# Transition Fronts for Monostable Reaction-Diffusion Equations

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**Time-dependent reaction-diffusion equation**

$$u_t = u_{xx} + f(t,u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}$$

with monostable Fisher-KPP type reaction

$$\begin{cases} f(t,0) = f(t,1) = 0, \quad f(t,u) \geq 0 \text{ in } \mathbb{R} \times [0,1] \\ \dfrac{f(t,u)}{u} \text{ is nonincreasing with respect to } u \in (0,1] \end{cases}$$

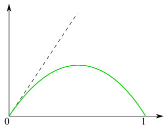



Figure : Function $f(t,\cdot)$

and there are two functions $f_\pm : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} f_\pm(0) = f_\pm(1) = 0, \quad f_\pm(u) > 0 \text{ in } (0,1) \\ \dfrac{f(t,u)}{f_\pm(u)} \underset{t \to \pm\infty}{\longrightarrow} 1 \text{ uniformly for } u \in (0,1) \end{cases}$$

- Typical case:
$$f(t,u)=\widetilde{\mu}(t)\,\widetilde{f}(u)$$
where $\widetilde{f}(0)=\widetilde{f}(1)=0$, $\widetilde{f}>0$ on $(0,1)$,

$\widetilde{f}(u)/u$ is nonincreasing with respect to $u\in(0,1]$

and $\widetilde{\mu}(t)\to\widetilde{\mu}_\pm>0$ as $t\to\pm\infty$.

In this case, $f_\pm(u)=\widetilde{\mu}_\pm\widetilde{f}(u)$.

- Remark: $f(t,u)>0$ for $(t,u)\in\mathbb{R}\times(0,1)$ with large $|t|$.

But the case $f(t,\cdot)=0$ for some $t$ in a compact set is not excluded.

**Propagating solutions connecting the unstable steady state $0$ and the stable steady state $1$ ?**

## Standard traveling fronts when $f = f(u)$ does not depend on $t$

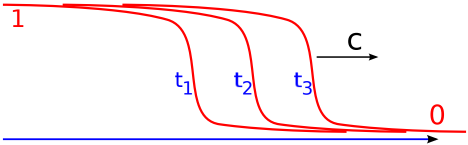
Homogeneous equation

$$u_t = u_{xx} + f(u)$$

Traveling fronts

$$u(t,x) = \varphi_c(x - ct)$$

with $\varphi_c : \mathbb{R} \to (0,1)$, $\varphi_c(-\infty) = 1$, $\varphi_c(+\infty) = 0$




Set of admissible speeds $\{c\} = [c^*, +\infty)$ with $c^* = 2\sqrt{f'(0)}$

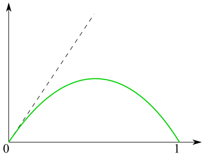
For each speed $c \geq c^*$, $\varphi_c$ is decreasing and unique up to shifts

Stability of the fronts

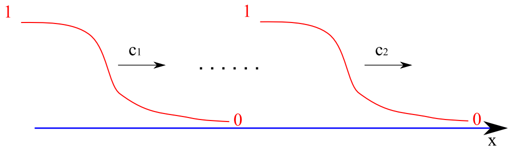
[Aronson, Weinberger] [Bramson] [Fisher] [Hamel, Nolen, Roquejoffre, Ryzhik] [Hamel, Roques] [Kamataka] [Kolmogorov, Petrovski, Piskunov] [Lau] [Sattinger] [Uchiyama]

## Other propagating fronts even in the homogeneous case $f = f(u)$

$$u_t = u_{xx} + f(u) \quad \text{with concave function } f$$




(a) Function $f$




(b) Fronts with changing speed

For any $c_2 > c_1 \geq 2\sqrt{f'(0)}$, there are some solutions $u(t,x)$ such that

$$\begin{cases} u(t,x) - \varphi_{c_1}(x - c_1 t) \to 0 & \text{as } t \to -\infty \\ u(t,x) - \varphi_{c_2}(x - c_2 t) \to 0 & \text{as } t \to +\infty \end{cases} \quad \text{uniformly in } x \in \mathbb{R}$$

