Existence and stability of a solution with a new prescribed behavior for a heat equation with a critical nonlinear gradient term

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Marseille
April 4-8, 2016

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Introduction: The equation

We consider the following PDE:

$$\begin{cases} \partial_t u &= \Delta u + \mu|\nabla u|^q + |u|^{p-1}u, \\ u(\cdot,0) &= u_0 \end{cases}$$

where:

- $p > 3$, $\mu > 0$, $q = q_c = \frac{2p}{p+1}$,
- $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,
- $u_0 \in W^{1,\infty}(\mathbb{R}^N)$.

History of the equation

- *Introduction*: Chipot-Weissler (1989), mathematical motivation ($\mu < 0$).
- *Population dynamics interpretation*: Souplet (1996).
- *Mathematical analysis*: Chipot, Weissler, Peletier, Kawohl, Fila, Quittner, Deng, Alfonsi, Tayachi, Souplet, Snoussi, Galaktionov, Vázquez, Ebde, Z., Nguyen, ...
- *Elliptic version* : Chipot, Weissler, Serrin, Zou, Peletier, Voirol, Fila, Quittner, Bandle ...

Two limiting cases

- When $\mu = 0$, this is the well-known *semilinear heat equation*:

$$\partial_t u = \Delta u + |u|^{p-1} u.$$

- When $\mu = +\infty$, we recover (after rescaling) the *Diffusive Hamilton-Jacobi equation*:

$$\partial_t u = \Delta u + |\nabla u|^q.$$

This is a critical case

Scaling: Only when $q = q_c$, we have "u solution $\Rightarrow u_\lambda$ solution", where

$$u_\lambda(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t),\ \forall \lambda > 0,\ \forall t > 0,\ x \in \mathbb{R}^N,$$

as for the equation without gradient term ($\mu = 0$).

This is a critical case (cont.)

When $\mu \in \mathbb{R}$ and $w(y,s)$ is the similarity variables version of $u(x,t)$:

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t),$$

we have for all $s \geq -\log T$ and $y \in \mathbb{R}^N$:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.$$

with

$$\alpha = \frac{q(p+1)}{2(p-1)} - \frac{p}{p-1} = \frac{(p+1)}{2(p-1)}(q - q_c) \text{ and } q_c = \frac{2p}{p+1}.$$

Therefore, we have 3 cases:

- subcritical when $q < q_c$: we have a “perturbation” of the semilinear heat equation;
- supercritical when $q > q_c$: we are in the Hamilton-Jacobi limit;
- critical when $q = q_c$: this is the aim of the talk.

Cauchy problem and blow-up solutions

- *Cauchy problem*: Wellposed in $W^{1,\infty}(\mathbb{R}^N)$ (fixed point argument, see Alfonsi-Weissler (1993), Souplet-Weissler (1999)).
- *Blow-up solutions*: If $T < \infty$, then $\lim_{t\to T} \|u(t)\|_{W^{1,\infty}(\mathbb{R}^N)} = \infty$.

  Definition: x_0 is a blow-up point if $\exists (t_n, x_n) \to (T, x_0)$ s.t. $|u(x_n, t_n)| \to \infty$ as $n \to \infty$.

Aim of the talk

Take

$$q = q_c.$$

We have 3 goals:

- construct a blow-up solution,
- determine its blow-up profile,
- prove its stability (with respect to perturbations in initial data).

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Case $\mu = 0$: the standard semilinear heat equation

- The (generic) profile is given by

$$(T-t)^{1/(p-1)}u(z\sqrt{(T-t)|\log(T-t)|},t) \sim f_0(z) \text{ as } t \to T,$$

where

$$f_0(x) = \left(p-1+b_0|x|^2\right)^{-1/(p-1)} \text{ and } b_0 = (p-1)^2/(4p).$$

See Galaktionov-Posashkov (1985), Berger-Kohn (1988), Herrero-Velázquez (1993). *The constructive existence proof* by Bricmont-Kupiainen (1994), Merle-Z. (1997) is based on:

  - The reduction of the problem to a finite-dimensional one;
  - The solution of the finite-dimensional problem thanks to the degree theory.

- Other profiles are possible.

