermite pseudo-spectral method

Vortex dynamics & interaction

- Data chosen (Bao, Du & Zhang, SIAM, J. Appl. Math., 07')

$$\varepsilon = 1, \qquad V(\vec{x},t) \equiv 1, \qquad \psi_0(\vec{x}) = \prod_{j=1}^{N} \phi_{n_j}(\vec{x} - \vec{x}_j), \quad m = \sum_{j=1}^{N} n_j$$

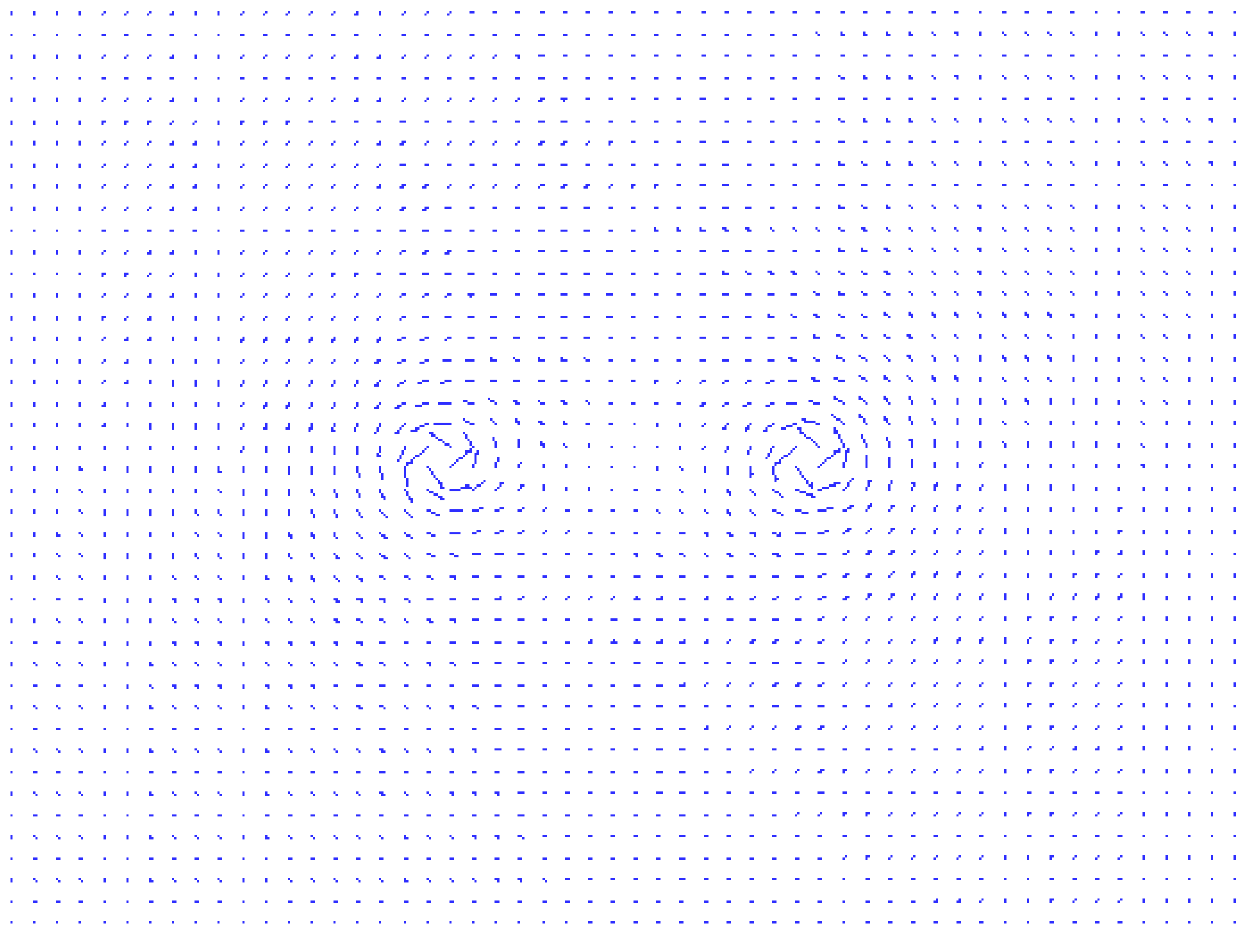
- Two vortices with like winding numbers

$$N = 2, \qquad n_1 = 1, \qquad n_2 = 1, \qquad \vec{x}_1 = (a,0), \qquad \vec{x}_2 = (-a,0)$$

  - For GLE: velocity density
  - For GPE: velocity density
  - Trajectory

- Summary

Frame 001 | 06 May 2005



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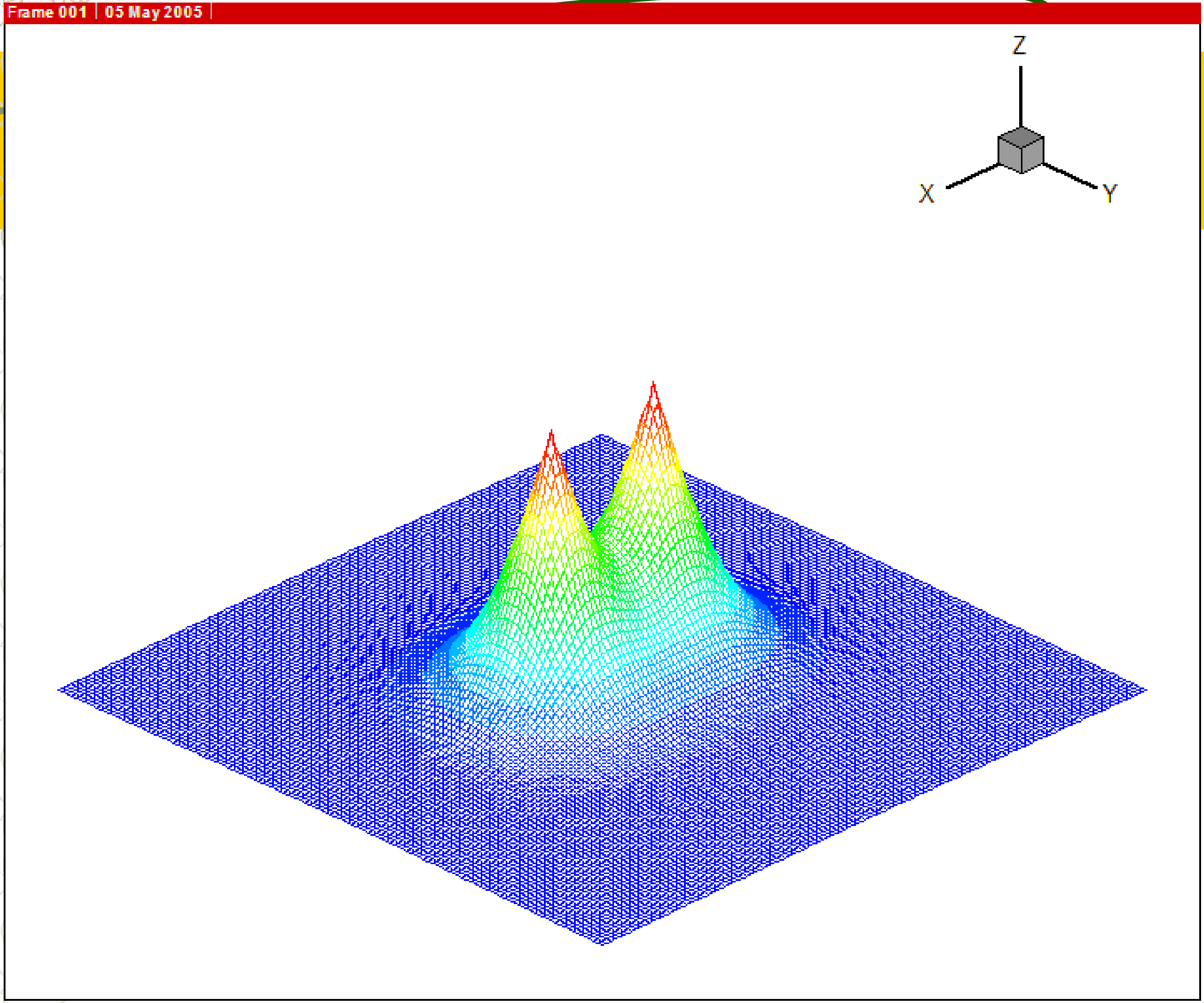