[Hamel, Nadirashvili]

## More general front-like solutions (for $u_t = u_{xx} + f(t, u)$)

**Definition [Berestycki, Hamel] (adapted to our equation)**

A *transition front* connecting $1$ and $0$ is a solution $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ for which there exists a family $(x_t)_{t\in\mathbb{R}}$ of real numbers such that

$$\begin{cases} u(t, x + x_t) \to 1 & \text{as } x \to -\infty \\ u(t, x + x_t) \to 0 & \text{as } x \to +\infty \end{cases} \quad \text{uniformly in } t \in \mathbb{R}$$

(the transition between $0$ and $1$ has a uniformly bounded width)

For a given transition front $u$, the real numbers $x_t$ are defined up to a bounded function and they are at a finite distance from any given level set: for every $0 < \alpha \leq \beta < 1$, there is $C = C(u, \alpha, \beta)$ such that

$$\forall\, t \in \mathbb{R}, \quad \{x \in \mathbb{R};\, \alpha \leq u(t, x) \leq \beta\} \subset [x_t - C, x_t + C]$$

Example: $u(t, x_t) = 1/2$

**Notion of global mean speed (for $u_t = u_{xx} + f(t,u)$)**

Definition [Berestycki, Hamel] (adapted to our equation)

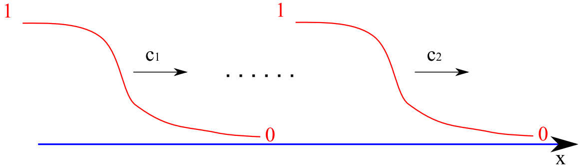
A transition front $u$ connecting $1$ and $0$ has a global mean speed $\gamma$ if

$$\frac{x_t - x_s}{t-s} \to \gamma \text{ as } t-s \to +\infty$$

If a given transition front $u$ has a global mean speed $\gamma$, then $\gamma$ is finite and does not depend on the precise choice of $(x_t)_{t\in\mathbb{R}}$

When $f = f(u)$, any standard traveling front $u(t,x) = \varphi_c(x-ct)$ is a transition front with global mean speed $c$

## Further notions of speeds




Transition fronts with $x_t = c_1 t$ for $t \leq 0$ and $x_t = c_2 t$ for $t \geq 0$

They are not standard traveling fronts

No global mean speed if $c_2 > c_1$

Need of further notions of asymptotic speeds to describe the solutions, even in the homogeneous case $f = f(u)$ !

**Definition of asymptotic speeds (for $u_t = u_{xx} + f(t,u)$)**

We say that a transition front connecting $1$ and $0$ has an *asymptotic past speed* $c_-$, respectively an *asymptotic future speed* $c_+$, if

$$\frac{x_t}{t} \to c_- \text{ as } t \to -\infty, \quad \text{respectively } \frac{x_t}{t} \to c_+ \text{ as } t \to +\infty$$

The asymptotic speeds, if any, are finite and do not depend on the precise choice of $(x_t)_{t\in\mathbb{R}}$

If a transition front has a global mean speed $\gamma$, then it has asymptotic past and future speeds $c_\pm = \gamma$

**Questions:**

- Set of transition fronts connecting $1$ and $0$ ?
- Conditions for a solution $0 < u(t,x) < 1$ to be a transition front ?
- Existence of asymptotic past and future speeds ?
- Set of admissible past and future speeds ?
- Asymptotic profiles as $t \to \pm\infty$ ?

