Remarks on some nonlocal dispersive equations

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Aim of the talk

- Motivations for studying nonlocal dispersive equations.
- The fKdV equation as a toy model.
- Comments on some full dispersion surface waves models (Whitham and FDKP) :
- Asymptotic models keeping the dispersion of the water waves system.
- Hope to get validity for larger frequency ranges.
- Shortcoming : loss of the "good" (for PDE methods) dispersive properties.

Why nonlocal dispersive equations ?

- First motivation.

  To study the influence of dispersion on the space of resolution, on the lifespan [1], the possible blow-up and on the dynamics of solutions to the Cauchy problem for "weak" dispersive perturbations of hyperbolic quasilinear equations or systems, as for instance various models of water waves or nonlinear optics.

1. Most of dispersive models are not derived from first principles but as asymptotic models in various regimes, and one does not expect a priori global well-posedness

- General idea : to investigate the "fight" between nonlinearity and dispersion. Usually this problem is attacked by fixing the dispersion (*eg* that of the KdV equation) and varying the nonlinearity (say $u^p u_x$ in the context of generalized KdV).
- Other possibility (probably more physically relevant) : to fix the quadratic nonlinearity (*eg* uu_x) and to vary (lower) the dispersion. In fact in many problems arising from Physics or Continuum Mechanics the nonlinearity is quadratic, with terms like $(u \cdot \nabla)u$ and the dispersion is in some sense weak. In particular the dispersion is not strong enough for yielding the dispersive estimates that allows to solve the Cauchy problem in relatively large functional classes (like the KdV or Benjamin-Ono equation in particular), down to the energy level for instance. [2]

2. And thus obtaining *global well-posedness* from the conservation laws.

Toy model (fKdV)

$$\partial_t u - D^\alpha \partial_x u + u\partial_x u = 0, \quad -1 < \alpha < 1, \tag{1}$$

where $x,\ t \in \mathbb{R}$, $\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$.

- $\alpha = 1$: Benjamin-Ono (or ILW). $\alpha = 2$: KdV.
- Extensively studied for $1 \leq \alpha \leq 2$ (Fonseca-Linares-Ponce, 2012-2013 : GWP).
- $\alpha = -1$: Burgers-Hilbert.
- $\alpha = -\frac{1}{2}$, reminiscent of the Whitham equation (see below). $\alpha = \frac{1}{2}$, see Whitham with surface tension.
- It turns out that only the case $0 < \alpha < 1$ improves the LWP theory (see below).

The Whitham equation (1967)

$$u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)u_x(y,t)dy = 0, \quad \text{or} \tag{2}$$

$$u_t + uu_x - Lu_x = 0, \tag{3}$$

where the Fourier multiplier operator L is defined by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi), \quad \text{where } p = \hat{k}.$$

In the original Whitham equation, the kernel k was given by

$$k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi}\right)^{1/2} e^{ix\xi} d\xi, \tag{4}$$

that is $p(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^{1/2}$.

- The dispersion is in this case that of the finite depth surface water waves without surface tension.
- With surface tension, one gets $p(\xi) = (1+\beta|\xi|^2)^{1/2}\left(\frac{\tanh \xi}{\xi}\right)^{1/2}, \quad \beta \geq 0.$

The Whitham equation makes a link with the

Second motivation :

- Lagrange (1781) first derived the water waves system (Euler with free boundary) and had the wonderful idea "to zoom" on a suitable regime of amplitude, wavelengths, wave steepness to derive simpler, asymptotic models (he derived the wave equation with the correct velocity $\sqrt{gh}$)...
- Boussinesq (1877) : Boussinesq system, "KdV" equation,..
- In those models (and in most of the classical ones), the nonlocal original dispersion relation is approximated, via Taylor expansion, leading to a differential operator. The full dispersion models somehow keep the original dispersion. Valid in all regimes : Boussinesq (or more nonlinear regimes), modulational,.. See David Lannes AMS book, 2013.

Back to fKdV. Basic questions for the Cauchy problem

- How the space of resolution of the Cauchy problem is enhanced when $0 < \alpha < 1$?
- Blow-up and what kind of blow-up ?
- Solitary waves.
- Structure of the solution when it is global (decomposition into solitary waves + dispersion ?).

The following quantities are conserved by the flow associated to fKdV,

$$M(u) = \int_{\mathbb{R}} u^2(x,t)dx, \tag{5}$$

and the Hamiltonian

$$H(u) = \int_{\mathbb{R}} \big(\frac{1}{2}|D^{\frac{\alpha}{2}}u(x,t)|^2 - \frac{1}{6}u^3(x,t)\big)dx. \tag{6}$$

By Sobolev $H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, and $H(u)$ is well-defined when $\alpha \geq \frac{1}{3}$ (energy critical).
Moreover, equation (1) is invariant under the scaling transformation

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1}t), \quad \forall \lambda > 0.$$

Straightforward computation : $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s+\alpha-\frac{1}{2}}\|u_\lambda\|_{\dot{H}^s}$, and thus the critical index corresponding to (1) is $s_\alpha = \frac{1}{2} - \alpha$. In particular, equation (1) is L^2-critical for $\alpha = \frac{1}{2}$.

Easy results

By standard compactness methods : the Cauchy problem associated to (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$.
Moreover, interpolation arguments or the following Gagliardo-Nirenberg inequality,

$$\|u\|_{L^3} \lesssim \|u\|_{L^2}^{\frac{3\alpha-1}{3\alpha}} \|D^{\frac{\alpha}{2}} u\|_{L^2}^{\frac{1}{3\alpha}}, \quad \alpha \geq \frac{1}{3},$$

combined with the conserved quantities M and H defined in (5) and (6) implies the existence of global weak solution in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ as soon as $\alpha > \frac{1}{2}$ and for small data in $H^{\frac{1}{4}}(\mathbb{R})$ when $\alpha = \frac{1}{2}$. More precisely[3] :

Theorem

Let $\frac{1}{2} < \alpha < 1$ *and* $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$. *Then* (1) *possesses a global weak solution in* $L^\infty([0,T]; H^{\frac{\alpha}{2}}(\mathbb{R}))$ *with initial data* u_0. *The same result holds when* $\alpha = \frac{1}{2}$ *provided* $\|u_0\|_{L^2}$ *is small enough.*

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3. We recall that we exclude the value $\alpha = 1$ which corresponds to the Benjamin-Ono equation for which much more complete results are known.

Moreover, (Ginibre and Velo 1991) a Kato type local smoothing property holds, implying global existence of weak L^2 solutions :

Theorem

Let $\frac{1}{2} < \alpha < 1$ *and* $u_0 \in L^2(\mathbb{R})$. *Then* (1) *possesses a global weak solution in* $L^\infty([0,\infty); L^2(\mathbb{R})) \cap l^\infty L^2_{loc}(\mathbb{R}; H^{\frac{\alpha}{2}}_{loc}(\mathbb{R}))$ *with initial data* u_0.

- However, the case $0 < \alpha < \frac{1}{2}$ is more delicate and the previous results are not known to hold. In particular the Hamiltonian H together with the L^2 norm do not control the $H^{\frac{\alpha}{2}}(\mathbb{R})$ norm anymore. Note that the Hamiltonian does not make sense when $0 < \alpha < \frac{1}{3}$ (energy supercritical).

The local theory (F. Linares-D. Pilod-JCS SIMA 2014)

Theorem

Let $0 < \alpha < 1$. Define $s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$ and assume that $s > s(\alpha)$. Then, for every $u_0 \in H^s(\mathbb{R})$, there exists a positive time $T = T(\|u_0\|_{H^s})$ (which can be chosen as a nonincreasing function of its argument), and a unique solution u to (1) satisfying $u(\cdot, 0) = u_0$ such that

$$u \in C([0,T] : H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbb{R})). \tag{7}$$

Moreover, for any $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{R})$ such that the flow map data-solution

$$S^s_{T'} : \mathcal{U} \longrightarrow C([0,T']; H^s(\mathbb{R})), \; u_0 \longmapsto u, \tag{8}$$

is continuous.