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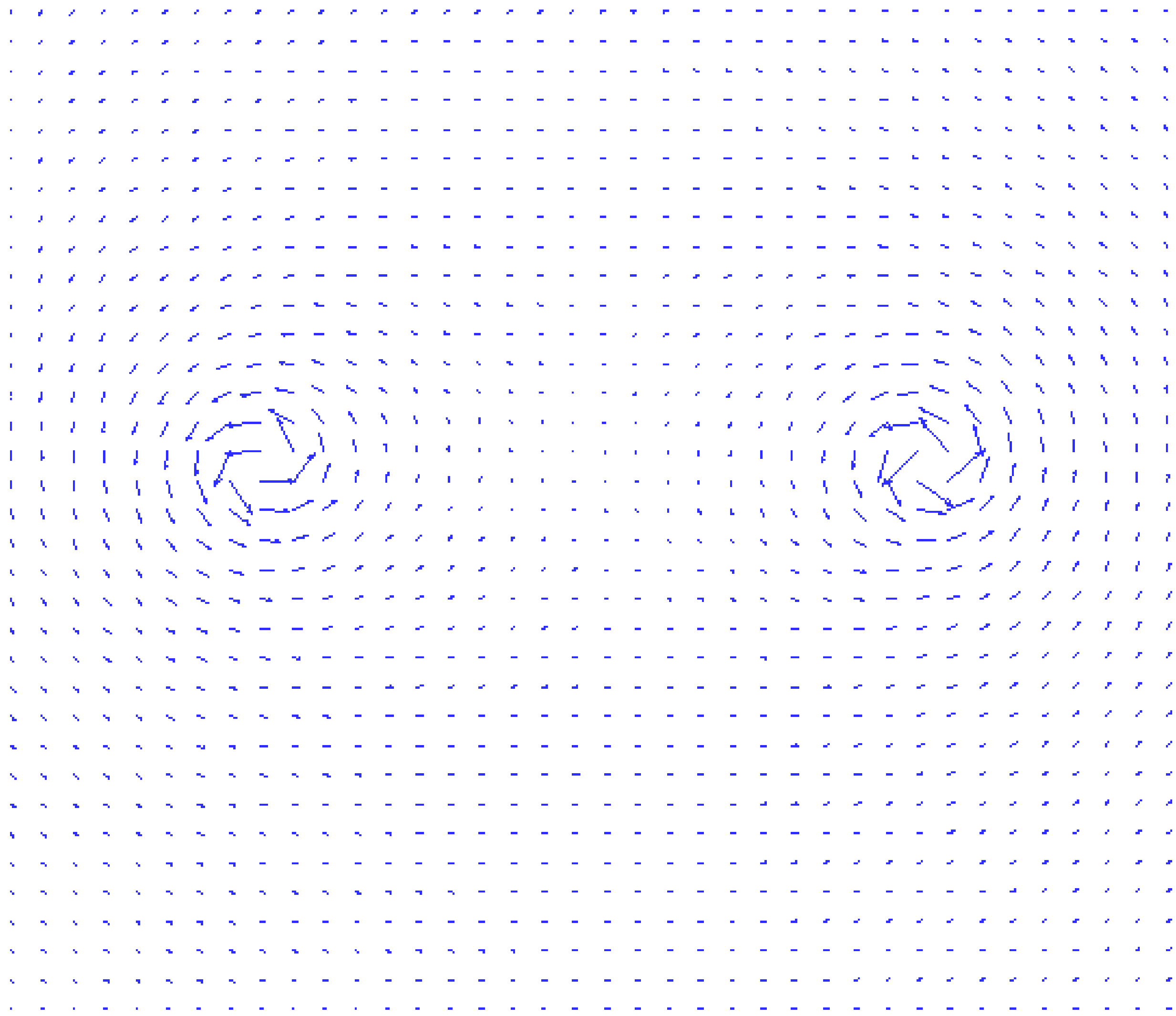

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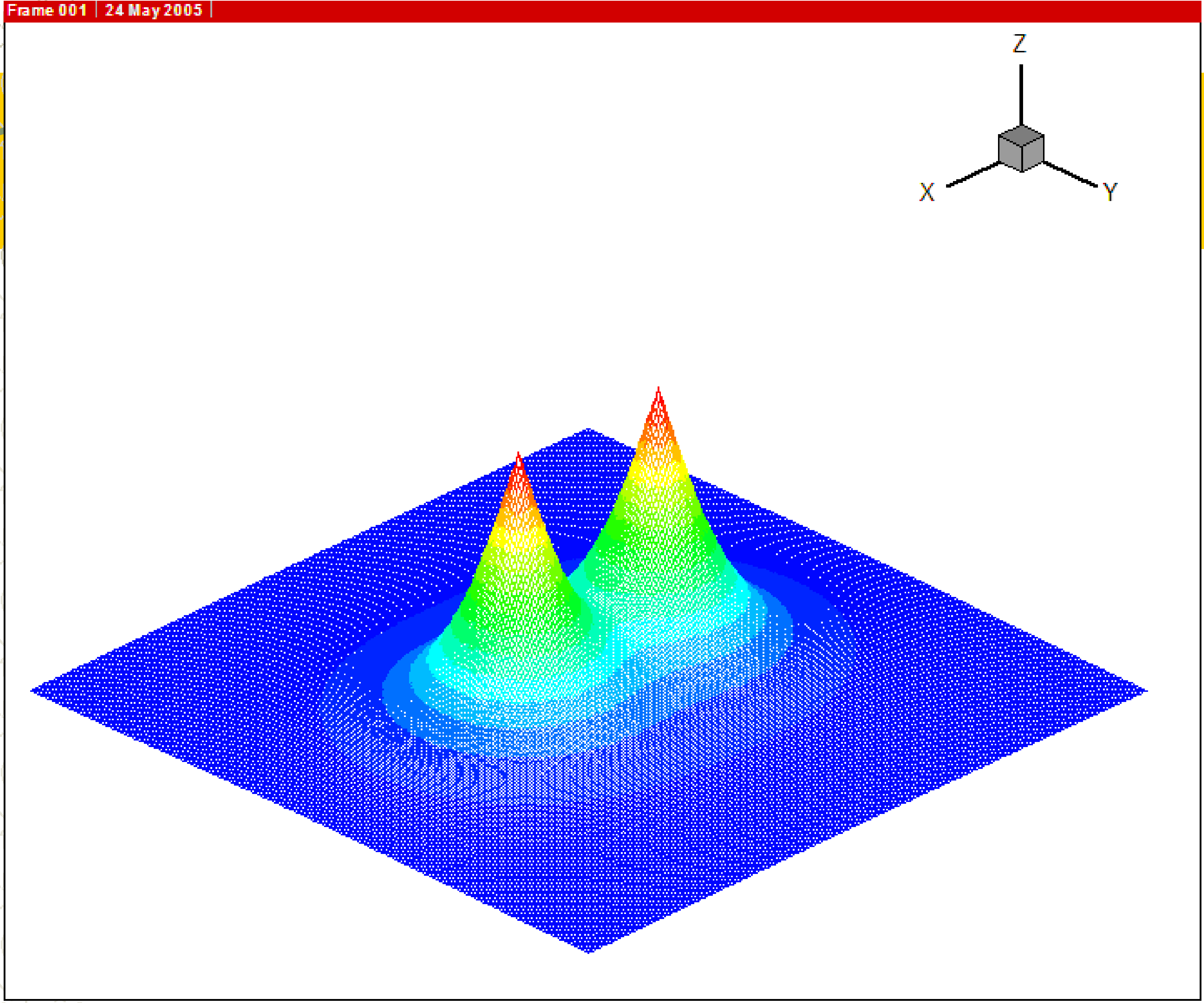

