Deformation of F-injectivity

Linquan Ma, Karl Schwede, Kazuma Shimomoto

Higher Dimensional Birational Geometry and Characteristic p

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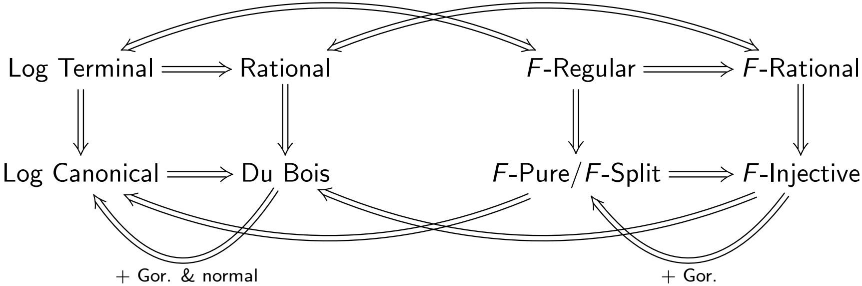
Overview

Throughout the talk, all rings and schemes contain a field.

In characteristic 0, all rings and schemes are essentially finite type over $\mathbb{C}$.

Singularities in characteristic $p > 0$ and zero



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$$0 \to R \to \oplus R_{x_i} \to \cdots \to \oplus R_{x_1 \cdots \widehat{x_i} \cdots x_n} \to R_{x_1 \cdots x_n} \to 0,$$

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Another way to understand the Frobenius action on local cohomology:

$$H^i_I(R) \to H^i_I(R^{1/p}) \cong H^i_I(R).$$

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Quite obviously, F-split implies F-injective, the converse is true if R is F-finite and Gorenstein. F-injective implies reduced.

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Definition/Theorem (Schwede) Let X be a reduced scheme of characteristic 0. Suppose $X \subseteq Y$ such that Y is smooth. Take an embedded resolution of (Y, X) with E the reduced pre-image of X. Then $R\pi_* O_E \cong \underline{\Omega}_X^0$.

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Definition X is Du Bois if $O_X \to \underline{\Omega}^0_X$ is an isomorphism.

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If we pick a counter-example $X = Spec(R, m)$ with minimal dimension, we can embed $h^i(\underline{\Omega}^0_X)$ into $H^{i+1}_m(R)$. Thus this will contradict the injectivity of Frobenius on local cohomology if $h^i(\underline{\Omega}^0_X) \neq 0$.

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In general, both conjectures are wide open.

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The deformation for F-singularity has been studied intensely: F-rationality deforms and this is quite easy to prove. In general, F-split does not deform (Fedder-Singh), F-regularity does not deform (Singh).

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Sketch: Reduce to X projective. Suffices to prove for all j, $H^0(h^j(\underline{\omega}^\bullet_X) \otimes L^n) \to H^0(h^j(\omega^\bullet_X) \otimes L^n)$ is injective for L ample and $n \gg 0$

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2. Holds if R/xR is F-injective and Cohen-Macaulay on $Spec - \{m\}$ (HMS).
3. Holds if R/xR is F-split (HMS). Hence R/xR F-split $\Rightarrow$ R F-injective.

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Proof strategy We will show R/xR has (dense) F-injective type $\Rightarrow$ x is a surjective element after reduction mod $p \gg 0$.

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If R_{red} is Du Bois, then $H^i_m(R) \to H^i_m(R_{red})$ is surjective. In particular, $H^i_m(R/x^nR) \to H^i_m(R/xR)$ is surjective for every $i, n > 0$.

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If $H^i_m(R/x^nR) \to H^i_m(R/xR)$ is surjective for every $i, n > 0$, then for all $p \gg 0$ (p independent of n), $H^i_{m_p}(R_p/x^nR_p) \to H^i_{m_p}(R_p/xR_p)$ is surjective for every $i, n > 0$. Hence x is a surjective element after mod $p \gg 0$.

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To get the stronger result, we generalize HMS:
Step 4': If x is a surjective element, then R/xR F-injective implies $x^{p-1}F$ acts injectively on $H^i_m(R)$ for every i.

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$$\begin{array}{ccc} H^i_m(R) & \longrightarrow & H^i_m(R_{red}) \\ \downarrow & & \downarrow \cong \\ \mathbb{H}^i_m(\underline{\Omega}^0_R) & \xrightarrow{\cong} & \mathbb{H}^i_m(\underline{\Omega}^0_{R_{red}}) \end{array}$$

Surjectivity of local cohomology–continued

An obvious equivalent way to interpret the surjectivity: suppose R is Du Bois, $H^i_m(S) \to H^i_m(R)$ is surjective for every i and every thickening S of R (i.e., $S_{red} = R$).

Surjectivity of local cohomology–continued

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In characteristic $p > 0$, F-split will imply this surjectivity, but we construct an example F-injective local ring such that the surjective property fails (based on an example of Enescu-Hochster).

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Let $R = \frac{k[x,y,z]}{x^3+y^3+z^3} \# k[s,t]$ be the Segre product. This is the affine cone of $E \times \mathbb{P}^1$ where E is the elliptic curve $Proj \frac{k[x,y,z]}{x^3+y^3+z^3}$. Let m denote the unique homogeneous maximal ideal of R.

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Further question

We suspect, if R is Du Bois, assuming the weak ordinarity conjecture, then R_p has the surjective property on local cohomology (called F-full) for infinitely many $p > 0$.

Thank you!