Subcritical case: $q < q_c = \frac{2p}{p+1}$

Ebde and Z. (2011) could adapt the previous existence strategy and *find the same behavior as for* $\mu = 0$, since the gradient term is subcritical in size in similarity variables:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.$$

with

$$\alpha = \frac{(p+1)}{2(p-1)} (q - q_c) < 0.$$

Critical case: $q = q_c$, with $-2 < \mu < 0$ and $p - 1 > 0$ small

Exact self-similar blow-up solution by Souplet, Tayachi and Weissler (1996):

$$u(x,t) = (T-t)^{-1/(p-1)} W\left(\frac{|x|}{\sqrt{T-t}}\right)$$

where W satisfies the following elliptic equation:

$$W'' + \frac{N-1}{r}W' - \frac{1}{2}rW' - \frac{W}{p-1} + W^p + \mu|W'|^{q_c} = 0.$$

Critical case: $q = q_c$; A numerical result

A similar profile to the case $\mu = 0$ was discovered *numerically* by Van Tien Nguyen (2014):

$$(T-t)^{1/(p-1)}u(z\sqrt{(T-t)|\log(T-t)|},t) \sim f_0(z) \text{ as } t \to T,$$

where

$$f_\mu(x) = \left(p-1+b_\mu|x|^2\right)^{-1/(p-1)}$$

with

$$b_\mu > 0 \text{ and } b_0 = (p-1)^2/(4p),$$

the same as for $\mu = 0$.

Remark: We initially wanted to confirm this result, and ended by finding a *new* type of behavior.

Critical case: $q = q_c$; Our new profile

Theorem (Tayachi and Z.) *There exists a solution* $u(x,t)$ *s.t.:*

- *Simultaneous Blow-up: Both* u *and* ∇u *blow up as* $t \to T > 0$ *only at the origin;*
- *Blow-up Profile:*

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}, t) \sim \bar{f}_\mu(z) \text{ as } t \to T$$

with

$$\bar{f}_\mu(z) = \left(p-1+\bar{b}_\mu|z|^2\right)^{-\frac{1}{p-1}} \text{ with } \bar{b}_\mu = \frac{1}{2}(p-1)^{\frac{p-2}{p-1}}\left(\frac{(4\pi)^{\frac{N}{2}}(p+1)^2}{p\int_{\mathbb{R}^N}|y|^q e^{-|y|^2/4}dy}\right)^{\frac{p+1}{p-1}} \mu^{-\frac{p+1}{p-1}} > 0.$$

- *Final profile When* $x \neq 0$, $u(x,t) \to u(x,T)$ *as* $t \to T$ *with*

$$u(x,T) \sim \left(\frac{\bar{b}_\mu}{2}\frac{|x|^2}{|\log|x||^{\frac{p+1}{p-1}}}\right)^{-\frac{1}{p-1}} \text{ as } x \to 0.$$

Comments

The exhibited behavior is new in two respects:

- *The scaling law*: $\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}$ instead of the laws of the case $\mu = 0$, $\sqrt{(T-t)|\log(T-t)|}$ or $(T-t)^{\frac{1}{2m}}$ where $m \geq 2$ is an integer;
- *The profile function*: $\bar{f}_\mu(z) = \left(p - 1 + \bar{b}_\mu |z|^2\right)^{-\frac{1}{p-1}}$ is different from the profile of the case $\mu = 0$, namely $f_0(z) = \left(p - 1 + b_0|z|^2\right)^{-\frac{1}{p-1}}$, in the sense that $\bar{b}_\mu \neq b_0$.

Note in particular, that

$$\bar{b}_\mu \to \infty \text{ as } \mu \to 0.$$

Remark: Our solution is different already in the scaling from the numerical solution of Van Tien Nguyen, which is in the $\mu = 0$ style.

Idea of the proof

We follow *the constructive existence proof* used by Bricmont-Kupiainen (1994), Merle-Z. (1997) for the *standard semilinear heat equation*.

That method is based on:

- The reduction of the problem to a finite-dimensional one ($N + 1$ parameters);
- The solution of the finite-dimensional problem thanks to the degree theory.