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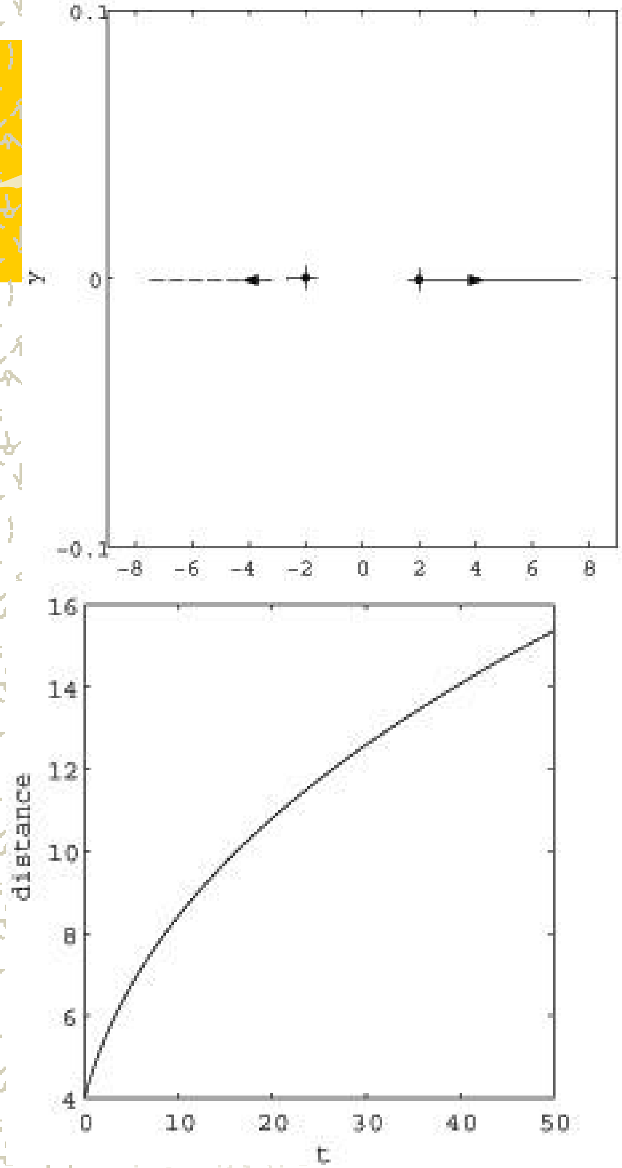
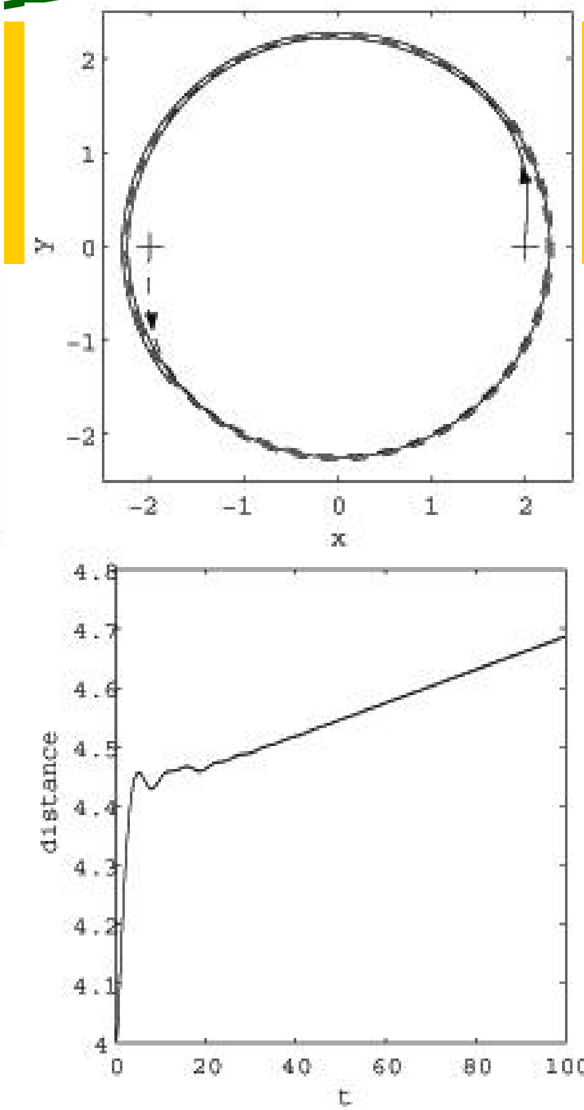
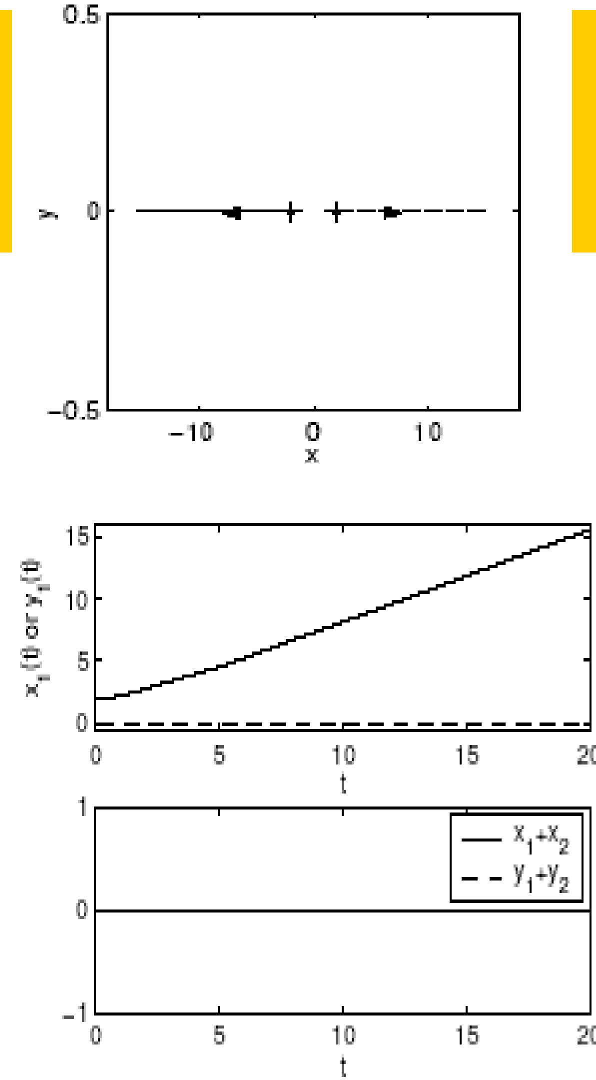


GLE



GPE



NLWE

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Summary of interaction

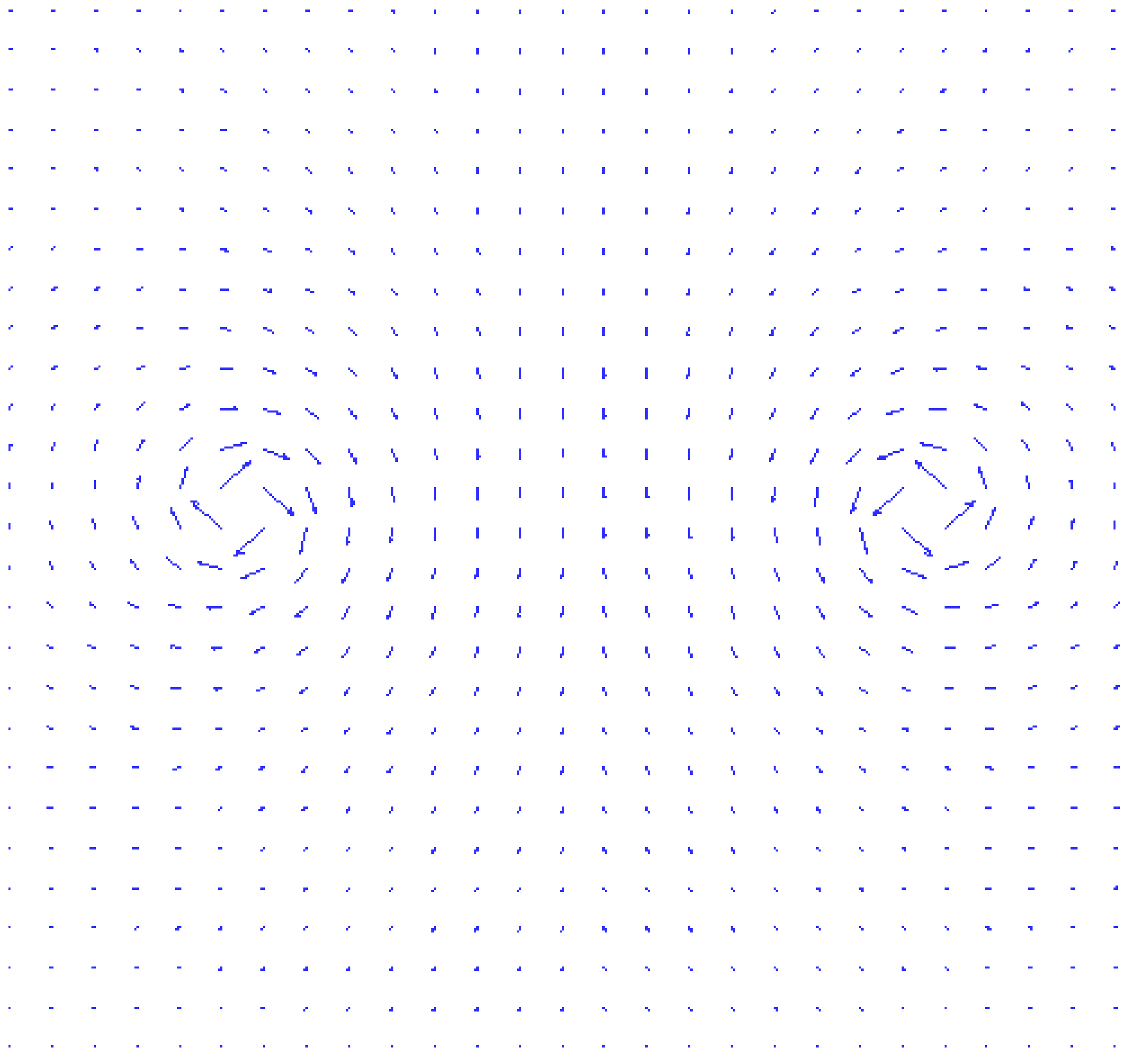
- For GLE (Bao, Du & Zhang, SIAP, 07')
  - Undergo a repulsive interaction
  - Core size of the two vortices will well separated
  - Energy decreasing: $E(\psi(\vec{x},t)) \to 2E(\psi_1(\vec{x})), \quad t \to \infty$
- For NLSE (Bao, Du & Zhang, SIAP, 07')
  - Behave like point vortices in ideal fluid
  - The two vortex centers move almost along a circle
  - Energy conservation & radiation
- For NLWE (Bao & Zhang, Physica D 07')
  - Similar as GLE but with different speed

