Propagation of regularity of solutions to the k-generalized Korteweg-de Vries equation

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IMPA

Recent Trends in Nonlinear Evolution Equations

CIRM, Luminy, April 5, 2016

In this talk we will discuss special regularity properties of solutions to the IVP associated to the k-generalized KdV equations.

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

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Outline

- Motivation
- Propagation of Regularity KdV equation
  - Idea of the Proofs
  - Remarks
- Extensions
- A different kind of Propagation

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Motivation

Linear Problem

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \quad x,\ t \in \mathbb{R}, \\ v(x,0) = v_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{2}$$

The solution of (2) is given by the unitary group $\{V(t)\}_{-\infty}^{\infty}$ defined via the Fourier transform as

$$v(x,t) = V(t)v_0(x) = \left(e^{it\xi^3}\widehat{v}_0\right)^{\vee}(x).$$

and satisfies

$$\|V(t)v_0\|_{H^s(\mathbb{R})} = \|v_0\|_{H^s(\mathbb{R})}.$$

Thus, if $v_0 \notin H^{s'}(\mathbb{R})$, $s' > s$, then for all $t \in \mathbb{R}$, $v(\cdot,t) \notin H^{s'}(\mathbb{R})$.

Nonlinear Problem

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad x,\ t \in \mathbb{R} \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{3}$$

If $s > s_0 (= -\frac{3}{4})$, there exists a unique solution v of the IVP (3)

$$u \in C([-T,T] : H^s(\mathbb{R})) \cap \ldots$$

with $u_0 \mapsto u(\cdot,t)$ continuous (smooth) for any $T > 0$.
Notice that if

$$u_0 \in H^{s'}(\mathbb{R}) \text{ with } u_0 \notin H^{s''}(\mathbb{R}), \quad \text{for } s' < s''$$

$$\Longrightarrow$$

$$u(\cdot,t) \in H^{s'}(\mathbb{R}) \text{ but } u(\cdot,t) \notin H^{s''}(\mathbb{R}).$$

Local smoothing

The solution of the linear problem can be written explicitly as

$$v(x,t) = V(t)v_0 = S_t * v_0(x) \tag{4}$$

where

$$S_t(x) = \frac{1}{\sqrt[3]{t}} A_i\Big(\frac{x}{\sqrt[3]{t}}\Big) \tag{5}$$

and A_i denotes the Airy function

$$Ai(x) = c \int_{-\infty}^{\infty} e^{ix\xi + i\xi^3/3}\, d\xi,$$

which satisfies the estimate

$$|Ai(x)| \le c \frac{e^{-cx_+^{3/2}}}{(1+x_-)^{1/4}}, \quad x_- = \max\{-x, 0\}.$$

Strichartz estimates

Solutions of the linear problem (2) satisfy

$$\|D^{\theta\alpha/2}V(t)v_0\|_{L^q_t L^p_x} \leq c\|v_0\|_{L^2} \tag{6}$$

where $(q,p) = (\frac{6}{\theta(\alpha+1)}, \frac{2}{1-\theta})$, $\theta \in [0,1]$ and $\alpha \in [0,1/2]$.

In particular, we have

$$\|D^{1/4}V(t)v_0\|_{L^4_t L^\infty_x} \leq c\|v_0\|_{L^2},$$

and

$$\|V(t)v_0\|_{L^8_t L^8_x} \leq c\|v_0\|_{L^2}.$$

The estimate (6) was established by Kenig, Ponce and Vega.

Kato smoothing effect

Suppose $\chi \in C^\infty(\mathbb{R})$, increasing, positive with $\chi' \in C_0^\infty(\mathbb{R})$, $\chi' \geq 0$. After multiplying the equation in (3) by $u\chi$ and integration by parts one gets

$$\frac{d}{dt}\int u^2\chi + \frac{3}{2}\int (\partial_x u)^2\chi' - \frac{1}{2}\int u^2\chi''' + \frac{1}{3}\int u^3\chi' = 0. \tag{7}$$

If $u_0 \in L^2(\mathbb{R})$, then the solution u of IVP (3) satisfies

$$u \in C([-T,T] : L^2(\mathbb{R})) \cap \ldots \quad \text{and} \quad \partial_x u \in L^2([-T,T] \times [-R,R]).$$

This result was extended by Constantin-Saut, Sjölin, Vega.