**Other assumptions: $T$-periodic reaction $f(t,u)$, pulsating fronts**

$$\begin{cases} u(t,x) = \varphi(t, x-ct) \\ \varphi(t,\xi) \text{ is } T\text{-periodic in } t \\ \varphi(t,-\infty)=1,\ \varphi(t,+\infty)=0 \end{cases}$$

The profile $t \mapsto u(t, ct+\cdot)$ is $T$-periodic in time

Existence of a continuum of speeds $[c^*, +\infty)$ [Fréjacques] [Hamel, Roques] [Liang, Zhao] [Nadin] [Nolen, Rudd, Xin] [Weinberger]

Time almost-periodic, uniquely ergodic media [Huang, Shen] [Nadin, Rossi] [Shen]

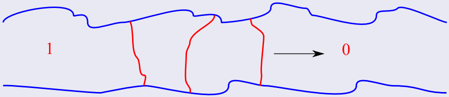
**Bistable reaction $f(t,u)$**

[Alikakos, Bates, Chen] [Contri] [Fang, Zhao] [Shen]

**Spatial dependence $u_t = u_{xx} + f(x,u)$: pulsating and transition fronts**

Berestycki, Ding, Ducrot, El Smaily, Fang, Giletti, Hamel, Heinze, Liang, Matano, Mellet, Nadin, Nadirashvili, Nolen, Roquejoffre, Roques, Ryzhik, Sire, Weinberger, Xin, Zhao, Zlatoš

## Another definition of generalised fronts, by H. Matano




Example: $u_t = u_{xx} + b(x)\,f(u)$

Define $\sigma_\xi b(\cdot) = b(\cdot + \xi)$ and assume that $\mathcal{H} = \overline{\{\sigma_\xi b\}}$ is compact in $L^\infty(\mathbb{R})$

A definition by H. Matano: $u$ is a generalised front if there exists a continuous mapping $w : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} u(t, x + \xi(t)) = w(\sigma_{\xi(t)} b, x) \\ w(z, s) \to 1 \text{ as } s \to -\infty \text{ uniformly w.r.t. } z \in \mathcal{H} \\ w(z, s) \to 0 \text{ as } s \to +\infty \text{ uniformly w.r.t. } z \in \mathcal{H} \end{cases}$$

For homogeneous equations, the profile of the solution is invariant in time

In the general case, a generalized front is a transition front connecting 1 and 0

Definition of wave-like solutions in random media, by W. Shen

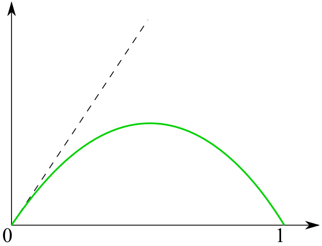
*I. Introduction: transition fronts and asymptotic speeds*

II. Homogeneous Fisher-KPP equation

III. Heterogeneous Fisher-KPP type equation

$$u_t = u_{xx} + f(u)$$

with concave function $f$




Standard traveling fronts $\varphi_c(x - ct)$ for $c \geq c^* = 2\sqrt{f'(0)}$

## Global mean speeds

**Theorem**

The set of admissible global mean speeds of transition fronts connecting $1$ and $0$ is equal to the interval $[c^*, +\infty)$.

Furthermore, if a transition front $u$ has a global mean speed $\gamma > c^*$, then it is a standard traveling front of the type $u(t,x) = \varphi_\gamma(x - \gamma t)$.

First part follows from [Aronson, Weinberger]

Second part follows from [Hamel, Nadirashvili] and further qualitative results (see later...)

## Asymptotic past and future speeds

**Theorem**

Transition fronts connecting 1 and 0 and having asymptotic past and future speeds $c_\pm$ exist if and only if

$$c^* \leq c_- \leq c_+ < +\infty$$

The sufficiency condition follows from the construction of [Hamel, Nadirashvili]

The necessity condition means that transition fronts always accelerate

## Asymptotic profiles

The necessity condition of the previous theorem is a consequence of a more general result:

**Theorem**

For any transition front connecting $1$ and $0$, there holds

$$c^* \le \liminf_{t\to-\infty} \frac{x_t}{t} \le \liminf_{t\to+\infty} \frac{x_t}{t} \le \limsup_{t\to+\infty} \frac{x_t}{t} < +\infty$$