Vortex dynamics & interaction

- Two vortices with opposite winding numbers

$$N = 2, \qquad n_1 = 1, \qquad n_2 = -1, \qquad \vec{x}_1 = (a,0), \qquad \vec{x}_2 = (-a,0)$$

  - For GLE: velocity density
  - For NLSE:
    - Case 1: velocity density
    - Case 2: velocity density
  - Trajectory
- Summary



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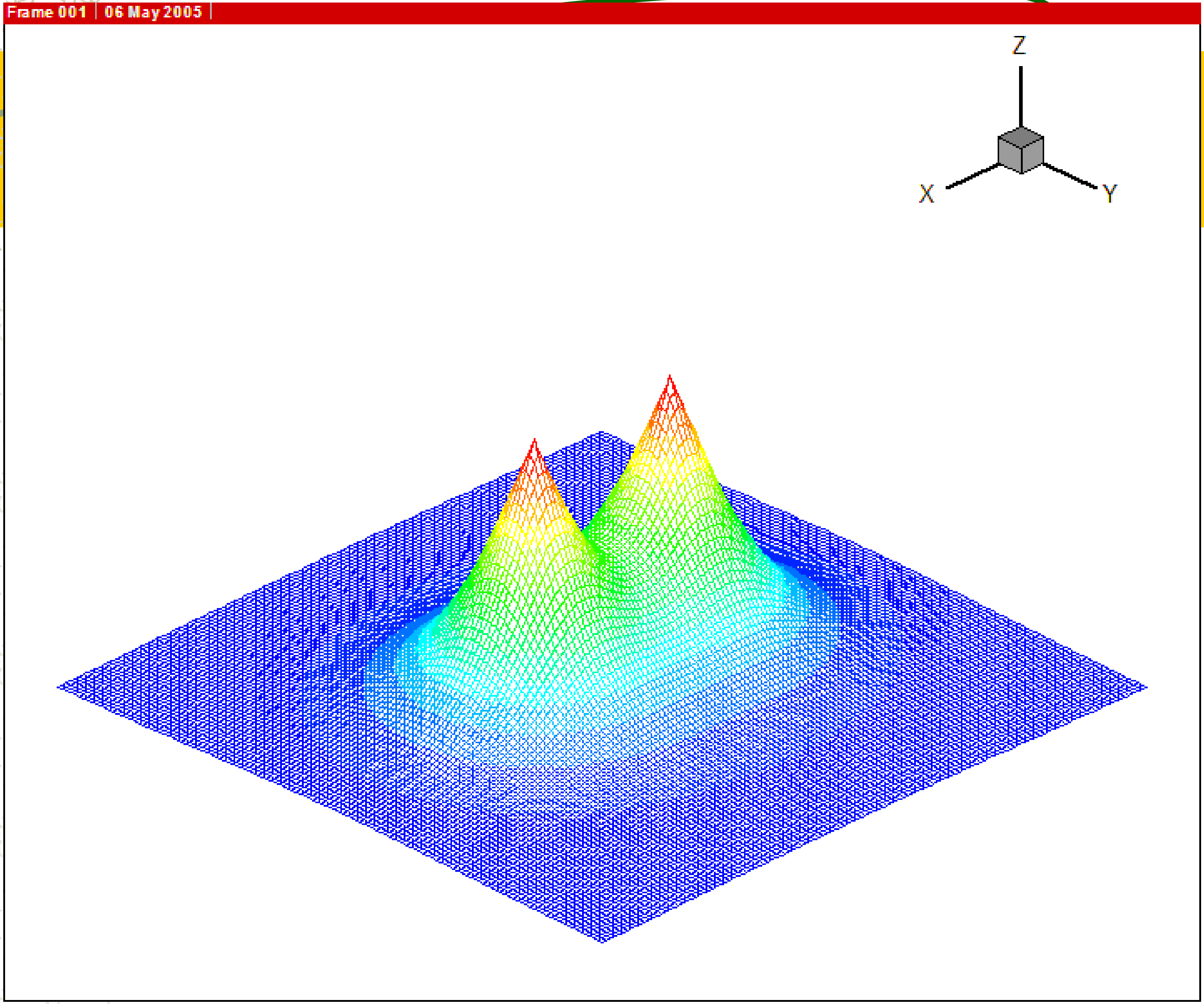

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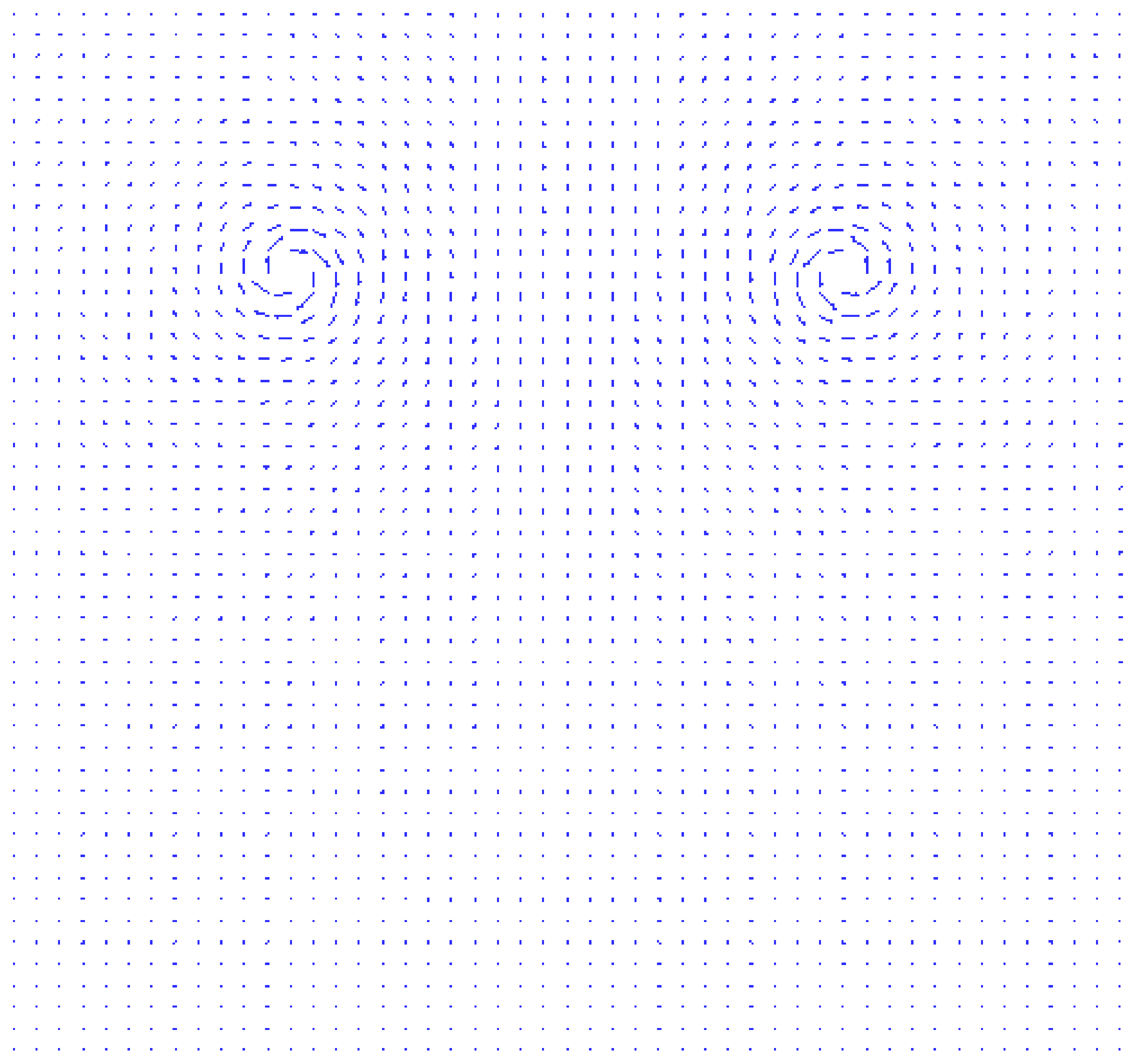