Kenig, Ponce and Vega proved that solutions of the linear problem (2) satisfy

$$\int_{-\infty}^{\infty} |\partial_x V(t) u_0(x)|^2 \, dt \equiv \int_{-\infty}^{\infty} |u_0(y)|^2 \, dy, \qquad \forall x \in \mathbb{R}.$$

(Homogeneous Smoothing Effect)

On the other hand, if we consider the inhomogeneous linear problem, the solution satisfies

$$\|\partial_x^2 \int_0^t V(t-t') g(\cdot, t') \, dt' \|_{L_x^\infty L_T^2} \leq c \|g\|_{L_x^1 L_T^2}.$$

(Inhomogeneous Smoothing Effect)

Suppose now that we have $\chi = \chi(x,t)$

$$\frac{d}{dt}\int u^2\chi - \underbrace{\int u^2\partial_t\chi}_{A} + \frac{3}{2}\int(\partial_x u)^2\partial_x\chi - \underbrace{\frac{1}{2}\int u^2\partial_x^3\chi}_{B} = 0$$

$$\int u^2\left(\partial_t\chi + \partial_x^3\chi\right)dx = 0$$

$\chi(x,t) = e^{a(t)x_+^\alpha}$

$$\begin{aligned}\partial_t\chi &= a'(t)x_+^\alpha\chi \\ \partial_x^3\chi &\simeq \alpha(\alpha-1)(\alpha-2)x_+^{3(\alpha-1)}(a(t))^3\chi\end{aligned}$$

$$3(\alpha-1) = \alpha \iff \alpha = 3/2 \text{ and } a(t) = \frac{a_0}{(1+27a_0^2t/4)^{1/2}}.$$

Theorem 1 (Isaza-L-Ponce (2014)). *Let* a_0 *be a positive constant. For any given data*

$$u_0 \in L^2(\mathbb{R}) \cap L^2(e^{a_0 x_+^{3/2}} dx),$$

the unique solution of the IVP (3) *satisfies that for any* $T > 0$

$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} e^{a(t)x_+^{3/2}} |u(x,t)|^2 dx \leq C^*$$

$C^* = C^*(a_0, \|u_0\|_2, \|e^{a_0 x_+^{3/2}/2} u_0\|_2, T)$, *with*

$$a(t) = \frac{a_0}{(1 + 27a_0^2 t/4)^{1/2}}.$$

We observe that this is sharp in the sense of following result by Escuriaza, Kenig, Ponce and Vega:

Theorem A (EKPV (2006)). *There exists* $c_0 > 0$ *such that if a solution*

$$u \in C([0,1] : H^4(\mathbb{R}) \cap L^2(|x|^2 dx))$$

of the IVP (3)*, satisfies*

$$u(\cdot,0),\;\; u(\cdot,1) \in L^2(e^{c_0 x_+^{3/2}} dx),$$

then $u \equiv 0$.

Above we used the notation: $x_+ = \max\{x; 0\}$.

Propagation of Regularity

Let us assume that we have a datum $u_0 \in H^{3/4^+}(\mathbb{R})$ whose restriction belongs to $H^l((b,\infty))$ for some $l \in \mathbb{Z}^+$ and $b \in \mathbb{R}$ we shall prove that the restriction of the corresponding solution $u(\cdot,t)$ belongs to $H^l((\beta,\infty))$ for any $\beta \in \mathbb{R}$ and any $t \in (0,T)$.

We start defining the class of solutions to the IVP (3) for which our results apply. We shall rely on the following well-posedness result:

Theorem B (Kenig-Ponce-Vega)**.** *If* $u_0 \in H^{3/4^+}(\mathbb{R})$*, then there exist* $T = T(\|u_0\|_{3/4^+,2}; k) > 0$ *and a unique solution of the IVP* (3) *such that*

$$\begin{aligned} &\text{(i)} && u \in C([-T,T] : H^{3/4^+}(\mathbb{R})),\\ &\text{(ii)} && \partial_x u \in L^4([-T,T] : L^\infty(\mathbb{R})), \quad (Strichartz),\\ &\text{(iii)} && \sup_x \int_{-T}^{T} |J^r \partial_x u(x,t)|^2\,dt < \infty \quad \textit{for} \quad r \in [0, 3/4^+],\\ &\text{(iv)} && \int_{-\infty}^{\infty} \sup_{-T\le t\le T} |u(x,t)|^2\,dx < \infty. \end{aligned} \tag{8}$$

Moreover, the map data-solution, $u_0 \to u(x,t)$ *is locally continuous (smooth) from* $H^{3/4+}(\mathbb{R})$ *into the class defined in* (8)*.*

Our first result is concerned with the propagation of regularity in the right hand side of the data for positive times.