Remarks

- Classical result : IVP associated to the Burgers equation is ill-posed in $H^{\frac{3}{2}}(\mathbb{R})$.
- When $\alpha = 1$, the exponent $s(\alpha)$ corresponds to $\frac{9}{8}$ obtained for the BO equation in Kenig-Koenig (2003). The index $s(\alpha)$ is very likely not optimal (see Molinet-Pilod, in progress).
- Molinet-S-Tzvetkov (2001) : for $0 < \alpha < 2$ the Cauchy problem is C^2-ill-posed [4] for initial data in any Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, and in particular the Cauchy problem cannot be solved by a Picard iterative scheme implemented on the Duhamel formulation (quasilinear type).
- Well-posedness in $H^{\frac{\alpha}{2}}(\mathbb{R})$ in the case $\frac{1}{2} \le \alpha < 1$, (which would imply global well-posedness by using the conserved quantities (5) and (6)) : still open.

  This conjecture is supported by the numerical simulations in C. Klein-S (see below) that suggest that the solution is global in this case, for arbitrary large initial data.

4. That is that the flow map cannot be C^2.

- Theorem 3 extends easily by perturbation to some non pure power dispersions. For instance, in the case of the Whitham equation with surface tension, it suffices to observe that

$$(1+\xi^2)^{1/2}\left(\frac{\tanh|\xi|}{|\xi|}\right)^{1/2} = |\xi|^{1/2} + R(|\xi|),$$

where $|R(|\xi|)| \leq |\xi|^{-3/2}$ for large $|\xi|$.

Solitary waves (mainly based on Linares-Pilod-S 2014-2015. See also Arnesen, arXiv 2015 for related results).
A (localized) solitary wave solution of (1) of the form $u(x,t) = Q_c(x - ct)$ must satisfy the equation

$$D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0, \tag{9}$$

where $c > 0$.
One does not expect solitary waves to exist when $\alpha < \frac{1}{3}$ since then the Hamiltonian does not make sense (see a formal argument in Kuznetsov-Zakharov 2000). In fact :

- Assume that $0 < \alpha \leq \frac{1}{3}$. Then (9) does not possesses any nontrivial solution Q_c in the class $H^{\frac{\alpha}{2}}(\mathbb{R}) \cap L^3(\mathbb{R})$ [5]. (The proof works as well for $\alpha < 0$).

Based on the identity

$$\int_{\mathbb{R}} (D^\alpha \phi) x \phi' dx = \frac{\alpha - 1}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} \phi|^2 dx,$$

5. This implies that the Hamiltonian is well defined.

The solitary waves are classically obtained following Weinstein approach by looking for the best constant $C_{p,\alpha}$ in the Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} |u|^{p+2} \leq C_{p,\alpha} \left(\int_{\mathbb{R}} |D^{\alpha/2} u|^2 \right)^{\frac{p}{2\alpha}} \left(\int_{\mathbb{R}} |u|^2 \right)^{\frac{p}{2\alpha}(\alpha-1)+1}, \quad \alpha \geq \frac{p}{p+2}. \qquad (10)$$

This amounts to minimize the functional

$$J^{p,\alpha}(u) = \frac{\left(\int_{\mathbb{R}} |D^{\alpha/2} u|^2\right)^{\frac{p}{2\alpha}} \left(\int_{\mathbb{R}} |u|^2\right)^{\frac{p}{2\alpha}(\alpha-1)+1}}{\int_{\mathbb{R}} |u|^{p+2}}. \qquad (11)$$

One obtains (see Frank-Lenzman 2010 and the references therein) :

Theorem

Let $\frac{1}{3} < \alpha < 1$. Then
(i) Existence : There exists a solution $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ of equation (9) such that $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ is a minimizer for $J^{p,\alpha}$.
(ii) Symmetry and Monotonicity : If $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ is a nontrivial solution of (9) with $Q \geq 0$, then there exists $x_0 \in \mathbb{R}$ such that $Q(\cdot - 0)$ is an even, positive and strictly decreasing in $|x - x_0|$.
(iii) Regularity and Decay : If $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ solves (9), then $Q \in H^{\alpha+1}(\mathbb{R})$. Moreover, we have the decay estimate $|Q(x)| + |xQ'(x)| \leq \frac{C}{1+|x|^{1+\alpha}}$, for all $x \in \mathbb{R}$ and some constant $C > 0$.

- Uniqueness issues have been addressed in Frank-Lenzman 2010 (in any dimension). They concern ground states solutions according to the following definition

Definition

Let $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})$ be an even and positive solution of (9) . If

$$J^{(p,\alpha)}(Q) = \inf \left\{ J^{(p,\alpha)}(u) \ : \ u \in H^{\frac{\alpha}{2}}(\mathbb{R}) \setminus \{0\} \right\},$$

then Q is a ground state solution.

- The main result in Frank-Lenzman 2010 implies in our case ($p = 1$) that the ground state is unique.
- Observe that the uniqueness (up to the trivial symmetries) of the solitary-waves of the Benjamin-Ono equation has been established by Amick-Toland 1991 (see Albert-Toland 1994, Albert 1995, Frank-Lenzmann 2011 for the ILW equation).
- Note that the method of proof of the existence Theorem does not yields any (orbital) stability result. One has to use instead a variant of the Cazenave-Lions method, that is obtain the solitary waves by minimizing the Hamiltonian with fixed L^2 norm. See Albert-Bona-S (1997) in the case $\alpha = 1$ (and for non-homogeneous symbols, as in the ILW equation) and Albert (1999) when $\alpha \geq 1$.

- One can extend Albert 1999 to the case $1/2 < \alpha < 1$ (Linares-Pilod-S 2015)

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} [|D^{\frac{\alpha}{2}} u|^2 - \frac{1}{3}u^3]dx \quad \text{and} \quad M(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx\,.$$

For $q > 0$ fixed, we set

$$I_q = \inf_{u \in H^{\frac{\alpha}{2}}(\mathbb{R})} \{E(u) \ : \ M(u) = q\}. \tag{12}$$

We will denote by G_q the set (possibly empty) of minimizers.

Linares-Pilod-S- ADE 2015 (similar results obtained independently by M.A. Arnesen, arxiv 2015)

- When $\frac{1}{2} < \alpha < 1$, G_q is not empty and orbitally stable for any $q > 0$.
- Any minimizer in G_q is (up to scaling) a ground state and thus (by Frank-Lenzman 2013) positive, radial and unique.
- Let $\frac{1}{2} < \alpha < 1$, $c > 0$ and $Q_c = cQ(c^{\frac{1}{\alpha}} \cdot)$, where Q is the ground state solution of (**??**). For every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^s(\mathbb{R})$, $s > s_\alpha = \frac{3}{2} - \frac{3\alpha}{8}$, satisfy

$$\|u_0 - Q_c\|_{\frac{\alpha}{2}} < \alpha\,, \tag{13}$$

then the corresponding solution u emanating from u_0 satisfies

$$\inf_{y\in\mathbb{R}} \|u(\cdot,t) - Q_c(\cdot + y)\|_{\frac{\alpha}{2}} < \epsilon \tag{14}$$

for all $t \in [0, T_s)$, where T_s is the maximal time of existence of u.

- This is so far a conditional stability result since the global well-posedness of the fKdV equation is still unknown (even for initial data close to a solitary wave).
- Questions : asymptotic stability, "multi-solitons",...
- Instability (by blow-up) is expected when $\frac{1}{3} < \alpha \leq \frac{1}{2}$ (see the simulations below) but not proven yet. Recall that for GKdV, instability was proven when $p > 4$ (Bona-Souganidis-Strauss (BSS) 1987) and when $p = 4$ (Martel-Merle 2001,...).

- Why BSS does not extend straightforwardly ?