Furthermore, if $c^* < \liminf_{t\to-\infty} x_t/t$, then the front has asymptotic past and future speeds $c_\pm$, with $c^* < c_- \le c_+$, and

$$u(t, x_t + \cdot) \to \varphi_{c_\pm} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$

**Conjecture**

Same conclusion without the condition $c^* < \liminf_{t\to-\infty} x_t/t$

## Sufficient condition for a solution $0 < u < 1$ to be a transition front

For any solution $0 < u(t,x) < 1$, one has $\max_{[-c|t|,c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for every $0 \le c < c^*$ [Aronson, Weinberger]

### Theorem

Let $0 < u(t,x) < 1$ be a solution such that

$$\exists\, c > c^*, \quad \max_{[-c|t|,c|t|]} u(t,\cdot) \to 0 \ \text{ as } t \to -\infty.$$

Then the limit

$$\lambda = -\lim_{x\to+\infty} \frac{\ln u(t,x)}{x}$$

exists independently of $t \in \mathbb{R}$ and satisfies $\lambda \in \big[0, \sqrt{f'(0)}\big)$.

Furthermore, $u$ is a transition front connecting $1$ and $0$ if and only if $\lambda > 0$.

Lastly, if $\lambda > 0$, then $u$ has asymptotic speeds $c_\pm$ given by

$$c^* < c_- = \sup\Big\{\gamma \ge 0,\ \lim_{t\to-\infty} \max_{[-\gamma|t|,\gamma|t|]} u(t,\cdot) = 0\Big\} \le c_+ = \lambda + \frac{f'(0)}{\lambda}$$

and it has asymptotic profiles $\varphi_{c_\pm}$.

**Transition fronts as superposition of standard traveling fronts**

Standard traveling fronts $\varphi_c(x-ct)$ and $\varphi_c(-x-ct)$ for $c \ge c^* = 2\sqrt{f'(0)}$

$$\left\{\begin{array}{ll} \varphi_c(\xi) \sim e^{-\lambda_c \xi} & \text{if } c > c^* \\ \varphi_{c^*}(\xi) \sim \xi\, e^{-\lambda_{c^*}\xi} & \end{array}\right. \quad \text{as } \xi \to +\infty \quad \text{(up to shift in } \xi)$$

with $\lambda_c = (c - \sqrt{c^2 - 4f'(0)})/2$ $(\lambda_{c^*} = c^*/2 = \sqrt{f'(0)})$

Spatially-uniform solution $\theta'(t) = f(\theta(t))$ s.t. $\theta(t) \sim e^{f'(0)t}$ as $t \to -\infty$

Set $\mathcal{X} = (-\infty, -c^*] \cup [c^*, +\infty) \cup \{\infty\}$

Set $\mathcal{M}$ of nonnegative Borel measures $\mu$ on $\mathcal{X}$ s.t. $0 < \mu(\mathcal{X}) < +\infty$

One-to-one map $\mu \mapsto u_\mu$ from $\mathcal{M}$ to the set of solutions $0 < u_\mu(t,x) < 1$ [Hamel, Nadirashvili]

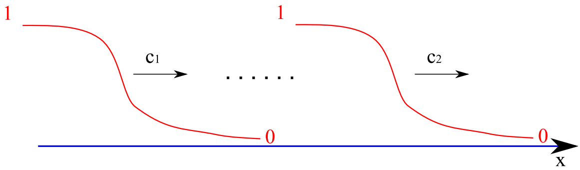
Furthermore, for a given $\mu \in \mathcal{M}$, calling $M = \mu(\mathcal{X}\backslash\{-c^*, c^*\})$,