Theorem 2 (Isaza-L-Ponce(2015)). *If* $u_0 \in H^{3/4^+}(\mathbb{R})$ *and for some* $l \in \mathbb{Z}^+,\ l \geq 1$ *and* $x_0 \in \mathbb{R}$

$$\| \partial_x^l u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2 dx < \infty,$$

then the solution of the IVP (3) *provided by Theorem B satisfies that for any* $v > 0$ *and* $\epsilon > 0$

$$\sup_{0\leq t\leq T} \int_{x_0+\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\, dx < c,$$

for $j = 0, 1, \dots, l$ *with* $c = c(l; \|u_0\|_{3/4^+,2}; \| \partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; T)$*.*

In particular, for all $t \in (0,T]$*, the restriction of* $u(\cdot,t)$ *to any interval* (x_0,∞) *belongs to* $H^l((x_0,\infty))$*.*
Moreover, for any $v \geq 0$*,* $\epsilon > 0$ *and* $R > 0$

$$\int_0^T \int_{x_0+\epsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\, dx dt < c,$$

with $c = c(l; \|u_0\|_{3/4^+,2}; \|\,\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; R; T)$*.*

Thus, this kind of regularity moves with infinite speed to its left as time evolves.

Remark 1. *It can be deduced from our proof of Theorem 2 that the inequality (2) can be more precise i.e. for $\delta > 0$ and $t \in (0,1)$ and* $j = 1, \dots, l$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\, dx \leq \frac{c}{t},$$

with

$$c = c(\|u_0\|_{3/4^+,2};\ \|\partial_x^j u_0\|_{L^2((x_0,\infty))};\ x_0;\ \delta).$$

(On question of K. Nakanishi)

Our second result describes the persistence properties and regularity effects, for positive times, in solutions associated with data having polynomial decay in the positive real line.

Theorem 3 (Isaza-L-Ponce (2015)). *If* $u_0 \in H^{3/4^+}(\mathbb{R})$ *and for some* $n \in \mathbb{Z}^+, \; n \geq 1$,

$$\| x^{n/2} u_0\|^2_{L^2((0,\infty))} = \int_0^\infty |x^n| \, |u_0(x)|^2 dx < \infty, \tag{9}$$

then the solution u *of the IVP* (3) *provided by Theorem B satisfies that*

$$\sup_{0 \leq t \leq T} \int_0^\infty |x^n| \, |u(x,t)|^2 \, dx \leq c \tag{10}$$

with $c = c(n; \|u_0\|_{3/4^+,2}; \| x^{n/2} u_0\|_{L^2((0,\infty))}; T)$.

Moreover, for any $\epsilon, \delta, R > 0, v \geq 0$, $m,\ j \in \mathbb{Z}^+$, $m + j \leq n$, $m \geq 1$,

$$\sup_{\delta \leq t \leq T} \int_{\epsilon - vt}^{\infty} (\partial_x^m u)^2(x,t)\, x_+^j\, dx + \int_\delta^T \int_{\epsilon - vt}^{R - vt} (\partial_x^{m+1} u)^2(x,t)\, x_+^{j-1}\, dxdt \leq c,$$

with $c = c(n; \|u_0\|_{3/4^+,2}; \|\, x^{n/2} u_0\|_{L^2((0,\infty))}; T; \delta; \epsilon; R; v)$.

As a direct consequence of Theorem 2 and Theorem 3, the above comments and the time reversible character of the equation in (3) one has:

Corollary 1. *Let* $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ *be a solution of the equation in* (3) *described in Theorem B. If there exist* $m \in \mathbb{Z}^+$, $\hat{t} \in (-T,T)$, $a \in \mathbb{R}$ *such that*

$$\partial_x^m u(\cdot, \hat{t}\,) \notin L^2((a,\infty)),$$

then for any $t \in [-T, \hat{t})$ *and any* $\beta \in \mathbb{R}$

$$\partial_x^m u(\cdot, t) \notin L^2((\beta,\infty)), \quad \textit{and} \quad x^{m/2}\, u(\cdot, t) \notin L^2((0,\infty)).$$

As a consequence of Theorem 2 and Theorem 3 one has that for an appropriate class of data the singularity of the solution travels with infinite speed to the left as time evolves. In the integrable cases $k = 1, 2$ this is expected as part of the so called resolution conjecture.