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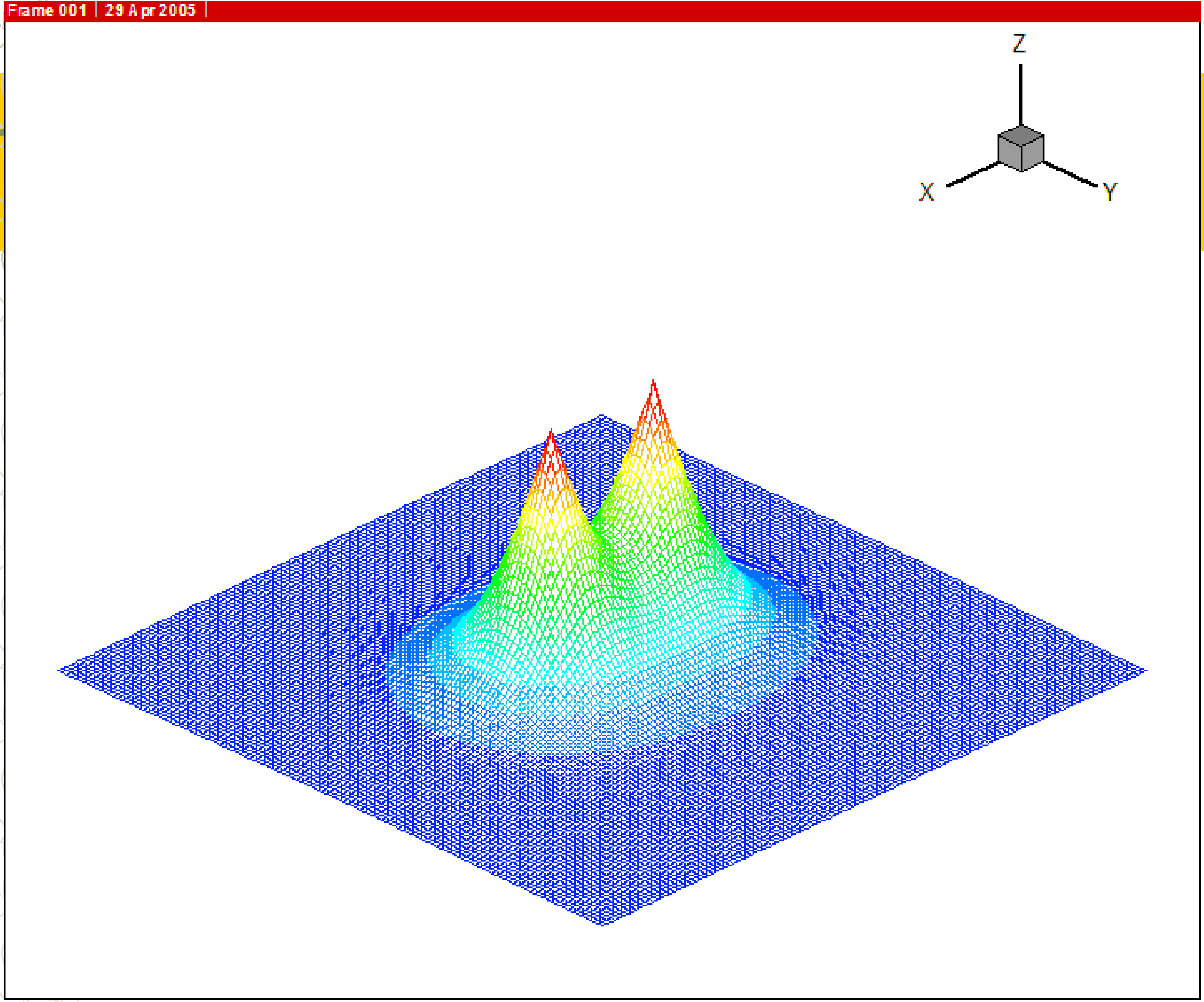

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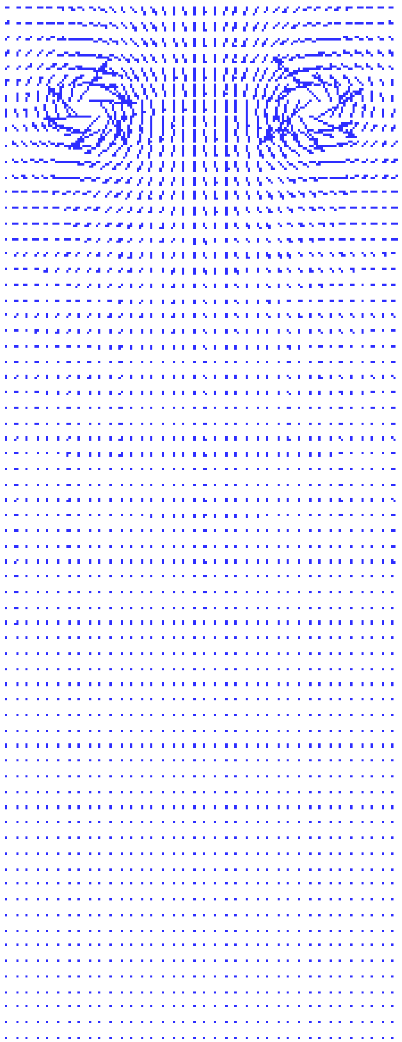

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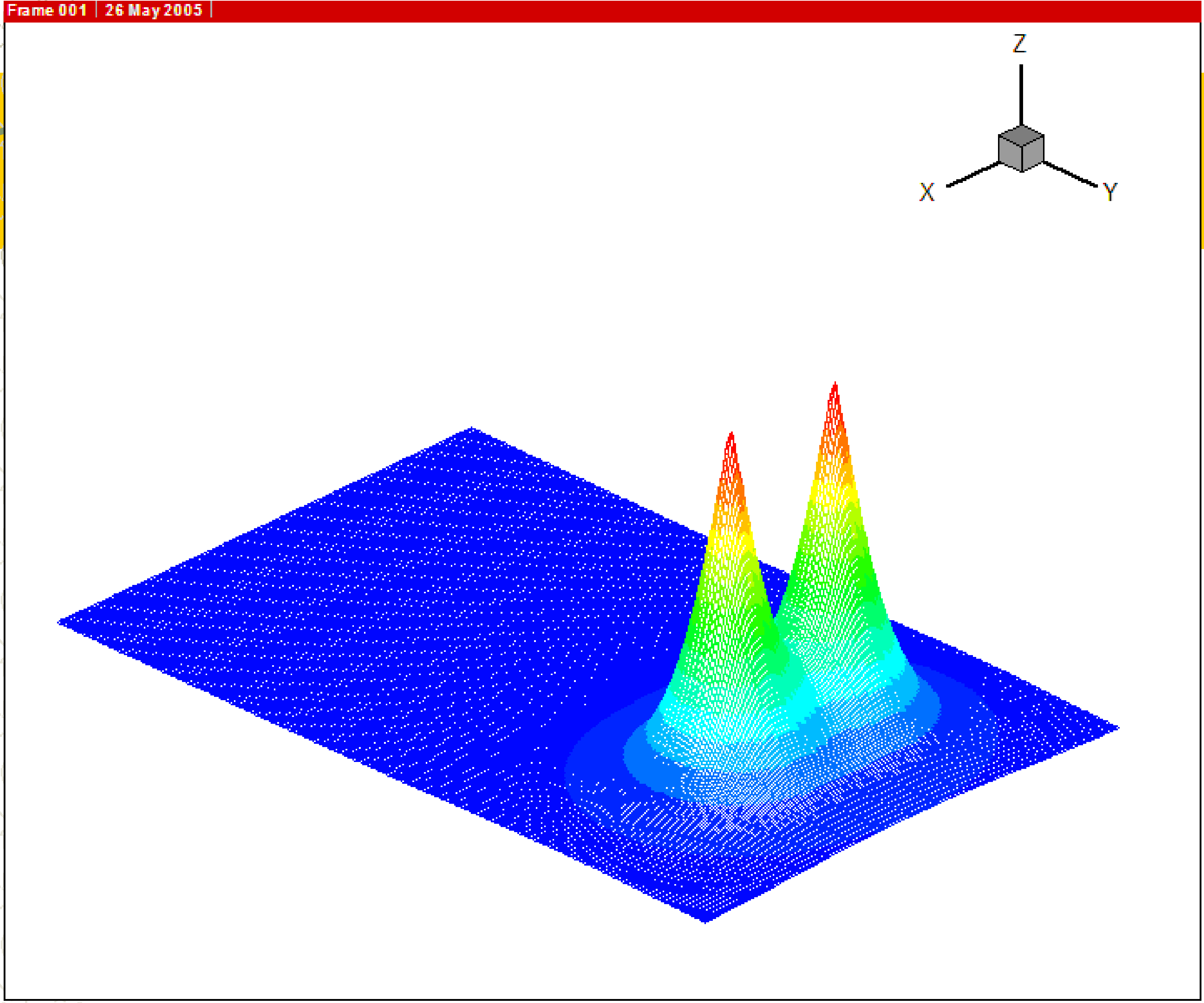