As in BSS the first step is to give a sense to the formal conserved quantity

$$I(u) = \int_{\mathbb{R}} u dx. \tag{15}$$

As in BSS one checks that if $u_0 \in H^s(\mathbb{R}), s \geq 1+\alpha$ is such that $\int_{-\infty}^{\infty} u_0(x)dx$ converges as a generalized Riemann integral, then $I(u(t))$ converges for any $t \in [0, T_s(u_0))$ and is constant, where $T_s(u_0)$ is the lifespan of the solution u of the corresponding Cauchy problem.
Again as in BSS one has to estimate how fast the tail of $I(u)$ near infinity grows with t. This cannot be deduced from BSS since

$$G_\alpha(x) = \int_{-\infty}^{\infty} e^{i(x\xi - \xi|\xi|^\alpha)} d\xi$$

is not a bounded function of x when $\alpha < 1$.

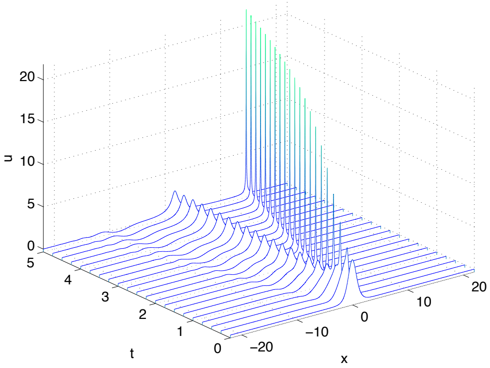
- Actually, (Sidi-Sulem-Sulem 1986), $G_\alpha(x) = O(x^{-(\alpha+2)})$ as $x \to +\infty$ and oscillates when $x \to -\infty$, growing as $|x|^{(1-\alpha)/2\alpha}$.
- To proceed as in BSS, one would need to impose a (one sided) decay property to u_0 insuring that the resulting solution of the Cauchy problem decays sufficiently to the left to compensate the growth of the fundamental solution (in progress...).

Blow-up issues (open)

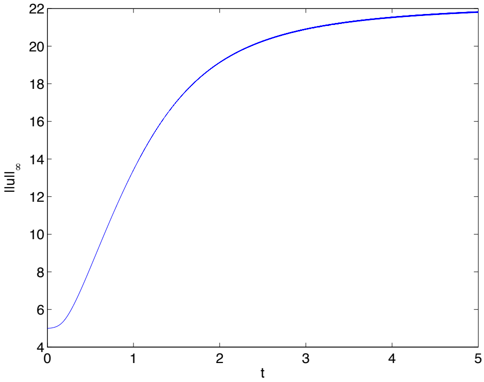
- Solutions of fKdV are conjectured to blow-up in finite time in the energy super critical case $0 < \alpha \leq 1/3$ and in the L^2 super critical case $1/3 < \alpha \leq 1/2$.
- They are conjectured to be global in the L^2 sub-critical case $\alpha > 1/2$.
- Scattering is expected (for small localized initial data) when $0 < \alpha < 1$ and actually even when $-1 < \alpha < 0$.
- See the following computations (Klein-S 2014).

Numerical results for the fKdV (Christian Klein-JCS 2015).

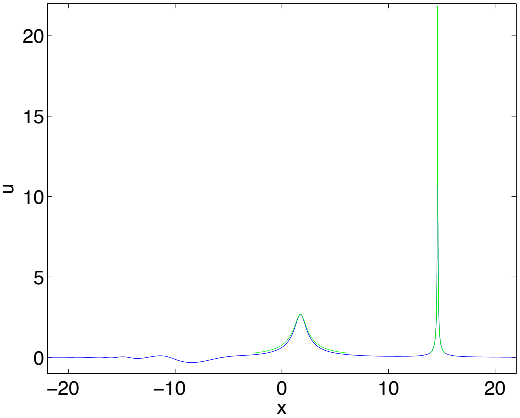
L^2-subcritical case $\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$



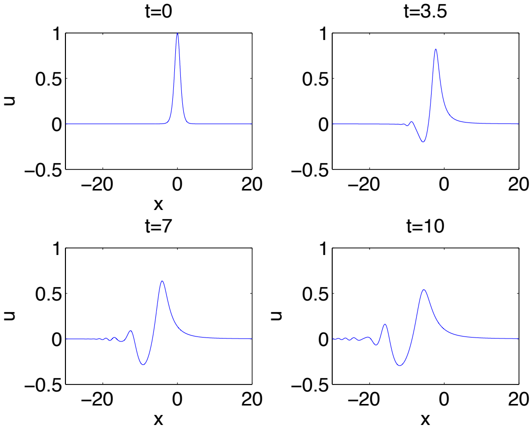
L^2-subcritical case $\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$. Evolution of the sup norm



$\alpha = 0.6$. $u_0 = 5\text{sech}^2 x$. Fitted soliton at humps in green



L^2 critical case $\alpha = 0.5$. $u_0 = \mathrm{sech}^2 x$.



L^2-critical case $\alpha = \frac{1}{2}$. $u_0 = 3\text{sech}^2 x$.

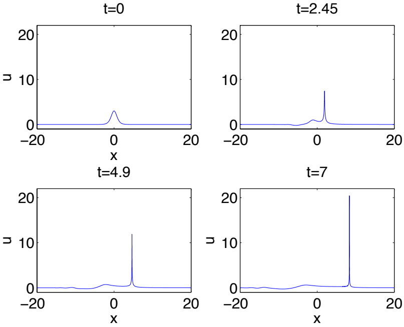
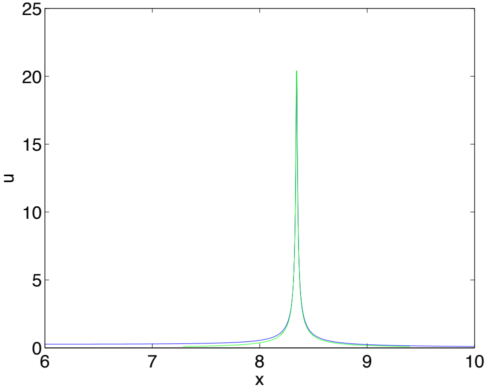
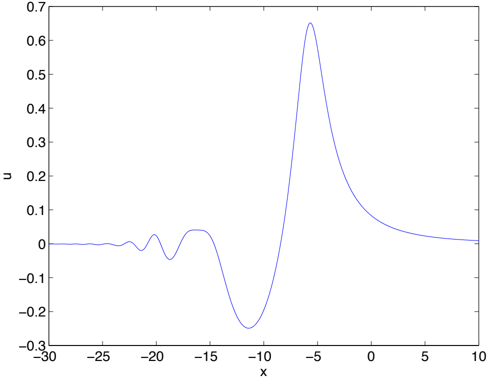


FIGURE 4. Solution to the fKdV equation (7) for $\alpha = 0.5$ and the initial data $u_0 = 3\text{sech}^2 x$ for several values of t.

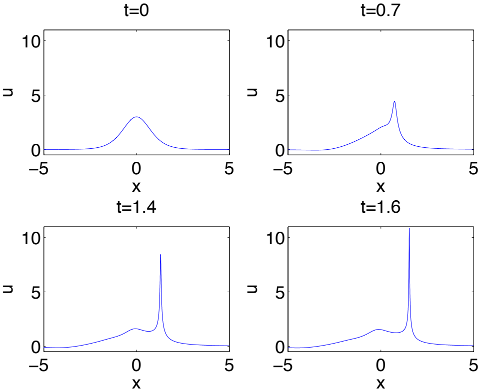
$\alpha = \frac{1}{2}$. $u_0 = 3\text{sech}^2 x$. Fit with rescaled soliton (green)



L^2-supercritical & Energy subcritical $\alpha = 0.45$. $u_0 = \mathrm{sech}^2 x$. $t = 10$



L^2-supercritical & Energy subcritical $\alpha = 0.45$. $u_0 = 3\text{sech}^2 x$



L^2-supercritical & Energy subcritical $\frac{1}{3} < \alpha = 0.45$ $u_0 = \text{sech}^2 x$

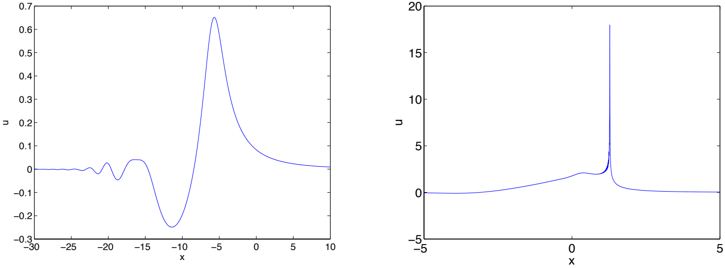


FIGURE 10. Solution to the fKdV equation (Cauchy 7) for $\alpha = 0.4$, on the left for the initial data $u_0 = \text{sech}^2 x$ at $t = 10$, on the right for the initial data $u_0 = 3\text{sech}^2 x$ at $t = 1.11$.