Consider the class $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx)$ $r, s > 0$. Isaza-L-Ponce (2014) showed that the solution flow associated to the KdV equation preserves this class if and only if $s \geq 2r$.

Corollary 2. *Given* $u_0 \in H^s(\mathbb{R})$, $s > 3/4^+$. *If the corresponding solution of the KdV equation satisfies that for some* $m > 0$

$$u(\cdot, t_1) \in L^2(|x_+|^m \, dx)$$

and

$$u(\cdot, t_2) \in L^2(|x_-|^m \, dx)$$

with $t_1 < t_2$, *then*

$$u \in C(\mathbb{R} : H^{2m}(\mathbb{R})).$$

Idea of the Proofs

We construct a class of real functions $\chi_{0,\epsilon,b}(x)$ for $\epsilon > 0$ and $b \geq 5\epsilon$ such that

$$\chi_{0,\epsilon,b} \in C^\infty(\mathbb{R}),\ \ \chi'_{0,\epsilon,b} \geq 0,$$

$$\chi_{0,\epsilon,b}(x) = \begin{cases} 0, & x \leq \epsilon, \\ 1, & x \geq b, \end{cases}$$

with

$$\operatorname{supp} \chi_{0,\epsilon,b} \subseteq [\epsilon, \infty), \quad \operatorname{supp} \chi'_{0,\epsilon,b}(x) \subseteq [\epsilon, b],$$

and

$$\chi'_{0,\epsilon,b}(x) \geq (b-3\epsilon)^{-1} 1_{[3\epsilon, b-2\epsilon]}(x),$$

Thus

$$\chi'_{0,\epsilon/3,b+\epsilon}(x) \geq c_j\, |\chi^{(j)}_{0,\epsilon,b}(x)|,\ \ \forall x \in \mathbb{R},\ \ \forall j \geq 1.$$

We shall use an induction argument. First, we shall prove (2) for $l = 1$ to illustrate our method.

Formally, take partial derivative with respect to x of the equation in (3) and multiply by $\partial_x u \chi_{0,\epsilon,b}(x+vt)$ to obtain after integration by parts the identity

$$\frac{1}{2}\frac{d}{dt}\int (\partial_x u)^2(x,t)\chi_0(x+vt)\,dx - \underbrace{v\int (\partial_x u)^2(x,t)\chi_0'(x+vt)\,dx}_{A_1}$$
$$+\frac{3}{2}\int (\partial_x^2 u)^2(x,t)\chi_0'(x+vt)\,dx - \underbrace{\frac{1}{2}\int (\partial_x u)^2(x,t)\chi_0'''(x+vt)\,dx}_{A_2}$$
$$+\underbrace{\int \partial_x(u\partial_x u)\partial_x u(x,t)\chi_0(x+vt)\,dx}_{A_3} = 0.$$

Remarks

Consider the IVP for the mKdV

$$\begin{cases} \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{11}$$

which is also an integrable system.

We will see that the statement the singularity of the solution travels to the left is not a precise one.

We recall a result that can be obtained as a consequence of the argument given by Bona and Saut.

Theorem 4. *There exists*

$$u_0 \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

so that the solution $u(\cdot, t)$ *of the IVP* (11) $u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap \ldots$ *satisfies*

$$\begin{cases} u(\cdot, t) \in C^1(\mathbb{R}), \;\; t > 0, \quad t \notin \mathbb{Z}^+, \\ u(\cdot, t) \in C^1(\mathbb{R}\backslash\{0\})\backslash C^1(\mathbb{R}), \quad t \in \mathbb{Z}^+. \end{cases} \tag{12}$$

The argument of Bona-Saut is based on the asymptotic decay of the Airy function and the well-posedness of the IVP (11) with data $u_0(x)$ in appropriate weighted Sobolev spaces.

This argument was simplified (for the case of two points in (12)) for the modified KdV equation by L-Scialom without relying in weighted spaces. A direct proof of Theorem 4 can be given following the approach used by L-Scialom.