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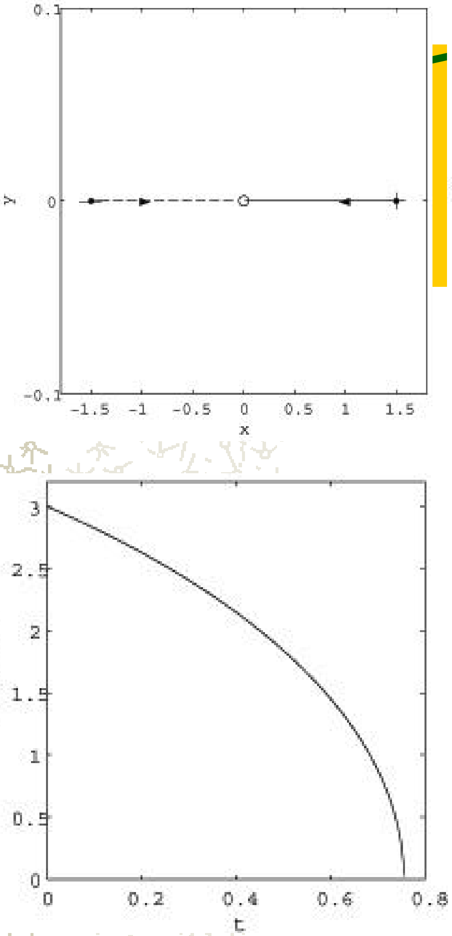
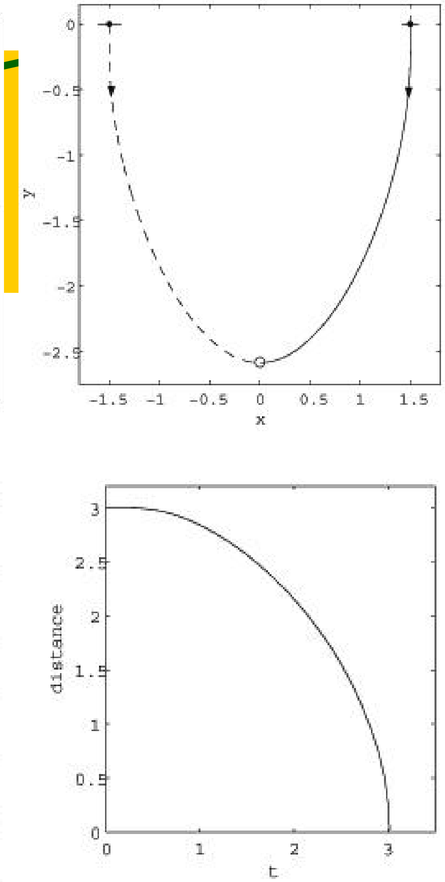
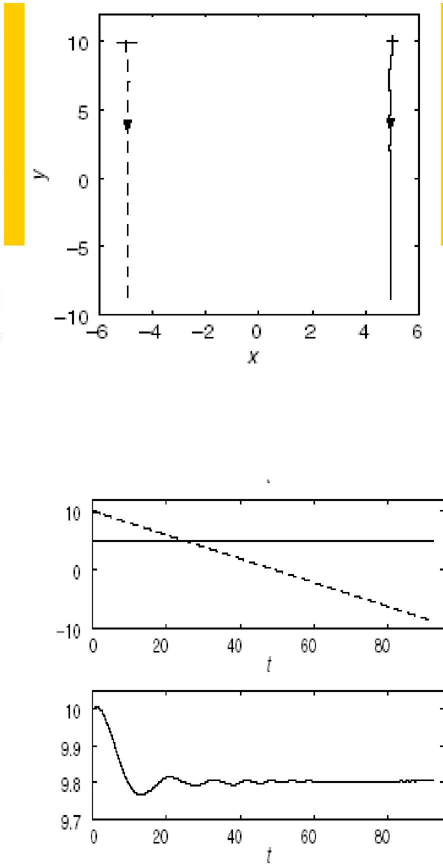