Energy supercritical $0 < \alpha < \frac{1}{3}. u_0 = \operatorname{sech}^2 x$

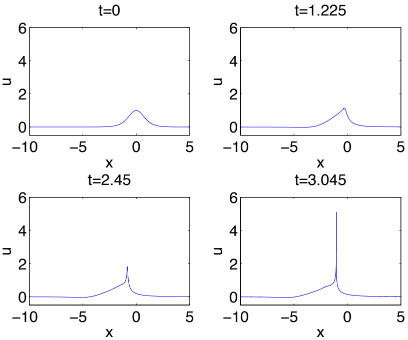
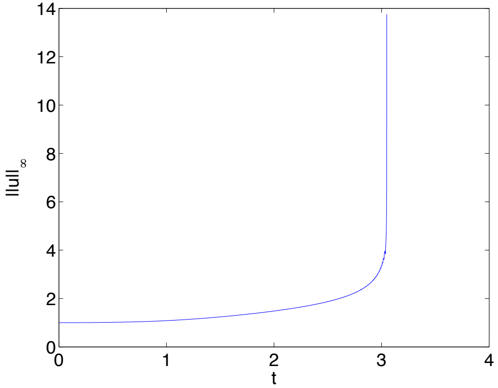
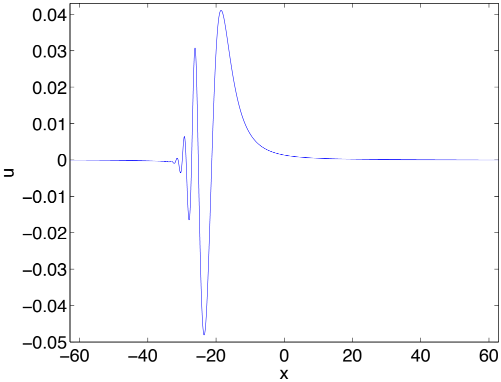


FIGURE 7. Solution to the fKdV equation (7) for $\alpha = 0.2$ and the initial data $u_0 = \operatorname{sech}^2 x$ for several values of t.

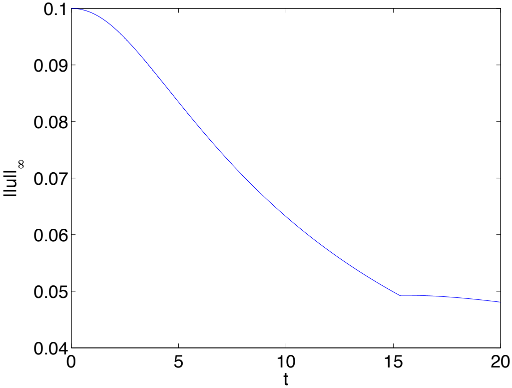
Energy supercritical $\alpha = 0.2. u_0 = \text{sech}^2 x.$ $||u||_\infty$



Energy supercritical $\alpha = 0.2. u_0 = 0.1\text{sech}^2 x$, $t = 20$



Energy supercritical $\alpha = 0.2. u_0 = 0.1\text{sech}^2 x, ||u||_\infty$



Conjectures for the fKdV equation

- $\alpha > 0.5$: solutions to the fKdV equations stay smooth for all t. For large t they decompose asymptotically into solitons and radiation.
- $0 < \alpha \leq 0.5$: solutions to the fKdV equations with initial data u_0 sufficiently small, but non-zero mass stay smooth for all t.
- $\alpha = 0.5$: solutions to the fKdV equations with initial data u_0 with negative energy and mass larger than the soliton mass blow up at finite time t^* (cf GKdV when $p = 4$).
- $1/3 < \alpha < 0.5$: solutions to the fKdV equations with the initial data u_0 and sufficiently large L_2 norm blow up at finite time t^* and finite $x = x^*$. A soliton-type hump separates from the initial hump and eventually blows up (cf GKdV when $p > 4$).
- $0 < \alpha < 1/3$ (energy super-critical) : solutions to the fKdV equations with the initial data u_0 and sufficiently large L_2 norm blow up at finite time t^* and finite $x = x^*$. The nature of blow-up is different from the previous one since no solitary waves exist in this case, the maximum of the initial hump evolves directly into a blow-up.

Another toy model : the fKP equation

(Linares-Pilod-S, in progress)

$$u_t + uu_x - D_x^\alpha u_x + \epsilon \partial_x^{-1} u_{yy} = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad -1 < \alpha < 2 \qquad (16)$$

where $\epsilon = 1$ corresponds to the fKP II equation and $\epsilon = -1$ to the fKP I equation.
$\left(D_x^\alpha f\right)^\wedge(\xi, \eta) = |\xi|^\alpha \widehat{f}(\xi, \eta)$.
When $\alpha = \pm 1/2$ it has some links with the full dispersion KP equation with surface tension (see below)
Hamiltonian :

$$H_\alpha(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|D_x^{\frac{\alpha}{2}} u|^2 - \epsilon \frac{1}{2}|\partial_x^{-1} u_y|^2 - \frac{1}{6}u^3\right). \qquad (17)$$

The corresponding energy space is

$$Y_\alpha = \{u \in L^2(\mathbb{R}^2) \; : \; D_x^{\frac{\alpha}{2}} u, \; \partial_x^{-1} u_y \in L^2(\mathbb{R}^2)\}.$$

One checks readily that the transformation

$$u_\lambda(x,y,t)=\lambda^\alpha u(\lambda x,\lambda^{\frac{\alpha+2}{2}}y,\lambda^{\alpha+1}t)$$

leaves (16) invariant.
Moreover, $|u_\lambda|_2=\lambda^{\frac{3\alpha-4}{4}}|u|_2$, so that $\alpha=\frac{4}{3}$ is the L^2 critical exponent.
Fractionary Gagliardo-Nirenberg inequality :

Lemma

Let $\frac{4}{5}<\alpha<1$. *For any* $f\in Y_\alpha$ *one has*

$$|f|_3^3\leq c|f|_2^{\frac{5\alpha-4}{\alpha+2}}\|f\|_{H_x^{\frac{\alpha}{2}}}^{\frac{18-5\alpha}{2(\alpha+2)}}|\partial_x^{-1}f_y|_2^{\frac{1}{2}},$$

where $\|\cdot\|_{H_x^{\frac{\alpha}{2}}}$ *denotes the natural norm on the space*

$$H_x^{\frac{\alpha}{2}}(\mathbb{R}^2)=\{f\in L^2(\mathbb{R}^2)\;:\;D_x^{\frac{\alpha}{2}}f\in L^2(\mathbb{R}^2)\}.$$

This implies obviously the embedding $Y_\alpha\hookrightarrow L^3(\mathbb{R}^2)$ if $\frac{4}{5}<\alpha<1$.

Non trivial properties of the linear group

- Strichartz estimates when $\alpha > \frac{1}{2}$ (by-product of Molinet-S-Tzvetkov 2007).
- Local smoothing "à la Kato" for fKP-II when $\alpha > \frac{1}{2}$ (combine arguments in S- 1993 (KP-II) and Ginibre-Velo 1991 (fKdV)) : for initial data in $L^2(\mathbb{R}^2)$, gain (locally) of $|D_x^{\alpha/2}u|_2$ and $|\partial_x^{-1}u_y|_2$.