$$\max\Big(\varphi_{c^*}\big(x - c^*t - c^*\ln\mu(c^*)\big),\ \varphi_{c^*}\big(-x - c^*t - c^*\ln\mu(-c^*)\big),$$
$$M^{-1}\int_{\mathbb{R}\backslash[-c^*,c^*]}\varphi_{|c|}\big((\mathrm{sgn}\, c)x - |c|t - |c|\ln M\big)\,d\mu(c)$$
$$+ M^{-1}\theta(t + \ln M)\,\mu(\infty)\Big)$$
$$\le\ u_\mu(t,x)\ \le\ \varphi_{c^*}\big(x - c^*t - c^*\ln\mu(c^*)\big) + \varphi_{c^*}\big(-x - c^*t - c^*\ln\mu(-c^*)\big)$$
$$+ M^{-1}\int_{\mathbb{R}\backslash[-c^*,c^*]} e^{-\lambda_{|c|}((\mathrm{sgn}\, c)x - |c|t - |c|\ln M)}\,d\mu(c)$$
$$+ M^{-1}e^{f'(0)(t + \ln M)}\mu(\infty)$$

For any solution $0 < u(t,x) < 1$, one has $\max_{[-c|t|,c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for every $0 \le c < c^*$ [Aronson, Weinberger]

If $\max_{[-c|t|,c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for some $c > c^*$, then $u = u_\mu$ [Hamel, Nadirashvili]

*Example*




Measure $\mu = m_1\delta_{c_1} + m_2\delta_{c_2}$ with $c^* \le c_1 < c_2$

$$\begin{cases} u(t,x) - \varphi_{c_1}(x - c_1 t) \to 0 & \text{as } t \to -\infty \\ u(t,x) - \varphi_{c_2}(x - c_2 t) \to 0 & \text{as } t \to +\infty \end{cases} \quad \text{uniformly in } x \in \mathbb{R}$$

**Theorem**

Let $u_\mu$ be the solution associated with a measure $\mu \in \mathcal{M}$.

Then $u_\mu$ is a transition front connecting $1$ and $0$ if and only if the support of $\mu$ is a compact subset of $[c^*, +\infty)$.

In such a case, $u_\mu$ is decreasing with respect to $x$.

- Right-moving fronts $\varphi_c(x - ct)$ are decreasing in $x$ and connect $1$ at $-\infty$ to $0$ at $+\infty$
- Left-moving fronts $\varphi_c(-x - ct)$ are increasing in $x$ and connect $0$ at $-\infty$ to $1$ at $+\infty$
- Faster fronts are flatter, so, for the transition zone between $1$ and $0$ to be uniformly bounded, it is reasonable to expect that the fronts $\varphi_c(x - ct)$ involved in $u_\mu$ have bounded speeds
- Sufficiency condition when $\mathrm{supp}(\mu)$ is a compact subset of $(c^*, +\infty)$, with other arguments, by Zlatoš.

## Theorem

Let $\mu \in \mathcal{M}$ be a measure such that

$$c^* \le c_- := \min\left(\mathrm{supp}(\mu)\right) \le \max\left(\mathrm{supp}(\mu)\right) =: c_+ < +\infty.$$

- The transition front $u_\mu$ has an asymptotic past speed equal to $c_-$ and an asymptotic future speed equal to $c_+$
- The positions $(x_t)_{t\in\mathbb{R}}$ satisfy

$$\begin{cases} \limsup_{t\to\pm\infty} |x_t - c_\pm t| < +\infty & \text{if } \mu(c_\pm) > 0 \\ \lim_{t\to\pm\infty} (x_t - c_\pm t) = -\infty & \text{if } \mu(c_\pm) = 0 \end{cases}$$

- If $c_- > c^*$, then

$$u_\mu(t, x_t + \cdot) \to \varphi_{c_\pm} \text{ in } C^2(\mathbb{R}) \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$

- If $c_- = c^*$ and $\mu(c^*) > 0$, then

$$u_\mu(t, c^* t + c^* \ln \mu(c^*) + \cdot) \to \varphi_{c^*} \text{ in } C^2(\mathbb{R}) \text{ as } t \to -\infty$$

## Corollary

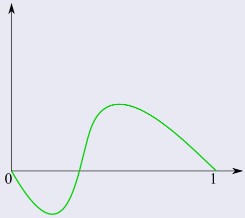
There are solutions $0 < u(t,x) < 1$ such that

$$\forall\, t \in \mathbb{R}, \quad u(t,-\infty) = 1, \quad u(t,+\infty) = 0$$

and which are not transition fronts connecting $1$ and $0$.