Critical case: $q = q_c$; Stability of the constructed solution

Thanks to the interpretation of the $(N+1)$ parameters of the finite-dimensional problem in terms of the blow-up time (in $\mathbb{R}$) and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

Theorem (Tayachi and Z.: Stability)

The constructed solution is stable with respect to perturbations in inital data in $W^{1,\infty}(\mathbb{R}^N)$.

Remark: I don't give the proof in this talk.

Applications: Perturbed Hamilton-Jacobi Equation

Corollary (Tayachi and Z.)

After an appropriate scaling, our results yield stable blow-up solutions for the following Viscous Hamilton-Jacobi equation:

$$\partial_t v = \Delta v + |\nabla v|^q + \nu |v|^{p-1} v;$$

with

$$\nu > 0,\ 3/2 < q < 2,\ p = \frac{q}{2-q}.$$

The solution and its gradient blow up simultaneously, only at one point.

Of course, the blow-up profile is given after an appropriate scaling.

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A formal approach to find the ansatz ($N = 1$)

Following the standard semilinear heat equation case, we work in similarity variables:

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t).$$

We need to *find a solution* for the following equation defined for all $s \geq s_0$ and $y \in \mathbb{R}^N$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^q,$$

such that

$$0 < \epsilon_0 \leq \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\epsilon_0} \text{ (type 1 blow-up).}$$

Idea 1: Look for a (non trivial) stationary solution: already successful by Souplet, Tayachi and Weissler (1996), for p close to 1 (self-similar solution in the $u(x,t)$ setting).

Idea 2: Since $w \equiv \kappa \equiv (p-1)^{-\frac{1}{p-1}}$ is a trivial solution, let us look for a solution w such that

$$w \to \kappa, \quad \text{as} \quad s \to \infty.$$

Inner expansion

We write

$$w = \kappa + \overline{w},$$

and *look for* $\overline{w}$ such that

$$\overline{w} \to 0 \text{ as } s \to \infty.$$

The equation to be satisfied by $\overline{w}$ is the following:

$$\partial_s \overline{w} = \mathcal{L}\overline{w} + \overline{B}(\overline{w}) + \mu|\nabla \overline{w}|^{q_c},$$

where $q_c = \frac{2p}{p+1}$,

$$\mathcal{L}v = \partial_y^2 v - \frac{1}{2} y \partial_y v + v,$$

and

$$\overline{B}(\overline{w}) = |\overline{w} + \kappa|^{p-1}(\overline{w} + \kappa) - \kappa^p - p\kappa^{p-1}\overline{w}.$$

Note that $\overline{B}$ is quadratic:

$$\left|\overline{B}(\overline{w}) - \frac{p}{2\kappa}\overline{w}^2\right| \leq C|\overline{w}^3|.$$

The linear operator

Note that $\mathcal{L}$ is self-adjoint in $D(\mathcal{L}) \subset L^2_\rho(\mathbb{R})$ where

$$L^2_\rho(\mathbb{R}) = \left\{ f \in L^2_{loc}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} (f(y))^2 \rho(y) dy < \infty \right\}$$

and

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}.$$

The spectrum of $\mathcal{L}$ is explicitly given by

$$spec(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \;\middle|\; m \in \mathbb{N} \right\}.$$

All the eigenvalues are simple, and the eigenfunctions h_m are (rescaled) Hermite polynomials, with

$$\mathcal{L} h_m = \left(1 - \frac{m}{2}\right) h_m.$$

In particular, for $\lambda = 1,\ \frac{1}{2},\ 0$, the eigenfunctions are $h_0(y) = 1,\ h_1(y) = y$ and $h_2(y) = y^2 - 2$.