The energy critical value $\alpha = \frac{4}{5}$ is obviously related to the non existence of localized solitary waves $u(x, y, t) = \phi(x - ct, y)$. One has by Pohozaev type arguments :

Proposition

Assume that $0 < \alpha \leq \frac{4}{5}$ *when* $\epsilon = -1$ *or that* α *is arbitrary when* $\epsilon = 1$. *Then fKP does not possess non trivial solitary waves in the space* $Y_\alpha \cap L^3(\mathbb{R}^2)$.

Existence of solitary waves (fKP I) :

- $\alpha > \frac{4}{5}$ (minimization of $||u||^2_{Y^\alpha}$ under the constraint $\int_{\mathbb{R}^2} u^3 = \lambda$.)
- Existence of a (conditionally orbitally stable) set of minimizers of the Hamiltonian with fixed L^2 norm when $\alpha > \frac{4}{3}$.
- Extension of the corresponding proofs for the generalized KP I equation (de Bouard-S 1997).

Blow-up issues

- For the generalized KP-I equation

$$u_t + u^p u_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \tag{18}$$

finite blow-up occurs when $p > 4/3$ (S-1993, Liu 2001).

- No corresponding results for fKP I.

$$u_t+(u_x)+\epsilon u u_x+T_\epsilon u_x=0, \quad T_\epsilon=\left(\frac{\tanh\sqrt{\epsilon}D}{\sqrt{\epsilon}D}\right)^{1/2}, \quad 0<\epsilon\ll 1. \quad (19)$$

Rigorous results on the Whitham equation

- Well posedness of the Cauchy problem in $H^s(\mathbb{R}), s>3/2$ on the correct time scale $1/\epsilon$: trivial (skew adjoint perturbation of $u_t+\epsilon u u_x$).
- The Whitham equation does possess solitary waves that are formally orbitally stable (Ehrnström-Groves-Wahlen 2012). The proof uses in a crucial way that the dispersion relation of the Whitham equation approaches the KdV one for small frequencies.
- Existence of periodic travelling waves Ehrnström-Kalish 2009, 2013 ; stability issues : Hur-Johnson 2014, 2014, Kalish et al 2014 (numerics) and references therein.

Some ideas on Ehrnström, Groves and Wahlen proof.

- A family of SW is found using a constrained minimization principle and concentration-compactness methods for noncoercive functionals. The SW are approximated by (scalings of) the corresponding solutions to the long wave limit equation (KdV).
- Write Whitham $(\epsilon = 1)$ as

$$u_t + 2uu_x + Lu_x = 0, \quad \mathcal{F}(Lf)(\xi) = \left(\frac{\tanh(\xi)}{\xi}\right)^{1/2}.$$

- Equation for a solitary wave $u(x - \nu t)$ (vanishing at infinity) :

$$Lu - \nu u + u^2 = 0.$$

- Weakly nonlinear ansatz :

$$u(x) = \mu w(\mu^2 x), \quad \mu = \frac{1}{2}\int_{\mathbb{R}} u^2 dx \ll 1$$

- The Whitham solitary waves are approximated by (suitably scaled) solutions of the "solitary wave" version of KdV

$$\frac{1}{6}w'' - \nu_{lw}w + w^2 = 0,$$

with solutions

$$D_{lw} = \{w_{KdV}(\cdot + y); y \in \mathbb{R}\}, \quad w_{KdV}(x) = \left(\frac{3}{2}\right)^{\frac{2}{3}} \operatorname{sech}^2\left(\left(\frac{3}{2}\right)^{\frac{1}{3}} x\right).$$

- Whitham SW are formally local minimizers of the functional $\mathcal{E} : H^1(\mathbb{R}) \to \mathbb{R}$,

$$\mathcal{E}(u) = -\frac{1}{2}\int_{\mathbb{R}} uLu - \int_{\mathbb{R}} u^2 dx := \mathcal{L}(u) + \mathcal{N}(u).$$

with $\mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2$ held fixed.
- L is smoothing and thus $\mathcal{E}$ not coercive.
- Use a method of Buffoni (2004) for capillary-gravity SW, see also Groves-Wahlen (2011).
- Consider a fixed ball

$$U = \{u \in H^1(\mathbb{R}); ||u||_1 < R\},$$

and look for small amplitude solutions in the set

$$U_\mu = \{u \in U; \mathcal{Q}(u) = \mu\},$$

where $0 < \mu \ll 1$.

- In particular one looks at the minimizing sequences for $\mathcal{E}$ over U_μ which do not approach the boundary of U.

Theorem

(Ehrnstrom-Groves-Wahlen 2012) There exists $\mu_* > 0$ such that for each $\mu \in (0, \mu_*)$ such that
(i) The set $\mathcal{D}_\mu$ of minimizers of $\mathcal{E}$ over U_μ is not empty and the estimate $||u||_1^2 = O(\mu)$ holds uniformly over $u \in D_\mu$ and $\mu \in (0, \mu_*)$. Each element of D_μ is a Whitham SW; the wave speed ν is the Lagrange multiplier in this constrained variational principle.
(ii) Let $s < 1$ and suppose that $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}$ over U_μ with the property that

$$\sup_{n\in\mathbb{N}} ||u_n||_1 < R. \tag{20}$$

There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that a subsequence of $\{u_n(\cdot + x_n)\}_{n\in\mathbb{N}}$ converges in $H^s(\mathbb{R})$ to a function in D_μ.

Theorem

Convergence to KdV :

$$\sup_{u\in D_\mu}\inf_{y\in\mathbb{R}} ||\mu^{-\frac{2}{3}}u(\mu^{-\frac{1}{3}}(\cdot+y))-w_{KdV}||_1 \to 0$$

and

$$\sup_{u\in D_\mu}\left|\nu(u)-1-\mu^{\frac{2}{3}}\left(\frac{2}{3}\right)^{\frac{1}{2}}\right| = o(\mu^{\frac{2}{3}})$$

as $\mu \to 0$.

Proof in two steps :

- Construction of a minimizing sequence satisfying (20) by considering the corresponding problem for periodic waves and penalize the variational functional so that minimizing sequences do not approach the boundary of the corresponding domain in function space.

  A crucial step is to prove the subadditivity of $I_\mu = \{\mathcal{E}(u); u \in U_\mu\}$. This is where the long wave scaling is used.
- Apply concentration-compactness principle to show that *any* minimizing sequence satisfying (20) converges (up to subsequences and translations) in $H^s(\mathbb{R}), s < 1$ to a minimizer of $\mathcal{E}$ over U_μ.

"Shock" formation for Whitham ?

- For the fractionary KdV equation

$$u_t + \epsilon u u_x + \epsilon D^\alpha u_x = 0,$$

  a finite time blow up may occur in a $C^{1+\delta}$ norm when $-1 < \alpha < 0$, (Castro-Cordoba-Gancedo (2010). The proof is easily adapted to the Whitham equation (Lannes-JCS 2013).
- When $-1 < \alpha < -\frac{1}{3}$ and for Whitham, blow-up in finite time (for suitable smooth initial data) of the sup norm of the derivative (Vera Hur 2015) : there exists $T > 0$ with $|u(x,t)| < +\infty$, $x \in \mathbb{R}, t < T$, and $\lim_{t \to T_-} \partial_x u(x,t) = -\infty$. It would be interesting to check that the blow-up time is beyond the relevant time scale $O(1/\epsilon)$. The Whitham equation is not supposed to describe wave breaking !.

Ideas on the proof :

- The proof uses estimates (Ehrnström 2015) on the kernel

$$k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\tanh(\xi)}{\xi}} e^{-ix\xi} d\xi,$$

more precisely

$$k(x) \sim (2\pi|x|)^{-1/2} \quad \text{and} \quad k'(x) \sim -\frac{1}{2}\text{sgn}(x)(2\pi|x|^3)^{-1/2} \quad \text{as } |x| \to 0.$$

- For $x \in \mathbb{R}$, let $X(t;x)$ solves

$$\frac{dX}{dt}(t;x) = u(X(t;x),t), \quad X(0;x) = x,$$

where u is a local solution of Whitham.