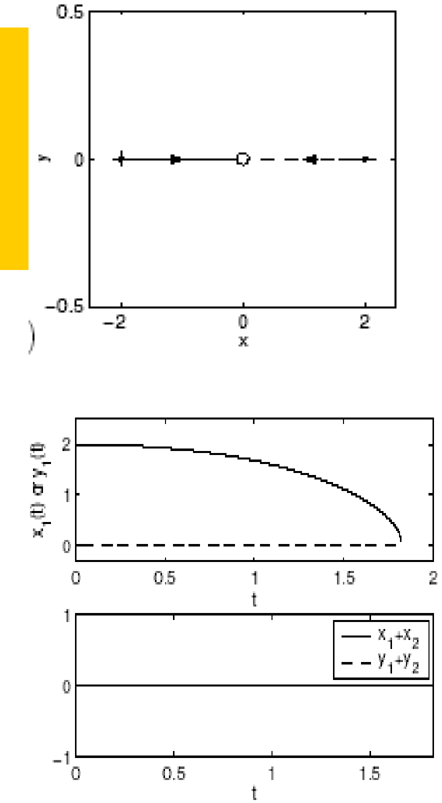
GLE



NLSE



NLSE

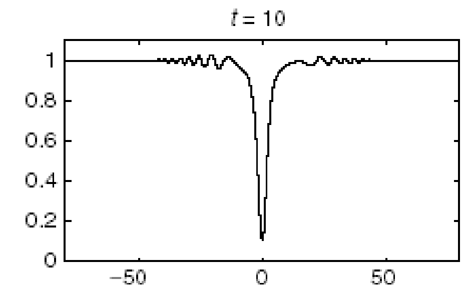


NLWE

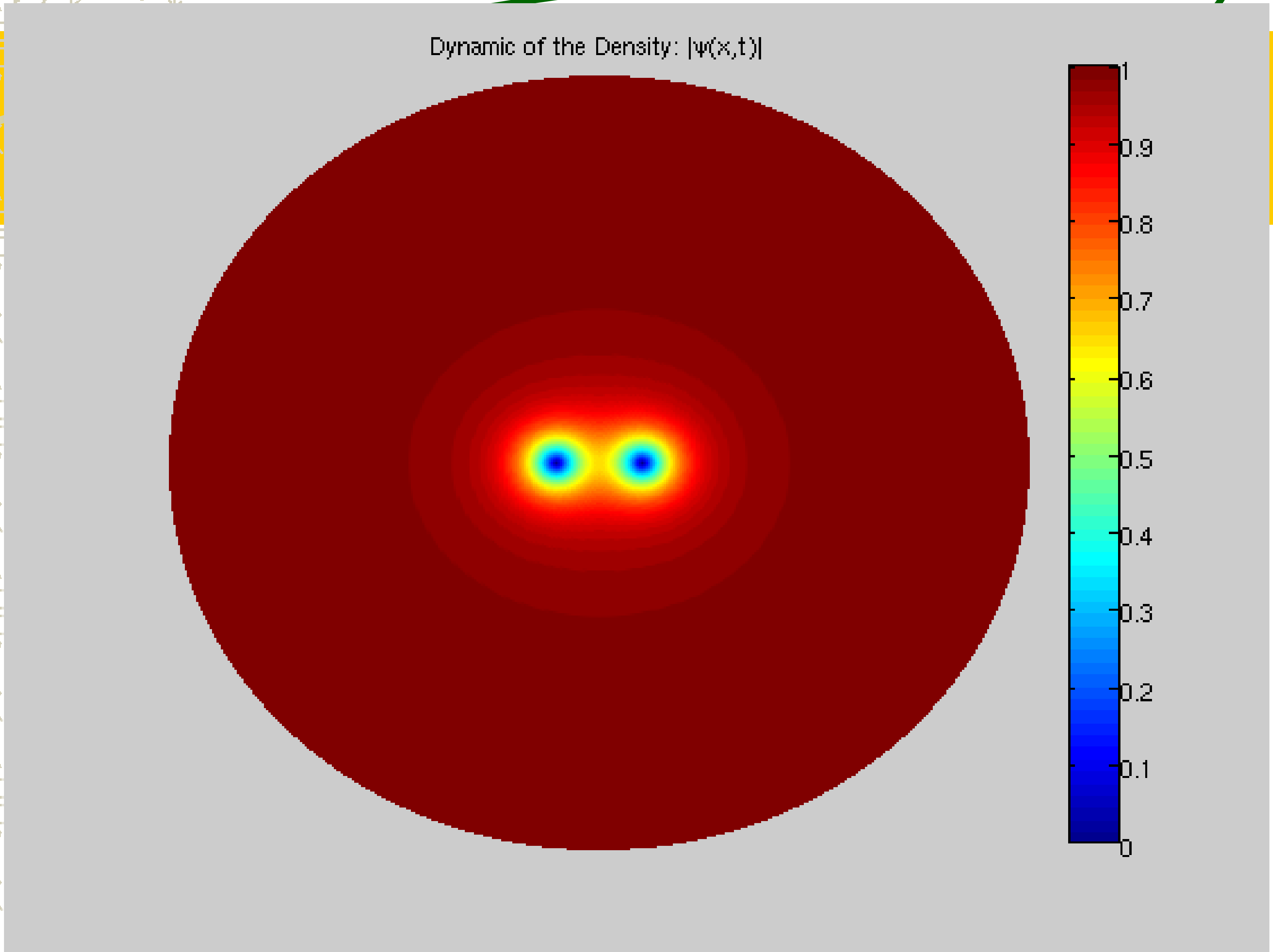
back

Summary of interaction

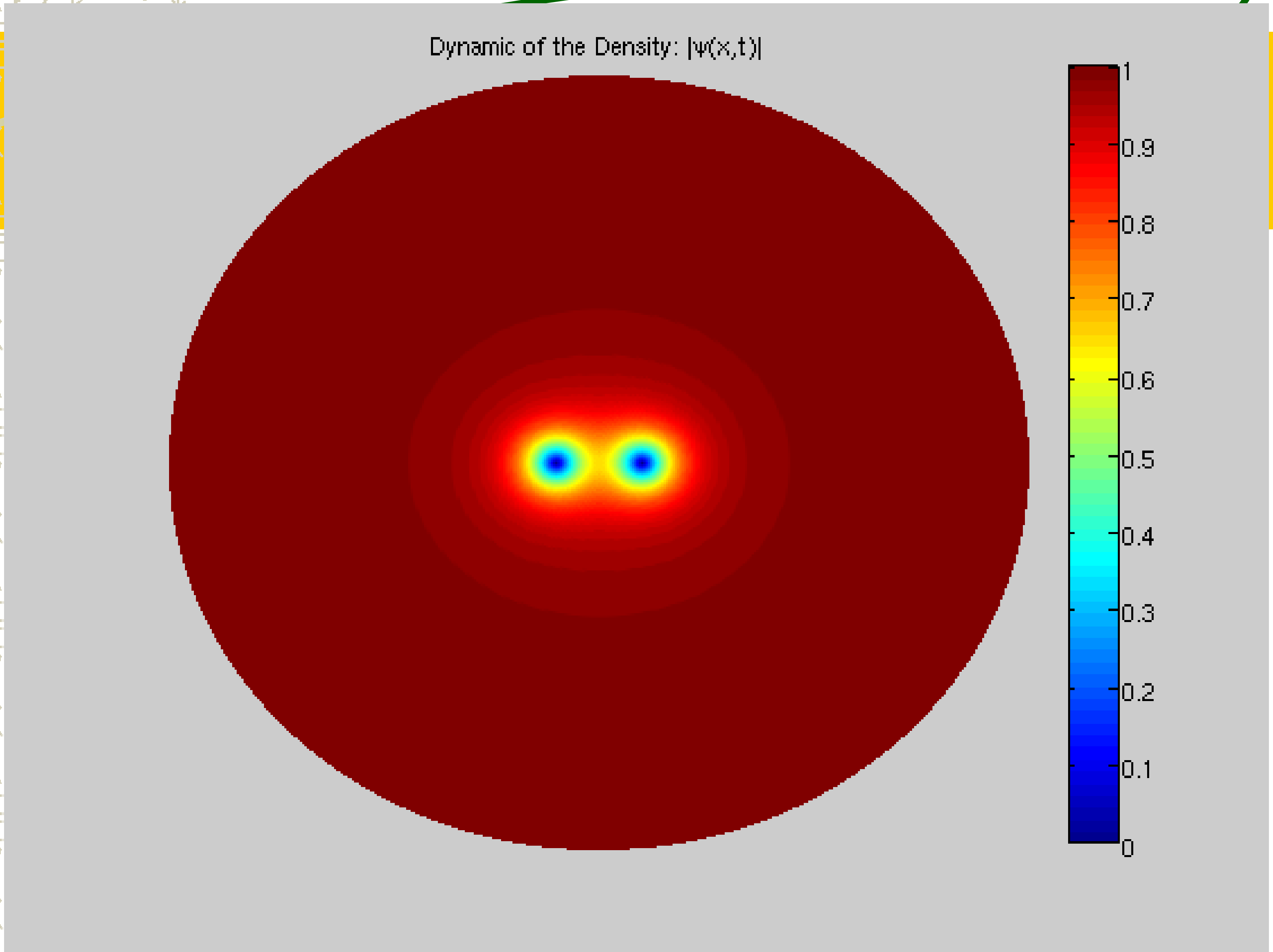
- For GLE (Bao, Du & Zhang, SIAP, 07')
  - Undergo an attractive interaction & merge at finite time
  - Energy decreasing: $E(\psi(\vec{x},t)) \to 0, \quad t \to \infty$



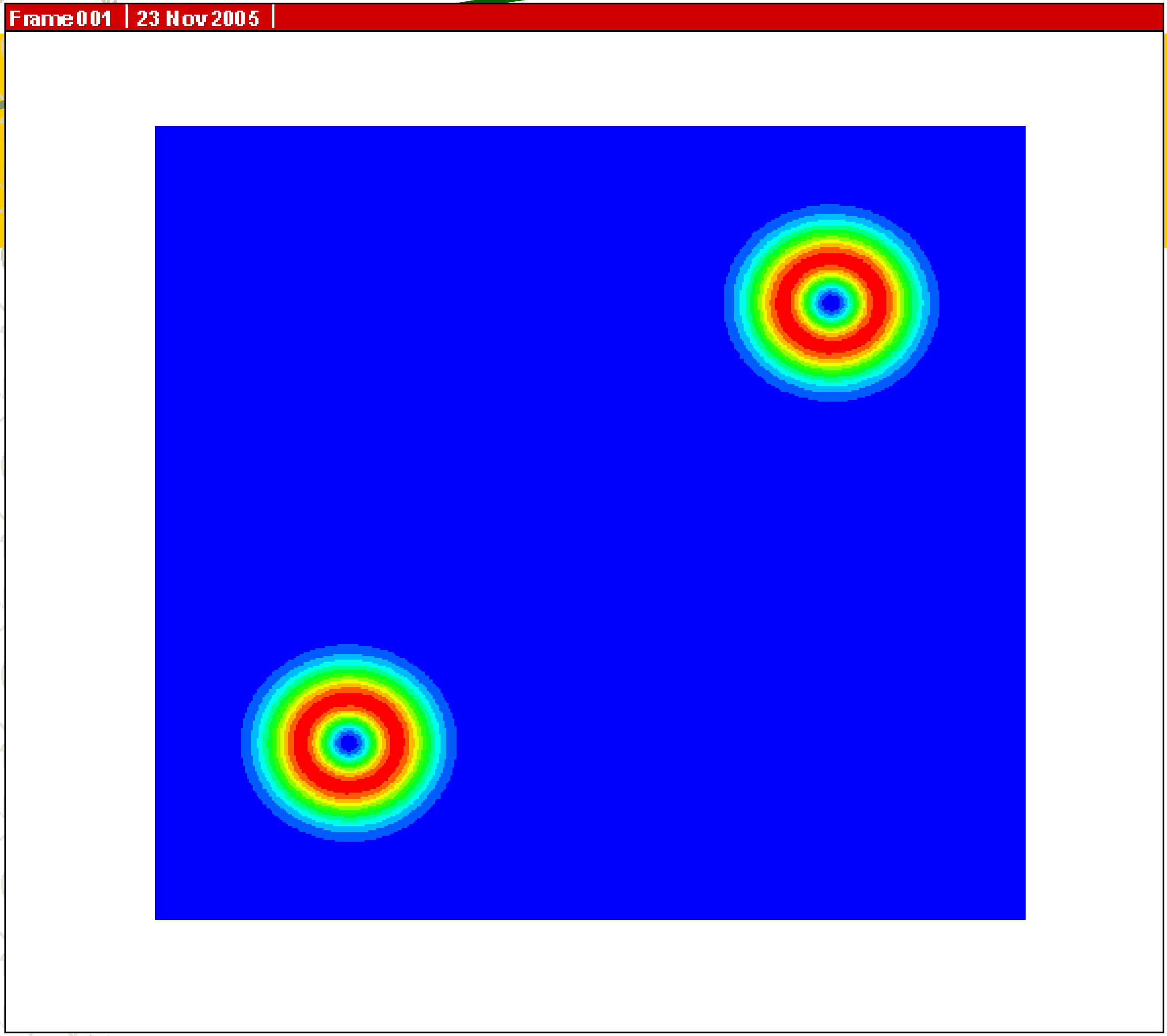
- For GPE (Bao, Du & Zhang, SIAP, 07')
  - depends on initial distance of the two centers
    - Small: merge at finite time & generate shock wave
    - Large: move almost paralleling & don't merge, solitary wave
  - Energy conservation & radiation
  - Sound wave generation
- For NLWE (Bao & Zhang, Physica D 07')
  - Undergo an attractive interaction & merge at finite time