Naturally, we expand $\overline{w}(y,s)$ according to the eigenfunctions of $\mathcal{L}$:

$$\overline{w}(y,s)=\sum_{m=0}^{\infty}\overline{w}_m(s)h_m(y).$$

Since h_m for $m\geq 3$ correspond to negative eigenvalues of $\mathcal{L}$, assuming $\overline{w}$ even in y, we may consider that

$$\overline{w}(y,s)=\overline{w}_0(s)h_0(y)+\overline{w}_2(s)h_2(y),$$

with

$$\overline{w}_0,\ \overline{w}_2\ \to 0 \text{ as } s\to\infty.$$

Plugging this in the equation to be satisfied by $\overline{w}$:

$$\partial_s\overline{w}=\mathcal{L}\overline{w}+\overline{B}(\overline{w})+\mu|\partial_y\overline{w}|^{q_c},$$

we first see that

$$\mu|\partial_y \overline{w}|^{q_c} = \mu 2^{q_c}|y|^{q_c}|\overline{w}_2|^{q_c},$$

then, projecting on h_0 and h_2, we get the following ODE system:

$$\overline{w}_0' = \overline{w}_0 + \frac{p}{2\kappa}\left(\overline{w}_0^2 + 8\overline{w}_2^2\right) + \tilde{c}_0|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

$$\overline{w}_2' = 0 + \frac{p}{\kappa}\left(\overline{w}_0\overline{w}_2 + 4\overline{w}_2^2\right) + \tilde{c}_2|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

where

$$1 < q_c = \frac{2p}{p+1} < 2, \quad \tilde{c}_0 = \mu 2^{q_c}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right), \quad \tilde{c}_2 = \mu q_c 2^{q_c-2}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right).$$

Note that the sign of $\tilde{c}_0$ and $\tilde{c}_2$ is the same as for μ.
Keeping only the main terms, one finds the following solution:

$$\overline{w}_0 \sim -\frac{\tilde{c}_0 B^{q_c}}{s^{\frac{q_c}{q_c-1}}} \ll \overline{w}_2 = -sign(\mu)\frac{B}{s^{\frac{1}{q_c-1}}} \text{ for some } B > 0.$$

Conclusion for the inner expansion

Recalling the ansatz

$$w(y,s) = \kappa + \overline{w}(y,s) = \kappa + \overline{w}_0(s)h_0(y) + \overline{w}_2(s)h_2(y) \text{with } h_2(y) = y^2 - 2,$$

we end-up with

$$w(y,s) = \kappa - sign(\mu)B\frac{y^2}{s^{2\beta}} + 2sign(\mu)B\frac{1}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right),$$

with

$$\beta = \frac{1}{2(q_c - 1)} = \frac{p+1}{2(p-1)} > \frac{1}{2}.$$

Remark: This expansion is valid in L^2_ρ and uniformly on compact sets by parabolic regularity. However, for y bounded, we see no shape: the expansion is asymptotically a constant.

Idea: *What if $z = \frac{y}{s^\beta}$ is the relevant space variable for the solution shape?*

Outer expansion

To have a *shape*, following the inner expansion, (*valid for* $|y|$ *bounded*),

$$w(y,s) = \kappa - sign(\mu)Bz^2 + 2sign(\mu)B\frac{1}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right) \quad \text{with } z = \frac{y}{s^\beta},$$

let us look for a solution of the following form (*valid for* $|z|$ *bounded*):

$$w(y,s) = \bar{f}_\mu(z) + \frac{a}{s^{2\beta}} + O(\frac{1}{s^\nu}),\ \nu > 2\beta,$$

with $z = \frac{y}{s^\beta}, \bar{f}_\mu(0) = \kappa$ and $\bar{f}_\mu$ bounded.
Plugging this ansatz in the equation,

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^{q_c},$$

then, keeping only the main order, we get

$$-\frac{1}{2} z \bar{f}_\mu'(z) - \frac{1}{p-1} \bar{f}_\mu(z) + (\bar{f}_\mu(z))^p = 0,$$

hence, $\bar{f}_\mu(z) = \left(p - 1 + \bar{b}_\mu |z|^2\right)^{-\frac{1}{p-1}}$, for some constant $\bar{b}_\mu \geq 0$.