Proof: $u = u_\mu$ with $\mathrm{supp}(\mu) \subset [c^*, +\infty)$ and $\mathrm{supp}(\mu)$ unbounded.

*Homogeneous bistable equation* $u_t = u_{xx} + f(u)$




Unique standard front $\varphi(x - ct)$ [Aronson, Weinberger] [Fife, McLeod]

Any transition front connecting $1$ and $0$ is equal to this front, up to shifts [Hamel]

But there are solutions $0 < u(t, x) < 1$ such that

$$\forall\, t \in \mathbb{R}, \quad u(t, -\infty) = 1, \quad u(t, +\infty) = 0$$

and which *are not* transition fronts. These solutions are close to the unstable zero $\theta$ on large space intervals as $t \to -\infty$
[Morita, Ninomiya]

## Some ingredients for the proof of the main theorems

**Proposition**

Let $\mu \in \mathcal{M}$ be supported in $[c^*, \gamma]$ for some $\gamma \in [c^*, +\infty)$.

Then, for every $(t, x) \in \mathbb{R}^2$,

$$\begin{cases} u_\mu(t, x+y) \geq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t,x)) + y\big) & \text{for all } y \leq 0, \\ u_\mu(t, x+y) \leq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t,x)) + y\big) & \text{for all } y \geq 0, \end{cases}$$

where $\varphi_\gamma^{-1} : (0,1) \to \mathbb{R}$ denotes the reciprocal of the function $\varphi_\gamma$.

In other words, $u_\mu(t, \cdot)$ is steeper than $\varphi_\gamma$.

Consequence: $(u_\mu)_x(t,x) \leq \varphi_\gamma'\big(\varphi_\gamma^{-1}(u_\mu(t,x))\big) < 0$.

### Proposition

Let $\mu \in \mathcal{M}$ be supported in $[\gamma, +\infty)$ for some $\gamma \in [c^*, +\infty)$.

Then, for every $(t,x) \in \mathbb{R}^2$,

$$\begin{cases} u_\mu(t, x+y) \le \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t,x)) + y\big) & \text{for all } y \le 0 \\ u_\mu(t, x+y) \ge \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t,x)) + y\big) & \text{for all } y \ge 0 \end{cases}$$

In other words, $u_\mu(t,\cdot)$ is less steep than $\varphi_\gamma$.

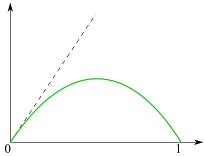
*I. Introduction: transition fronts and asymptotic speeds*

*II. Homogeneous Fisher-KPP equation*

III. Heterogeneous Fisher-KPP type equation

$$u_t = u_{xx} + f(t,u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}$$

$$\begin{cases} f(t,0) = f(t,1) = 0, \quad f(t,u) \geq 0 \text{ in } \mathbb{R} \times [0,1] \\ \dfrac{f(t,u)}{u} \text{ is nonincreasing with respect to } u \in (0,1] \end{cases}$$




and there are two functions $f_\pm : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} f_\pm(0) = f_\pm(1) = 0, \quad f_\pm(u) > 0 \text{ in } (0,1) \\ \dfrac{f(t,u)}{f_\pm(u)} \underset{t \to \pm\infty}{\longrightarrow} 1 \text{ uniformly for } u \in (0,1) \end{cases}$$

Notation: $\mu_\pm := f'_\pm(0) > 0, \quad \underline{\mu} := \min(\mu_-, \mu_+) > 0$

## Theorem (existence)

Let $c_\pm$ be any two real numbers such that

$$c_- \geq 2\sqrt{\mu_-} \quad \text{and} \quad c_+ \geq \kappa + \frac{\mu_+}{\kappa}$$

with

$$\kappa = \min\Big(\sqrt{\mu_+}, \frac{c_- - \sqrt{c_-^2 - 4\mu_-}}{2}\Big) > 0$$

Then there are some transition fronts $u$ connecting $1$ and $0$ with asymptotic past and future speeds $c_\pm$.