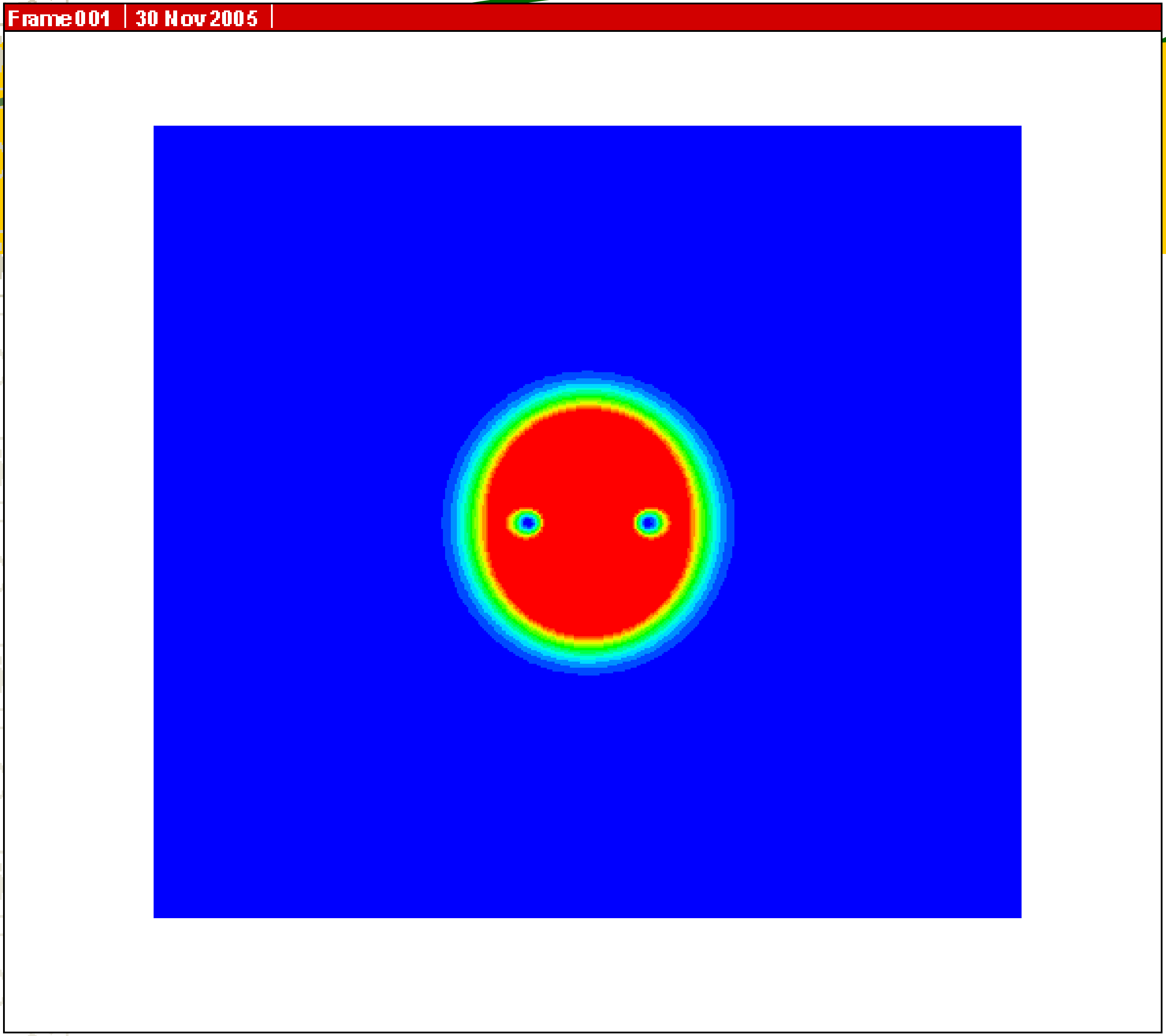
Vortex-pair on bounded domain



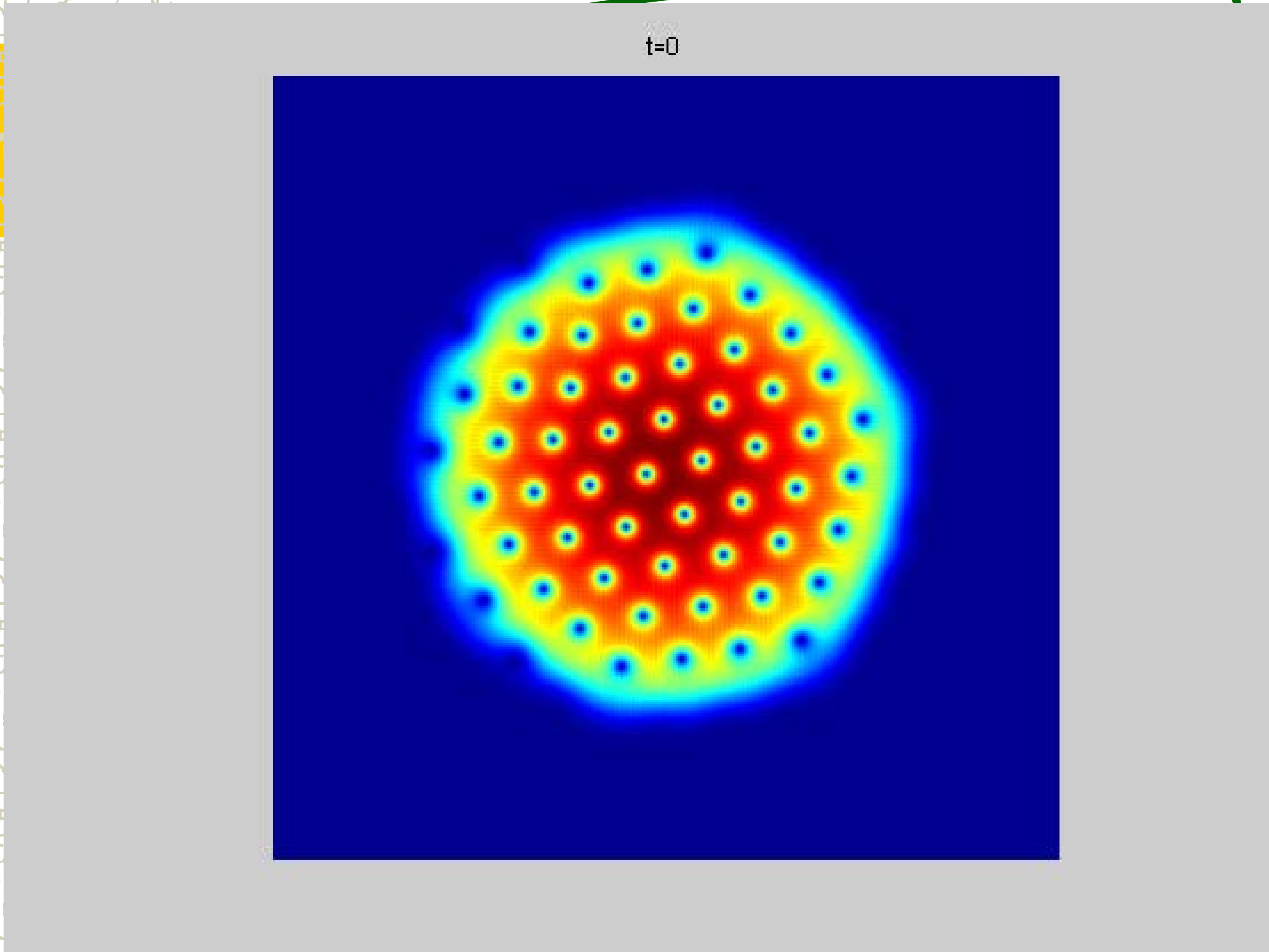
Vortex-dipole on bounded domain



`dark-tail’ vortex-pair in BEC – well-separate



`dark-tail’ vortex-dipole in BEC



`dark-tail' vortex lattice in BEC

Impurities Bound by Vortex lattice

PRL 116, 240402 (2016) PHYSICAL REVIEW LETTERS week ending 17 JUNE 2016

Hubbard Model for Atomic Impurities Bound by the Vortex Lattice of a Rotating Bose-Einstein Condensate

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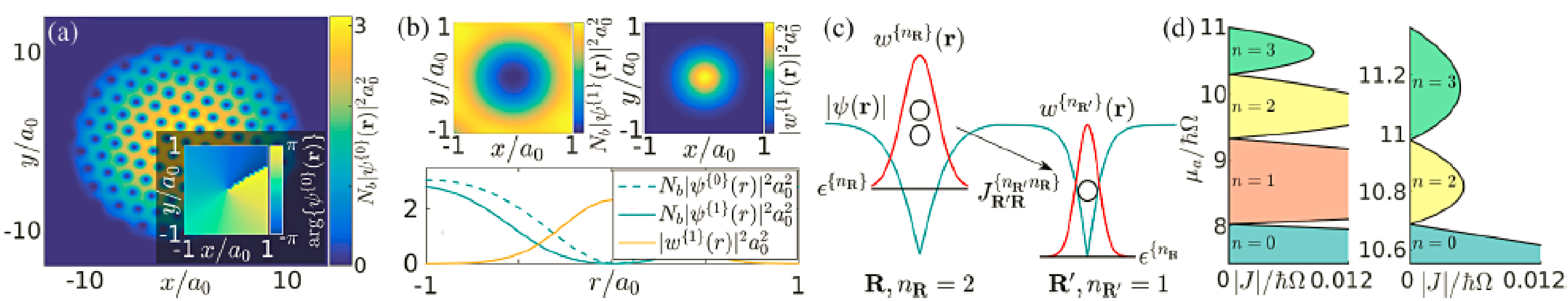
We investigate cold bosonic impurity atoms trapped in a vortex lattice formed by condensed bosons of another species. We describe the dynamics of the impurities by a bosonic Hubbard model containing occupation-dependent parameters to capture the effects of strong impurity-impurity interactions. These



Conclusions

- Analytical results for reduced dynamical laws
  - Mass center is conserved in GLE
  - Signed mass center is conserved in GPE
  - Solve analytically for a few types initial data
- Numerical results
  - Study numerically vortex dynamics in GLE, GPE, NLWE & NLSE
    - Stability of a vortex with different winding number
    - Interaction of vortex pair, vortex dipole, vortex tripole, ….
    - Vortex dynamics under non-uniform potential
    - Vortex interaction on bounded domain with different BCs
    - On bounded domain and applications in BEC