Matching asymptotics

For y bounded, both the inner expansion (*valid for $|y|$ bounded*)

$$w(y,s) = \kappa - sign(\mu)B\frac{y^2}{s^{2\beta}} + 2sign(\mu)B\frac{1}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right),$$

and the outer expansion (*valid for $|z|$ bounded*)

$$w(y,s) = \bar{f}_\mu(z) + \frac{a}{s^{2\beta}} + O(\frac{1}{s^\nu}) \text{ where } \nu > 2\beta, \quad z = \frac{y}{s^\beta}$$

and

$$\bar{f}_\mu(z) = \left(p - 1 + \bar{b}_\mu |z|^2\right)^{-\frac{1}{p-1}} = \kappa - \frac{\bar{b}_\mu \kappa}{(p-1)^2} z^2 + O(z^4),$$

have to agree. Therefore,

$$sign(\mu)B = \frac{\bar{b}_\mu \kappa}{(p-1)^2} \text{ and } a = 2sign(\mu)B.$$

Thus, since $B > 0$ and $\bar{b}_\mu > 0$ from the inner and the outer expansions, it follows that

$$\mu > 0, \quad \bar{b}_\mu = \frac{(p-1)^2}{\kappa}B \text{ and } a = 2B, \text{ with } B = \left[2^{q_c-2} q_c(q_c-1)\int_{\mathbb{R}} |y|^{q_c}\rho\right]^{-\frac{1}{q_c-1}} \mu^{-\frac{p+1}{p-1}}.$$

Conclusion of the formal approach

We have just derived the blow-up profile for $|y| \leq Ks^{\beta}$:

$$\varphi(y,s) = \bar{f}_{\mu}\left(\frac{y}{s^{\beta}}\right) + \frac{a}{s^{2\beta}} = \left(p - 1 + \bar{b}_{\mu}\frac{|y|^2}{s^{2\beta}}\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}},$$

where

$$\beta = \frac{p+1}{2(p-1)}, \quad \bar{b}_{\mu} = \frac{(p-1)^2}{\kappa}B \text{ and } a = 2B,$$

with

$$B = \left[2^{q_c - 2} q_c (q_c - 1) \int_{\mathbb{R}} |y|^{q_c} \rho\right]^{-\frac{1}{q_c - 1}} \mu^{-\frac{p+1}{p-1}}.$$

Strategy of the proof

We follow the strategy used by Bricmont and Kupiainen (1994) then Merle and Z. (1997) for the standard semilinear heat equation, based on:

- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

This strategy was later adapted for:

- the present equation with *subcritical* gradient exponent $q < q_c$ in Ebde and Z. (2011);
- the Ginzburg-Landau equation:

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u - \gamma u$$

  in Z. (1998) and Masmoudi and Z. (2008);
- the supercritical gKdV and NLS in Côte, Martel and Merle (2011);
- the semilinear wave equation

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u$$

  in Côte and Z. (2013), for the construction of a blow-up solution showing multi-solitons.

Construction of solutions of PDEs with prescribed behavior

More generally, we are in the framework of constructing a solution to some PDE with some *prescribed behavior*:

- NLS: Merle (1990), Martel and Merle (2006);
- KdV (and gKdV): Martel (2005), Côte (2006, 2007),
- water waves: Ming-Rousset-Tzvetkov (2013),
- Schrödinger maps: Merle-Raphaël-Rodniansky (2013),
- etc....

The strategy of the proof ($N = 1$)

We recall our aim: to construct a solution $w(y,s)$ of the equation in similarity variables:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^{q_c},$$

such that

$$w(y,s) \sim \varphi(y,s) \text{ where } \varphi(y,s) = \left(p - 1 + \bar{b}_\mu \frac{|y|^2}{s^{2\beta}}\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}}.$$

Idea: We linearize around $\varphi(y,s)$ by introducing

$$v(y,s) = w(y,s) - \varphi(y,s).$$

In that case, our aim becomes to construct $v(y,s)$ such that

$$\|v(s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

and $v(y,s)$ satisfies for all $s \geq s_0$ and $y \in \mathbb{R}$,

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s),$$

where

$$\mathcal{L}v = \partial_y^2 v - \frac{1}{2} y \partial_y v + v,$$

$$V(y,s) = p\,\varphi(y,s)^{p-1} - \frac{p}{p-1},$$

$$B(v) = |\varphi + v|^{p-1}(\varphi + v) - \varphi^p - p\varphi^{p-1}v,$$

$$G(\partial_y v) = \mu|\partial_y \varphi + \partial_y v|^{q_c} - \mu|\partial_y \varphi|^{q_c},$$

$$R(y,s) = -\partial_s \varphi + \partial_y^2 \varphi - \frac{1}{2} y \partial_y \varphi - \frac{\varphi}{p-1} + \varphi^p + \mu|\partial_y \varphi|^{q_c}.$$