Our method can be extended to $W^{s,p}$-setting. Indeed,

Theorem 5 (L-Ponce-Smith (?)). *Let $p \in (2, \infty)$ and $j \geq 1$, $j \in \mathbb{Z}^+$. There exists*

$$u_0 \in H^{3/4}(\mathbb{R}) \cap W^{j,p}(\mathbb{R}) \tag{13}$$

such that the corresponding solution

$$u \in C([-T, T] : H^{3/4}(\mathbb{R})) \cap \dots$$

of (3) satisfies that there exists $t \in [0, T]$ such that

$$u(\cdot, \pm t) \notin W^{j,p}(\mathbb{R}^+). \tag{14}$$

Remark. *It will follow from our proof that there exists u_0 as in (13) such that (14) holds in $\mathbb{R}^-$. Hence, the regularities in $W^{j,p}(\mathbb{R})$ for $p > 2$ do not propagate forward or backward in time to the right or to the left.*

Extensions

Results regarding propagation of regularity (similar to Theorem 2) have been extended for solutions of the IVP associated to

- the Benjamin-Ono equation

$$\partial_t v - \mathcal{H}\partial_x^2 v + v\partial_x v = 0 \tag{15}$$

  where $\mathcal{H}$ denotes the Hilbert transform.

- the (Kadomtsev-Petviashvilli) KP II equation

$$\partial_t w + \partial_x^3 w + \partial_x^{-1}\partial_y^2 w + w\partial_x w = 0 \tag{16}$$

  where

$$\widehat{\partial_x^{-1} f}(\xi) = -i\xi^{-1}\,\widehat{f}(\xi).$$

Hence, it is natural to ask if this propagation of regularity phenomenon is intrinsically related to the integrable character of the model.

Indeed, for the k-generalized dispersive BO equation,

$$\partial_t u + u^k \partial_x u - (-\partial_x^2)^{\alpha/2} \partial_x u = 0, \;\; k \in \mathbb{Z}^+, \;\; 1 \leq \alpha \leq 2,$$

which for $\alpha = 1$ corresponds to the k-generalized BO equation and $\alpha = 2$ to the k-generalized KdV equation, the propagation of regularities (as that presented in Theorem 2) is only known in the cases $\alpha = 1$ and $\alpha = 2$.

This fact seems to be more general. In particular, it is valid for solutions of the general quasilinear equation KdV type, that is,

$$\begin{cases} \partial_t u + a(u, \partial_x u, \partial_x^2 u)\, \partial_x^3 u + b(u, \partial_x u, \partial_x^2 u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{17}$$

where the functions $a, b : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ satisfy:

(H1) $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are C^∞ with all derivatives bounded in $[-M,M]^3$, for any $M > 0$,

(H2) given $M > 0$, there exists $\kappa > 0$ such that

$$1/\kappa \le a(x,y,z) \le \kappa \text{ for any } (x,y,z) \in [-M,M]^3,$$

and

$$\partial_z\, b(x,y,z) \le 0 \text{ for } (x,y,z) \in [-M,M]^3.$$

To establish the propagation of regularity in this case we shall follow the arguments and results obtained by Craig, Kappeler and Strauss. Under the hypotheses (H1) and (H2), they showed

Theorem C (CKS). *Let* $m \in \mathbb{Z}^+$, $m \geq 7$. *For any* $u_0 \in H^m(\mathbb{R})$, *there exist* $T = T(\|u_0\|_{7,2}) > 0$ *and a unique solution* $u = u(x,t)$ *of the IVP* (17) *satisfying,*

$$u \in L^\infty([0,T]; H^m(\mathbb{R})).$$

Moreover, for any $R > 0$

$$\int_0^T \int_{-R}^R (\partial_x^{m+1} u)^2(x,t)\, dxdt < \infty.$$

We need some (weak) continuous dependence of the solutions upon the data. Hence, we prove the following “refinement” of Theorem C.

Theorem 6 (L-Ponce-Smith (?)). *Let* $m \in \mathbb{Z}^+$, $m \geq 7$. *For any* $u_0 \in H^m(\mathbb{R})$ *there exist* $T = T(\|u_0\|_{7,2}) > 0$ *and a unique solution* $u = u(x,t)$ *of the IVP* (17) *such that*

$$u \in C([0,T] : H^{m-\delta}(\mathbb{R})) \cap L^\infty([0,T] : H^m(\mathbb{R})), \text{ for all } \delta > 0, \quad (18)$$

with

$$\partial_x^{m+1} u \in L^2([0,T] \times [-R,R]), \text{ for all } R > 0. \quad (19)$$

Moreover, the map data solution $u_0 \mapsto u(\cdot,t)$ *is locally continuous from* $H^m(\mathbb{R})$ *into* $C([0,T] : H^{m-\delta}(\mathbb{R}))$ *for any* $\delta > 0$.