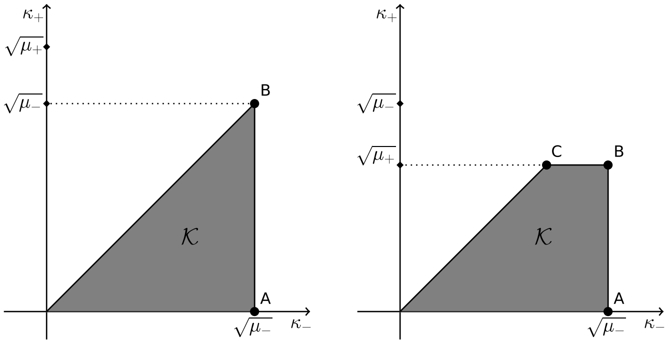
Furthermore, $u$ satisfies $u_x(t,x) < 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Lastly, in all cases, except possibly when $\mu_+ > \mu_-$ and $c_\pm$ satisfy $c_- = 2\sqrt{\mu_-}$ and $c_+ = \sqrt{\mu_-} + \mu_+/\sqrt{\mu_-}$, then

$$u(t, x_t + \cdot) \to \varphi^\pm_{c_\pm} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$, where $\varphi^\pm_{c_\pm}(x - c_\pm t)$ are standard traveling fronts connecting $1$ and $0$ for the limiting equations with nonlinearities $f_\pm$.

- $c_\pm \geq 2\sqrt{\mu_\pm}$
- Range of admissible past speeds $= [2\sqrt{\mu_-}, +\infty)$
- Range of admissible future speeds $= [2\sqrt{\mu_+}, +\infty)$ if $\mu_+ \leq \mu_-$
- Range of admissible future speeds $= \left[\sqrt{\mu_-} + \frac{\mu_+}{\sqrt{\mu_-}}, +\infty\right)$ if $\mu_+ > \mu_-$
- Equivalent formulation: $c_\pm = \kappa_\pm + \frac{\mu_\pm}{\kappa_\pm}$ with $\kappa_- \in (0, \sqrt{\mu_-}]$ and $\kappa_+ \in (0, \min(\kappa_-, \sqrt{\mu_+})]$




(c) Case $\mu_+ > \mu_-$ (d) Case $\mu_+ \leq \mu_-$

- If $\mu_+ > \mu_-$, then $c_+ > c_-$ (acceleration)
- If $\mu_+ \geq \mu_-$, then $c_+ \geq c_-$
- If $\mu_+ < \mu_-$, then $c_+$ may be less than $c_-$ (slow down)
- [Berestycki, Hamel] and [Nadin, Rossi]

  Other assumptions on $f(t,u)$

  Proof of the existence of fronts which would correspond to the case $\kappa_+ = \kappa_- \in (0, \sqrt{\underline{\mu}})$

## Theorem (optimality of the asymptotic speeds)

Assume that $f_-$ is concave and there is $\zeta \in L^1(-\infty, 0)$ such that

$$\sup_{s\in(0,1)} \Big|\frac{f(t,s)}{f_-(s)} - 1\Big| \le \zeta(t) \quad \text{for all } t < 0.$$

Let $u$ be any transition front connecting $1$ and $0$.

Then

$$\begin{cases} 2\sqrt{\mu_-} \le c_- := \liminf_{t\to-\infty} \dfrac{x_t}{t} \le \limsup_{t\to-\infty} \dfrac{x_t}{t} < +\infty \\ \kappa + \dfrac{\mu_+}{\kappa} \le c_+ := \liminf_{t\to+\infty} \dfrac{x_t}{t} \le \limsup_{t\to+\infty} \dfrac{x_t}{t} < +\infty. \end{cases}$$

Furthermore, if $c_- > 2\sqrt{\mu_-}$, then $u$ has asymptotic past and future speeds $c_\pm$.