Effect of the different terms

- The linear term: Its spectrum is given by $\{1-\frac{m}{2},\ \mid m\in\mathbb{N}\}$ and its eigenfunctions are Hermite polynomials with $\mathcal{L}h_m=(1-\frac{m}{2})h_m$.
  Note that we have two positive directions $\lambda=1,\ \frac{1}{2}$ and a null direction $\lambda=0$.
- The potential term V: it has two fundamental properties:
  - (i) $V(.,s)\to 0$ in $L^2_\rho(\mathbb{R})$ as $s\to\infty$. In practice, the effect of V in the blow-up area $(|y|\le Ks^\beta)$ is regarded as a perturbation of the effect of $\mathcal{L}$ (except on the null mode).
  - (ii) $V(.,s)\to -\frac{p}{p-1}$ as $s\to\infty$. and $\frac{|y|}{s^\beta}\to\infty$. Since $-\frac{p}{p-1}<-1$ and 1 is the largest eigenvalue of the operator $\mathcal{L}$, outside the blow-up area (i.e. for $|y|\ge Ks^\beta$), we may consider that the operator $\mathcal{L}+V$ has negative spectrum, hence, easily controlled.
- The nonlinear term in v: It is quadratic: $|B(v)|\le C|v|^2$,
- The nonlinear term in $\partial_y v$: It is sublinear: $\|G(\partial_y v)\|_{L^\infty(\mathbb{R})}\le\frac{C}{\sqrt{s}}\|\partial_y v\|_{L^\infty(\mathbb{R})}$.
- The rest term: It is small: $\|R(.,s)\|_{L^\infty}\le\frac{C}{s}$.

Remarks

- From the properties of the profile and the potential, the variable

$$z = \frac{y}{s^\beta}$$

plays a fundamental role, and our analysis will be different in the regions

$$|z| > K \text{ and } |z| < 2K.$$

- The linear operator will be predominant on all the modes, except on the null mode (i.e. with the eigenfunction $h_2(y)$) where the terms Vv and $G(\partial_y v)$ will play a crucial role.

Decomposition of $v(y,s)$ into “”inner” and “outer” parts

Consider a cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\, s^\beta}\right),$$

where $\chi_0 \in C^\infty\big([0,\infty),[0,1]\big)$, s.t. $\mathrm{supp}(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$ in $[0,1]$. Then, introduce

$$v(y,s) = v_{inner}(y,s) + v_{outer}(y,s),$$

with

$$v_{inner}(y,s) = v(y,s)\chi(y,s) \text{ and } v_{outer}(y,s) = v(y,s)\big(1-\chi(y,s)\big).$$

Note that

$$\mathrm{supp}\; v_{inner}(s) \subset B(0,2Ks^\beta),\;\; \mathrm{supp}\; v_{outer}(s) \subset \mathbb{R}^N \setminus B(0,Ks^\beta).$$

Remark: $v_{outer}(y,s)$ is easily controlled, because $\mathcal{L}+V$ has a negative spectrum (less than $1-\frac{p}{p-1}+\epsilon<0$).

Decomposition of the "inner" part

We decompose v_{inner}, according to the sign of the eigenvalues of $\mathcal{L}$:

$$v_{inner}(y,s) = \sum_{m=0}^{2} v_m(s)h_m(y) + v_-(y,s),$$

where v_m is the projection of v_{inner} (and not v on h_m, and $v_-(y,s) = P_-(v_{inner})$ with P_- being the projection on the negative subspace $E_- \equiv Span\{h_m \mid m \geq 3\}$ of the operator $\mathcal{L}$.

Remark: $v_-(y,s)$ is easily controlled because the spectrum of $\mathcal{L}$ restricted to E_- is less than $-\frac{1}{2}$.

It remains then to control v_0, v_1 and v_2.