- Let $v_n(t;x) = (\partial_x^n u)(X((t;x),t), \quad n = 0, 1, 2, ...$ and analyze the ODE system satisfied by the $v_n's$.

- In the context of water waves, the Whitham equation writes

$$u_t + (\mathcal{T}_\epsilon)^{1/2}\, u_x + \epsilon u u_x = 0 \qquad (21)$$

where $\epsilon \ll 1$ and $\mathcal{T}_\epsilon = \frac{\tanh \sqrt{\epsilon}|D|}{\sqrt{\epsilon}|D|}$, $D = -i\partial_x$.

- It should be a good approximation for water waves on time scales of order $O(\frac{1}{\epsilon})$, (with error $O(t\epsilon^2)$).
- In the relevant regime (KdV) one does not expect wave breaking, so the blow-up time found by V. Hur should be larger that $O(\frac{1}{\epsilon})$.

According to the previous results and the computations below, the Whitham equation seems to have three different regimes that need more investigations :

- A pure radiation regime for small enough initial data (supported by the dispersive estimates obtained by B. Melinand 2015 for the underlying linear group).
- An "hyperbolic" (shocklike) regime for large enough initial data.
- A solitonic regime in the KdV limit.

Numerical simulations of the Whitham equation ($\epsilon = 1$)

Klein-JCS. Physica D 2015 (see also Lannes-JCS 2013 for other simulations)

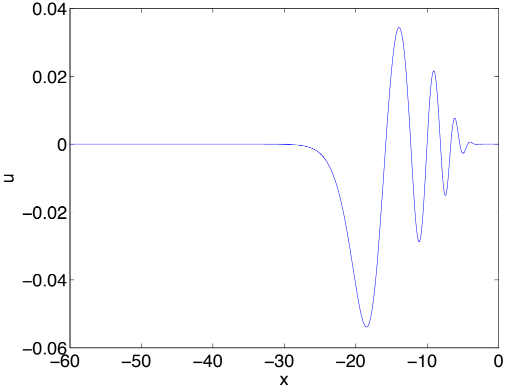
See also the fractional KdV (fKdV) equation

$$u_t + uux - |D|^\alpha u_x = 0,\ \alpha = -1/2$$

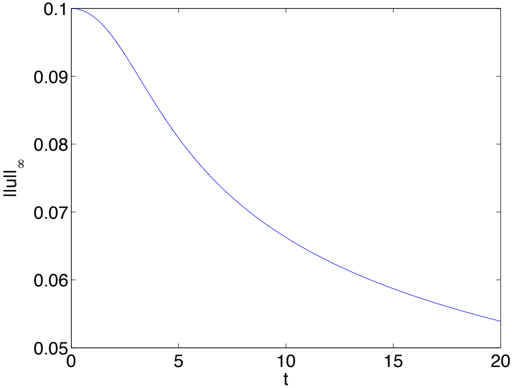
that has the same dispersion for large frequencies.

- Recall that fKdV does not possess solitary wave solutions when $\alpha \leq 1/3$.

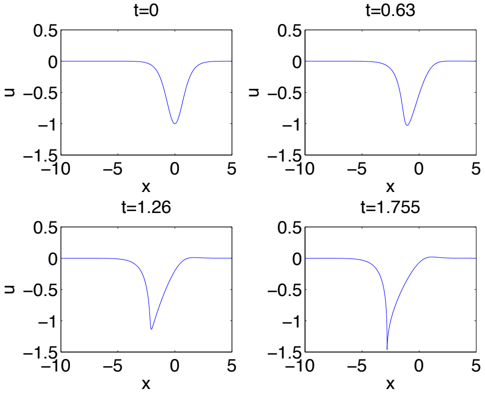
Whitham, $u_0 = -0.1\text{sech}^2 x$, $t = 20$



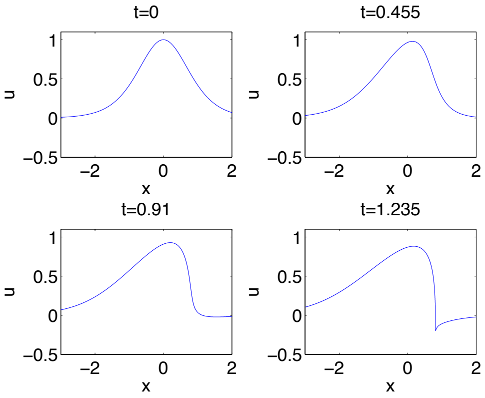
Whitham, $u_0 = -0.1\text{sech}^2 x$, $t = 20$



Whitham, $u_0 = -\text{sech}^2 x$



Whitham, $u_0 = \text{sech}^2 x$

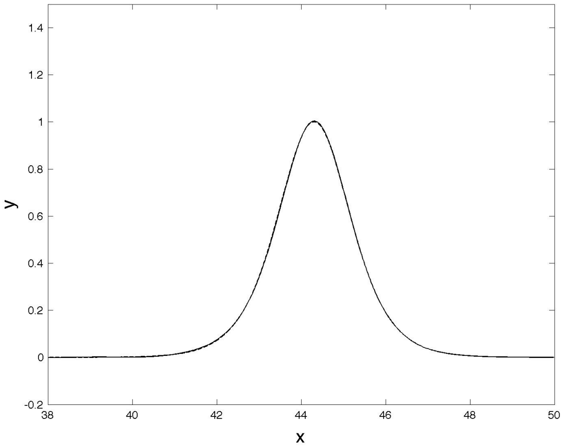


Conjectures for the Whitham equations

Consider smooth initial data $u_0 \in L_2(\mathbb{R})$ with a single hump. Then

- solutions to the Whitham equation and to fKdV equations with $-1 < \alpha < 0$ for initial data u_0 of sufficiently small mass stay smooth for all t and will be radiated away.
- solutions to the Whitham equation (21) and to the fKdV equation with $\alpha = -1/2$ for negative initial data u_0 of sufficiently large mass will develop a cusp at $t^* > t_c$ of the form $|x - x^*|^{1/3}$. The sup norm of the solution remains bounded at the blow-up point.
- solutions to the Whitham equation (21) and to the fKdV equation with $\alpha = -1/2$ for positive initial data u_0 of sufficiently large norm mass will develop a cusp at $t^* < t_c$ of the form $|x - x^*|^{1/2}$.

Comparison at the same time $t = 0.11$ of the solution to the KdV equation (dash) and the Whitham equation with different values of ϵ ($\mu = 0.01, \epsilon = 0.1, \epsilon = 1$) Initial condition is a KdV solitary wave Lannes-JCS 2013.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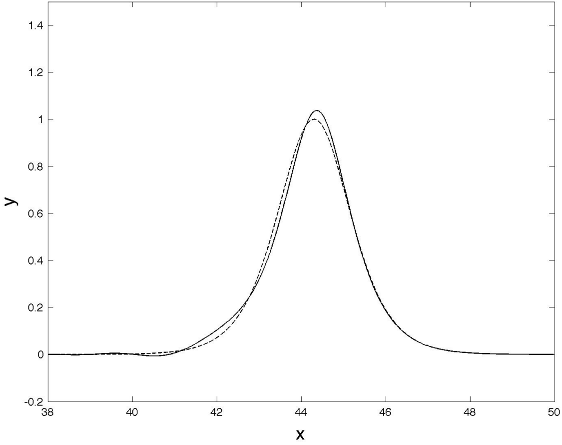
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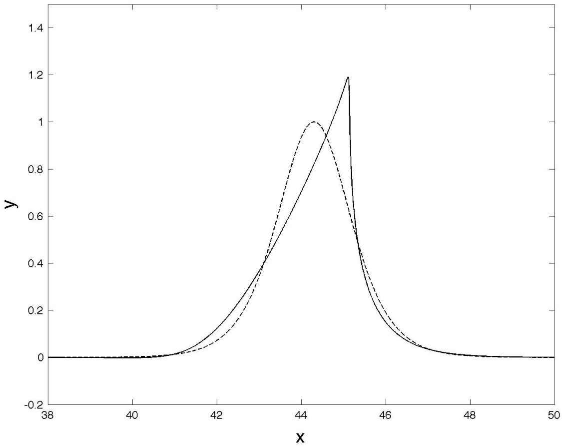
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Full dispersion KP equations