Lastly, if $c_- > 2\sqrt{\mu_-}$ and there is $\widetilde{\zeta} \in L^1(0, +\infty)$ such that

$$\sup_{s\in(0,1)} \Big|\frac{f(t,s)}{f_+(s)} - 1\Big| \le \widetilde{\zeta}(t) \quad \text{for all } t > 0,$$

then convergence to the limiting profiles $\varphi^{\pm}_{c_\pm}$.

*Example*: $f(t,u)=\widetilde{\mu}(t)\,\widetilde{f}(u)$ with $\widetilde{\mu}-\widetilde{\mu}(\pm\infty)\in L^1(\mathbb{R}_\pm)$

**Corollary**

Same assumptions as in the previous theorem.

Transition fronts connecting $1$ and $0$ and having asymptotic past and future speeds $c_\pm$ exist if and only if $c_-$ and $c_+$ satisfy

$$c_-\geq 2\sqrt{\mu_-}\quad\text{and}\quad c_+\geq\kappa+\frac{\mu_+}{\kappa}$$

with

$$\kappa=\min\Big(\sqrt{\mu_+},\,\frac{c_--\sqrt{c_-^2-4\mu_-}}{2}\Big)>0.$$

Transition fronts connecting $1$ and $0$ and having a global mean speed $\gamma$ exist if and only if $\mu_+\leq\mu_-$ and $\gamma\geq 2\sqrt{\mu_-}$.

## A sufficient condition for an entire solution to be a transition front

**Theorem**

Same assumptions as in the previous theorem.

Let $0 < u(t,x) < 1$ be an entire solution such that

$$\exists\, c > 2\sqrt{\mu_-}, \quad \max_{[-c|t|,c|t|]} u(t,\cdot) \to 0 \ \text{ as } t \to -\infty.$$

Then the following limit exists independently of $t \in \mathbb{R}$:

$$\lambda = -\lim_{x\to+\infty} \frac{\ln u(t,x)}{x} \in \big[0, \sqrt{\mu_-}\big).$$

Furthermore, $u$ is a transition front connecting $1$ and $0$ if and only if $\lambda > 0$.

Lastly, if $\lambda > 0$, then $u$ has asymptotic speeds $c_\pm$ given by

$$\begin{cases} 2\sqrt{\mu_-} < c_- = \sup\Big\{\gamma \geq 0,\ \lim_{t\to-\infty} \max_{[-\gamma|t|,\gamma|t|]} u(t,\cdot) = 0\Big\} \\ c_+ = \min(\lambda, \sqrt{\mu_+}) + \dfrac{\mu_+}{\min(\lambda, \sqrt{\mu_+})} \end{cases}$$

and it has asymptotic profiles $\varphi^{\pm}_{c_\pm}$.

**Time-dependent diffusivity**

$$u_t = \sigma(t)u_{xx} + f(t,u)$$

with $0 < a \le \sigma(t) \le b < +\infty$, $\sigma(t) \to \sigma_\pm$ as $t \to \pm\infty$ and $\sigma - \sigma_- \in L^1(\mathbb{R}_-)$

## Corollary

Transition fronts connecting $1$ and $0$ having asymptotic past and future speeds $c_\pm$ exist if and only

$$c_- \ge 2\sqrt{\sigma_-\mu_-} \quad \text{and} \quad c_+ \ge \kappa + \frac{\sigma_+\mu_+}{\kappa}$$

where

$$\kappa = \min\left(\sqrt{\sigma_+\mu_+}, \frac{\sigma_+}{\sigma_-} \times \frac{c_- - \sqrt{c_-^2 - 4\sigma_-\mu_-}}{2}\right)$$