Control of v_2

This is delicate, because it corresponds to the direction $h_2(y)$, the null mode of the linear operator $\mathcal{L}$.
Projecting the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s)$$

on $h_2(y)$, and recalling that $\mathcal{L}h_2 = 0$, we need to refine the contributions of Vv and $G(\partial_y v)$ to the linear term (this is delicate), and write:

$$v_2'(s) = -\frac{2}{s}v_2(s) + O\left(\frac{1}{s^{4\beta}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right).$$

Working in the slow variable $\tau = \log s = \log|\log(T-t)|$,

we see that

$$\frac{d}{d\tau}v_2 = -2v_2 + O\left(\frac{1}{s^{4\beta-1}}\right) + O\left(s\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right),$$

which shows a negative eigenvalue.
Conclusion: v_2 can be controlled as well.

We are left only with two components v_0 and v_1: A finite dimensional problem.

Dealing with v_0 and v_1

These remaining components correspond repectively to the projections along $h_0(y) = 1$ and $h_1(y) = y$, the *positive* directions of $\mathcal{L}$. Projecting the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s)$$

on $h_m(y)$ with $m = 0,\ 1$, we write

$$v_0'(s) = v_0(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right),$$

$$v_1'(s) = \frac{1}{2}v_1(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right).$$

Since all the other components are easy to control, we may assume that

$$v(y,s) = v_0(s)h_0(y) + v_1(s)h_1(s) = v_0(s) + v_1(s)y,$$

ending with a "baby" problem, *which is two-dimensional*, with initial data at $s = s_0$ given by

$$v_0(s_0) = d_0, \quad v_1(s_0) = d_1,$$

Solution of the baby problem

Recall the baby problem:

$$v_0'(s) = v_0(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(v_0(s)^2\right) + O\left(v_1(s)^2\right),$$
$$v_1'(s) = \frac{1}{2}v_1(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(v_0(s)^2\right) + O\left(v_1(s)^2\right),$$

with initial data at $s = s_0$ given by

$$v_0(s_0) = d_0, \quad v_1(s_0) = d_1.$$

This problem can be easily solved by contradiction, based on *Index Theory*:

There exist a particular value $(d_0, d_1) \in \mathbb{R}^2$ *such that the "baby" problem has a solution* $(v_0(s), v_1(s))$ *which converges to* $(0,0)$ *as* $s \to \infty$.

Conclusion for the full problem

For the full problem (which is *infinite-dimensional*), recalling that

$$v(y,s) = v_{inner}(y,s) + v_{outer}(y,s) = \sum_{m=0}^{2} v_m(s)h_m(y) + v_-(y,s) + v_{outer}(y,s),$$

and that all the three other components correspond to negative eigenvalues, hence easily converging to zero, we have the following statement:

Consider the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s)$$

equipped with initial data at $s = s_0$*:*

$$\psi_{s_0,d_0,d_1}(y) = \Big(d_0 h_0(y) + d_1 h_1(y)\Big)\chi(2y, s_0).$$

Then, there exists a particular value (d_0, d_1) *such that the corresponding solution* $v(y,s)$ *exists for all* $s \geq s_0$ *and* $y \in \mathbb{R}$*, and satisfies*

$$\|v(y,s)\|_{L^\infty(\mathbb{R})} \to 0 \ \text{ as } s \to \infty.$$

End of the proof of the existence proof

Introducing

$$T = e^{-s_0},$$

and recalling that

$$v(y,s) = w(y,s) - \varphi(y,s) \text{ and } u(x,t) = (T-t)^{-\frac{1}{p-1}} w\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right),$$

and

$$\varphi(y,s) = \bar{f}_\mu(\frac{y}{s^\beta}) + \frac{a}{s^{2\beta}},$$

we derive the existence of $u(x,t)$, a solution to the equation

$$\partial_t u = \Delta u + \mu|\nabla u|^{q_c} + |u|^{p-1}u,$$

such that

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}, t) \sim \bar{f}_\mu(z) \text{ as } t \to T.$$

Using refined parabolic regularity estimates, we derive that:

- $u(x,t)$ blows up only at the origin;
- the final profile satisfies $u(x,T) \sim \left(\frac{2}{\bar{b}_\mu}|x|^{-2}\,|\log|x||^{\frac{p+1}{p-1}}\right)^{\frac{1}{p-1}}$ as $x \to 0$.

Thank you for your attention.