Whitham equations are also 1D version of the Full Dispersion Kadomtsev-Petviashvili (FDKP) equations introduced by D. Lannes (2013) and studied in Lannes-JCS (2013).

$$\partial_t u + c_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \tag{22}$$

where $c_{WW}(\sqrt{\mu}k)$ is the phase velocity of the linearized water waves system, namely

$$c_{WW}(\sqrt{\mu}k) = \left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2}$$

and

$$|D^\mu| = \sqrt{D_1^2 + \mu D_2^2}, \quad D_1 = \frac{1}{i}\partial_x, \quad D_2 = \frac{1}{i}\partial_y.$$

Denoting by h a typical depth of the fluid layer, a a typical amplitude of the wave, λ_x and λ_y typical wave lengths in x and y respectively, the relevant regime here is when

$$\mu \sim \frac{a}{h} \sim \left(\frac{\lambda_x}{\lambda_y}\right)^2 \sim \left(\frac{h}{\lambda_x}\right)^2 \ll 1.$$

When adding surface tension effects, one has to replace (22) by

$$\partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \qquad (23)$$

with

$$\tilde{c}_{WW}(\sqrt{\mu}k) = (1+\beta\mu k^2)^{\frac{1}{2}}\left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2},$$

where $\beta > 0$ is a dimensionless coefficient measuring the surface tension effects,

- The idea is to overcome the unphysical properties (see below) of the classical KP equation due to the awful approximation of the dispersion at $\xi_1 = 0$...

Shortcoming of the KP equation

- The term $\partial_x^{-1} u_{yy}$ implies a physically irrelevant constraint on u. Roughly speaking, one has in some sense $\int_{-\infty}^{\infty} u(x,y,t)dx = 0, \forall y \in \mathbb{R}$.
  See L. Molinet-N. Tzvetkov-JCS (2007) for a discussion of this issue.
- It also prevents to get the optimal error estimate with the full water waves system. In fact, see D. Lannes (2002), D. Lannes -JCS (2006), one gets

  $$||U_{\text{Euler}} - U_{KP}|| = o(1), (O(\sqrt{\epsilon}) \text{ with some additional constraint})$$

  instead of $O(\epsilon^2 t)$ in the KdV (Boussinesq) regime.
- A solution is to introduce weakly transverse Boussinesq systems leading to optimal error estimates (Lannes-JCS 2006). Introducing Full dispersion KP equations is an alternate choice.

Advantages/disadvantages of the full dispersion KP equation

- Enlarges the validity of the model and relaxes somehow the zero-mass constraint.
- The classical K I and KP II equations are recovered formally by keeping the first order term in the expansion with respect to μ of the nonlocal operators.
- Shortcoming from a PDE point of view : lack of the nice dispersive properties....

The linearized equation

For purely gravity waves, the linearized equation writes

$$\partial_t u + \mathcal{P}(D_1, D_2)u = 0, \qquad (24)$$

where $\mathcal{P} = \mathcal{P}_\epsilon = \mathcal{P}(D_1, D_2)$ is the Fourier multiplier defined as

$$\mathcal{P}(D_1, D_2) = c_{WW}(\sqrt{\epsilon}|D^\epsilon|)(1 + \epsilon \frac{D_2^2}{D_1^2})^{1/2}\partial_x.$$

The symbol of $p(\xi_1, \xi_2)$ of $\mathcal{P}$ can be written

$$p(\xi_1, \xi_2) = \frac{i}{\epsilon^{1/4}} \left(\tanh[\sqrt{\epsilon}(\xi_1^2 + \epsilon\xi_2^2)^{\frac{1}{2}}]\right)^{\frac{1}{2}} (\xi_1^2 + \epsilon\xi_2^2)^{\frac{1}{4}} \operatorname{sgn} \xi_1; \qquad (25)$$

since it is real valued, it is clear that the linearized equation defines a unitary group in all Sobolev spaces $H^s(\mathbb{R}^2), s \in \mathbb{R}$.

The symbol $p(\xi_1, \xi_2)$ is not continuous at the origin. It remains however bounded, which is not the case for the symbol of the linear KP equation,

$$p_{KP}(\xi_1, \xi_2) = i\Big(\xi_1 + \frac{\epsilon}{2}\frac{\xi_2^2}{\xi_1} - \frac{\epsilon}{6}\xi_1^3\Big),$$

which grows to infinity as $\xi_1 \to 0$ if $\xi_2 \neq 0$.

- Expanding the FDKP symbol :

$$\begin{aligned} p(\xi_1,\xi_2) &= i\left(\xi_1 + \frac{\epsilon}{2}\frac{\xi_2^2}{\xi_1} - \frac{\epsilon}{6}\xi_1^3\right) + O(\epsilon^2) \\ &= p_{KP}(\xi_1,\xi_2) + O(\epsilon^2). \end{aligned}$$

- The expansion $p = p_{KP} + O(\epsilon^2)$ is only formal in the above computations. Due to the singularity in $1/\xi_1$, it can only be made rigorous when this singularity is controlled by a cancellation of the solution u at low frequencies in x, or equivalently, under a zero mass constraint, typically, $u \in \partial_x H^s(\mathbb{R}^2)$.

The zero mass constraint (occurs already in the linear problem)

- Well-known facts on the classical KP I/II equations :

$$\partial_t u + \partial_x^3 u \pm \partial_x^{-1}\partial_y^2 u = 0. \tag{26}$$

The linear evolution is given in Fourier variables by

$$\widehat{S_\pm(t)u_0}(\xi_1,\xi_2) = \hat{u}(\xi_1,\xi_2,t) = \exp\{it\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right)\}\widehat{u}_0(\xi_1,\xi_2),$$

Unitary group in any Sobolev space $H^s(\mathbb{R}^2)$, $s \geq 0$. On the other hand, even for smooth initial data, say in the Schwartz class, the relation

$$u_{xt} = u_{tx}$$

holds true only in a very weak sense, *e.g.* in $\mathcal{S}'(\mathbb{R}^2)$, if u_0 does not satisfy the constraint $\hat{u}_0(0,\xi_2) = \int_{-\infty}^{\infty} u_0(x,y)dx = 0$ for any $\xi_2 \in \mathbb{R}$ and $y \in \mathbb{R}$.

- In particular, even for smooth localized u_0, the mapping

$$\hat{u}_0 \mapsto \partial_t \hat{u} = i\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right) \exp\{it\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right)\}\hat{u}_0(\xi)$$

cannot be defined with values in a Sobolev space if u_0 does not satisfy the zero mass constraint. For instance, if u_0 is a gaussian, $\partial_t u$ is not even in L^2.

- Linear evolution of the FDKP equation :

$$\widehat{S_{FD}(t)}u_0(\xi_1,\xi_2) = \exp\{it\big(\tanh(\sqrt{\mu}(\xi_1^2+\mu\xi_2^2))^{1/2}\big)^{1/2}(\xi_1^2+\mu\xi_2^2)^{1/4}\text{sign } \xi_1\}\widehat{u}_0(\xi_1,\xi_2),$$

  and $u_0 \mapsto \partial_t \tilde{S}(t)u_0$ is continuous from $H^s(\mathbb{R}^2)$ to $H^{s-1/2}(\mathbb{R}^2)$, for any $s \geq 0$.
- For gravity-capillary waves :

$$\widehat{\tilde{S}_{FD}(t)}u_0(\xi_1,\xi_2) = \exp\Big\{it\big(\tanh(\sqrt{\mu}(\xi_1^2+\mu\xi_2^2))^{1/2}\big)^{1/2} \times (1+\beta\mu(\xi_1^2+\mu\xi_2^2))^{1/2}(\xi_1^2+\mu\xi_2^2)^{1/4}\text{sign } \xi_1\Big\}\widehat{u}_0(\xi_1,\xi_2),$$

  and the map $u_0 \mapsto \partial_t \tilde{S}(t)u_0$ is continuous from $H^s(\mathbb{R}^2)$ to $H^{s-3/2}(\mathbb{R}^2)$, for any $s \geq 0$.
- Note finally that in both FDKP cases (with or without surface tension), $\partial_t u \in H^\infty(\mathbb{R}^d)$ if for instance u_0 is in the Schwartz space, say a gaussian.