Theorem 7 (L-Ponce-Smith(?)). *Let* $n, m \in \mathbb{Z}^+$*,* $n > m \geq 7$*. If* $u_0 \in H^m(\mathbb{R})$ *and for some* $x_0 \in \mathbb{R}$

$$\partial_x^j u_0 \in L^2((x_0, \infty)) \text{ for } j = m+1, \dots, n.$$

Then the solution of the IVP (17) *provided by Theorem* 6 *satisfies that for any* $\epsilon > 0$*,* $v > 0$*, and* $t \in [0, T)$

$$\int_{x_0+\epsilon-vt}^{\infty} |\partial_x^j u(x,t)|^2 \, dx \leq c(\epsilon; v; \|u_0\|_{m,2}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))} : l = m+1, \dots, n), \tag{20}$$

for $j = m+1, \dots, n$*.*

Theorem 7 tells us that the propagation phenomenon described in Theorem 2 still holds in solutions of the quasilinear problem (17).

This result and those in KdV, BO, KPII equations seem to indicate that the propagation of regularity phenomena can be established in systems where Kato smoothing effect can be proved by integration by parts directly in the differential equation.

A different kind of propagation of regularity

Next we consider the propagation of regularities in solutions to some related dispersive models.

We choose the IVP associated to the Benjamin-Bona-Mahony (BBM) equation

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u - \partial_x^2\partial_t u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{21}$$

We recall the local well-posedness for the IVP (21) obtained by Bona and Tzvetkov

Theorem D. *Let* $s \geq 0$. *For any* $u_0 \in H^s(\mathbb{R})$ *there exist* $T = T(\|u_0\|_{s,2}) > 0$ *and a unique solution* u *of the IVP* (21)

$$u \in C([0,T] : H^s(\mathbb{R})) \equiv X_T^s.$$

Moreover, the map data-solution $u_0 \mapsto u(\cdot, t)$ *is locally continuous from* $H^s(\mathbb{R})$ *into* X_T^s.

For the IVP (21) we prove that

Theorem 8 (L-Ponce-Smith (?)). *Let* $u_0 \in H^s(\mathbb{R})$, $s \geq 0$. *If for some* $k \in \mathbb{Z}^+ \cup \{0\}$, $\theta \in [0,1)$, *and* $\Omega \subseteq \mathbb{R}$ *open*

$$u_0\big|_{\Omega} \in C^{k+\theta},$$

then the corresponding solution $u \in X_T^s$ *of the IVP* (21) *provided by Theorem D satisfies that*

$$u(\cdot,t)\big|_{\Omega} \in C^{k+\theta} \text{ for all } t \in [0,T].$$

Moreover,

$$u, \partial_t u \in C([0,T] : C^{k+\theta}(\Omega)).$$

Remarks.

- Theorem 8 tells us that in the time interval $[0, T]$ in the $C^{k+\theta}$ setting no singularities can appear or disappear in the solution $u(\cdot, t)$.

  In particular, one has the following consequence of Theorem 8 and its proof:

  Corollary 3. *Let* $u_0 \in H^s(\mathbb{R})$, $s \geq 0$. *If for* $a < x_0 < b$, $k \in \mathbb{Z}^+ \cup \{0\}$ *and* $\theta \in [0, 1)$

$$u_0\big|_{(a,x_0)},\ \ u_0\big|_{(x_0,b)} \in C^{k+\theta} \text{ and } u_0\big|_{(a,b)} \notin C^{k+\theta},$$

  then the corresponding solution $u \in X_T^s$ *of the IVP* (21) *provided by Theorem D satisfies*

$$u(\cdot, t)\big|_{(a,x_0)},\ \ u(\cdot, t)\big|_{(x_0,b)} \in C^{k+\theta} \text{ and } u(\cdot, t)\big|_{(a,b)} \notin C^{k+\theta}.$$

- Theorem 2, Theorem 4, Theorem 8, and Corollary 3 show that solutions of the BBM equation and the KdV equation exhibit a quite different behavior regarding the propagation of regularities.

Further results

We also proved similar type of results for the Degasperis-Procesi equation

$$\partial_t u - \partial_x^2 \partial_t u + 4u\partial_x u = 3\partial_x u \partial_x^2 u + u\partial_x^3 u,\ x \in \mathbb{R},\ t > 0,$$

and the 1D Brinkman model

$$\partial_t \rho = \partial_x \big(\rho (1 - \partial_x^2)^{-1} \partial_x (\rho^2)\big), \quad x \in \mathbb{R},\ t > 0.$$

Thanks for your attention