The nonlinear problem

Since (22) and (23) are skew-adjoint perturbations of the Burgers equation, one establishes by standard methods the following result which is valid for both gravity and capillary-gravity waves but of course does not take advantage of the dispersion. No zero mass constraint is needed.

Theorem

Let $s > 2$ *and* $u_0 \in H^s(\mathbb{R}^2)$. *There exist* $T(||u_0||_s, \epsilon) = O(\frac{1}{\epsilon})$ *and a unique solution* $u \in C([0, T(||u_0||_s, \epsilon)], H^s(\mathbb{R}^2))$ *of* (22) *with initial data* u_0. *Moreover,*

$$|u(\cdot, t)|_2 = |u_0|_2, \quad t \in [0, T(||u_0||_s, \epsilon)].$$

As for the classical KP I/II equations, the FDKP equation conserves the Hamiltonian

$$\mathfrak{H}_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |H_\epsilon(D)u|^2 + \frac{\epsilon}{4}\int_{\mathbb{R}^2} u^3, \tag{27}$$

where

$$H_\epsilon(D) = \left(\frac{\tanh(\sqrt{\epsilon}|D^\epsilon|)}{\sqrt{\epsilon}|D^\epsilon|}\right)^{1/4}\left(1+\epsilon\frac{D_2^2}{D_1^2}\right)^{1/4} = \left(\frac{\tanh(\sqrt{\epsilon}|D^\epsilon|)}{\sqrt{\epsilon}}\right)^{1/4}\frac{|D^\epsilon|^{1/4}}{|D_1|^{1/2}};$$

the conservation of $\mathfrak{H}_\epsilon(u)$ is indeed a direct consequence of the fact that FDKP can be written under the form

$$\partial_t u + \partial_x(\delta\mathfrak{H}_\epsilon(u)) = 0, \tag{28}$$

where $\delta\mathfrak{H}_\mu(u)$ denotes the variational derivative of $\mathfrak{H}_\mu(u)$.

Observe that unlike the Cauchy problem, the Hamiltonian for the FDKP equation requires a constraint to be well defined. This constraint however is weaker than for the classical KP equations (In the sense that the order of vanishing of the Fourier transform at the frequency $\xi_1 = 0$ is weaker than the corresponding one for the KP equations).

Namely the "energy space" associated to the FDKP equation (without surface tension) is

$$E = \{u \in L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2), \quad |D^2|^{1/4}|D_1|^{-1/2}u, \quad |D_2|^{1/2}|D_1|^{-1/2}u \in L^2(\mathbb{R}^2)\}.$$

Again, one finds the standard KP Hamiltonian by expanding formally $H_\epsilon(D)$ in powers of ϵ, namely

$$\mathfrak{H}_\epsilon(u) = \mathfrak{H}_{KP}(u) + O(\epsilon^2)$$

with

$$\mathfrak{H}_{KP}(u) = \frac{1}{2}\int_{\mathbb{R}^2} u^2 + \frac{\epsilon}{4}\int_{\mathbb{R}^2} [|\partial_y \partial_x^{-1} u|^2 - \frac{1}{3}|\partial_x u|^2 + u^3] dxdy.$$

Replacing $\mathfrak{H}_\epsilon(u)$ by $\mathfrak{H}_{KP}(u)$ in (28), the resulting equation is the KP II equation

$$\partial_t u + \partial_x u + \frac{\epsilon}{2}\partial_x^{-1}\partial_y^2 u + \frac{\epsilon}{6}\partial_x^3 u + \epsilon\frac{3}{2} u u_x = 0. \tag{29}$$

- With surface tension, the Hamiltonian $\widetilde{\mathfrak{H}}_\epsilon(u)$ is found by replacing $H_\epsilon(D)$ in the Hamiltonian $\mathfrak{H}_\epsilon(u)$ by

$$\widetilde{H}_\epsilon(D) = \left(\frac{(1+\beta\epsilon|D^\epsilon|^2)\tanh(\sqrt{\epsilon}|D^\epsilon|)}{\sqrt{\epsilon}|D_\epsilon|}\right)^{1/4}\left(1+\epsilon\frac{D_2^2}{D_1^2}\right)^{1/4}$$
$$= \left(\frac{(1+\beta\epsilon|D^\epsilon|^2)\tanh(\sqrt{\epsilon}|D^\epsilon|)}{\sqrt{\epsilon}}\right)^{1/4}\frac{|D^\epsilon|^{1/4}}{|D_1|^{1/2}}.$$

The corresponding energy space is

$$\tilde{E} = \{u \in L^2(\mathbb{R}^2)\cap L^3(\mathbb{R}^2),$$
$$|D_1|^{1/4}u, \quad |D_2|^{3/4}|D_1|^{-1/2}u, \quad |D_2|^{1/2}|D_1|^{-1/2} \in L^2(\mathbb{R}^2)\}$$

and the KP I (if $\beta > 1/3$) or KP II (if $\beta < 1/3$) Hamiltonian is found by a formal expansion with respect to ϵ,

$$\widetilde{\mathfrak{H}}_{KP}(u) = \frac{1}{2}\int_{\mathbb{R}^d} u^2 + \frac{\epsilon}{4}\int_{\mathbb{R}^2}[|\partial_y\partial_x^{-1}u|^2 + (\beta-\frac{1}{3})|\partial_x u|^2 + u^3]dxdy.$$

If u_0 satisfies an appropriate constraint, $u(\cdot, t)$ satisfies the constraint on $[0, T]$ and the Hamiltonian is conserved. More precisely,

Theorem

Assume that $s > 2$ and $u_0 \in H^s(\mathbb{R}^2) \cap E$ (resp. $u_0 \in H^s(\mathbb{R}^2) \cap \tilde{E}$). Then the solution u in Proposition 11 remains in E (resp. $\tilde{E}$) on $[0, T]$ and the Hamiltonian is conserved on $[0, T]$.

- In order to prove the conservation of the Hamiltonian we introduce $Y^s = \{f \in H^s(\mathbb{R}^2) \cap E; |D^\epsilon||D_1|^{-1} f \in L^2(\mathbb{R}^2)\}$. An argument using the Duhamel formula proves that Y^s is invariant by the FDKP flow and we conclude by proving that Y^s is dense in $H^s(\mathbb{R}^2) \cap E$.

Solitary waves.

- KP II has no localized solitary waves, contrary to KP I. (A. de Bouard-JCS 1997).
- Similar issues are unknown for the FDKP equations.
- It is unlikely that solitary waves exist for the FDKP equation without or with weak surface tension ($\beta < \frac{1}{3}$).
- When $\beta > \frac{1}{3}$, localized solitary waves are likely exist approximating the KP-I ground state in the long wave limit ($\sqrt{\epsilon}|\xi| \to 0.$).

Numerical simulations (D. Lannes-JCS 2013) suggest :

- Confirmation of shock formation for the Whitham equation. The blow-up time tends to infinity as $\epsilon \to 0$.
- Formation of a "lump like" solution for FDKP with large surface tension when $\epsilon \ll 1$.

Ongoing work (Linares-Pilod-S).

- Extend Ernström *et al* to the FDKP equation with strong surface tension :

$$\partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \qquad (30)$$

with

$$\tilde{c}_{WW}(\sqrt{\mu}k) = (1+\beta\mu k^2)^{\frac{1}{2}}\left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2},$$

- When $\beta > \frac{1}{3}$, the long wave limit of (30) is the usual KP I equation which admits localized (ground states) solitary waves (de Bouard-S 1997) or explicit lump solution, (Manakov et al 1977).