Bubbling blow-up in critical parabolic problems

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We consider three parabolic problems, popular in the literature, with "bubbling" blow-up phenomena in finite and/or infinite time.

1. The Sobolev critical semilinear heat equation in $\mathbb{R}^n$

$$u_t = \Delta u + u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in } \Omega \times (0, T) \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T)$$

$\Omega \subset \mathbb{R}^n$, $n \geq 3$, $u > 0$, $T \leq +\infty$.

2. The harmonic map flow from $\mathbb{R}^2$ into S^2

$$u_t = \Delta u + |\nabla u|^2 u, \quad |u| = 1 \quad \text{in } \Omega \times (0, T) \tag{2}$$

$$u = \varphi \quad \text{on } \partial\Omega \times (0, T), \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

Here $\Omega \subset \mathbb{R}^2$.

3. The Keller-Segel equation in $\mathbb{R}^2$

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad u > 0 \quad \text{in } \mathbb{R}^2 \times (0, T) \tag{3}$$

$$v = (-\Delta)^{-1} u := \frac{1}{2\pi} \log \frac{1}{|\cdot|} * u$$

$$u_t = \nabla \cdot (u \nabla (\log u - (-\Delta)^{-1} u))$$

The three problems have a **Lyapunov functional** (decreasing along trajectories)

$$\boxed{u_t = \Delta u + u^p, \quad p = \frac{n+2}{n-2}} \tag{1}$$

$$E_1(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int u^{p+1},$$

$$\partial_t E_1(u(\cdot,t)) = -\int |u_t|^2$$

$$\boxed{u_t = \Delta u + |\nabla u|^2 u, \quad |u| = 1} \tag{2}$$

$$E_2(u) = \frac{1}{2}\int |\nabla u|^2,$$

$$\partial_t E_2(u(\cdot,t)) = -\int |u_t|^2$$

$$\boxed{u_t = \nabla \cdot (u \nabla (\log u - (-\Delta)^{-1} u))} \tag{3}$$

$$E_3(u) = \int u \log u - \frac{1}{2} \int u(-\Delta)^{-1} u$$

$$\partial_t E_3(u(\cdot, t)) = -\int u |\nabla (\log u - (-\Delta)^{-1} u)|^2$$

The three problems have a continuous, blowing-up family of **energy invariant steady states** in entire space.

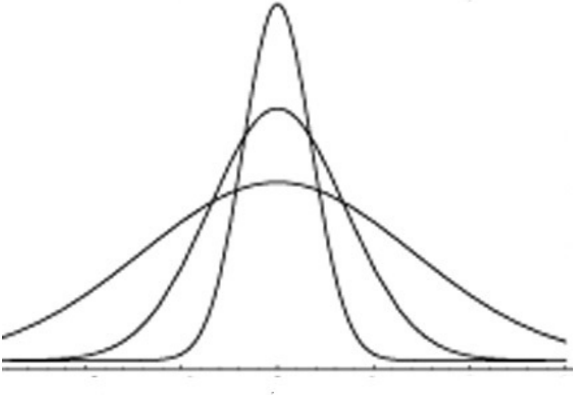
$$\boxed{\Delta u + u^{\frac{n+2}{n-2}} = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n}$$

$$U(x) = \alpha_n \left(\frac{1}{1+|x|^2} \right)^{\frac{n-2}{2}},$$

$$\boxed{U_{\lambda,x_0}(x) = \frac{1}{\lambda^{\frac{n-2}{2}}} U\left(\frac{x-x_0}{\lambda} \right) = \alpha_n \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}}.}$$

We have

$$E_1(U_{\lambda,x_0}) = E(U) \quad \text{for all} \quad \lambda, x_0.$$



$$U_{\lambda,x_0}(x) = \alpha_n \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}, \quad \lambda \downarrow 0$$

$$\boxed{\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \quad \text{in } \mathbb{R}^2}$$

$$U(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

the **1-corrotational harmonic map**.

$$\boxed{U_{\lambda, x_0, Q}(x) = QU\left(\frac{x - x_0}{\lambda}\right)}$$

with Q a linear orthogonal transformation of $\mathbb{R}^3$.

$$E_2(U_{\lambda, x_0, Q}) = E(U) \quad \text{for all} \quad \lambda, x_0.$$

$$\boxed{\Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u) = 0 \quad \text{in } \mathbb{R}^2}$$

$$U(x) = \frac{8}{(1 + |x|^2)^2} \quad x \in \mathbb{R}^2,$$

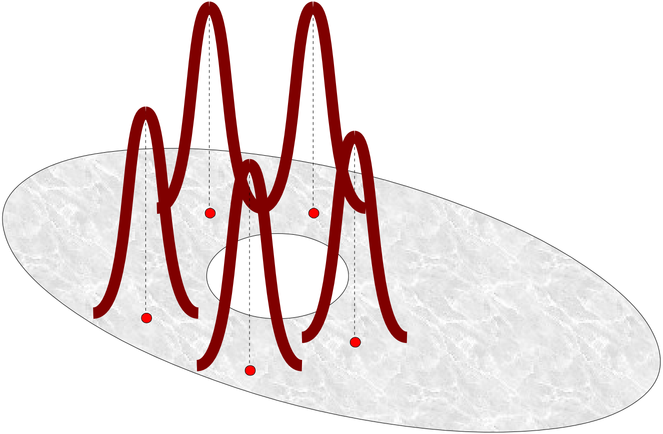
$$\boxed{U_{\lambda, x_0}(x) = \lambda^{-2} U\left(\frac{x - x_0}{\lambda}\right) = \frac{8\lambda^2}{(\lambda^2 + |x - x_0|^2)^2}}$$

$$E_3(U_{\lambda, x_0}) = E(U) \quad \text{for all} \quad \lambda, x_0.$$

Observe that

$$\int_{\mathbb{R}^2} U_{\lambda, x_0}(x)\, dx = 8\pi.$$

A **bubbling solution** $u(x,t)$ of Problem (1), (2) or (3) as $t \uparrow T \leq +\infty$ is one that blows-up by resembling about one or more points a steady state with a **time dependent** scaling parameter $\lambda(t) \to 0$ as $t \uparrow T$.



$$u_t = \Delta u + u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in } \Omega \times (0, +\infty) \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, +\infty)$$

We look for a solution that at main order as $t \to +\infty$ looks like

$$u(x,t) \approx \sum_{j=1}^{k} U_{\lambda_j(t), x_j(t)}(x) = \sum_{j=1}^{k} \alpha_n \left(\frac{\lambda_j(t)}{\lambda_j(t)^2 + |x - x_j(t)|^2} \right)^{\frac{n-2}{2}}$$

• A result: Galaktionov and King (2003) A globally defined radial solution $u(r,t)$, $r = |x|$ of (1) when $\Omega = B(0,1)$ blows-up like

$$u(r,t) \approx \alpha_n \left(\frac{\lambda_j(t)}{\lambda_j(t)^2 + r^2} \right)^{\frac{n-2}{2}}$$

where $\lambda(t) \to 0$ as $t \to +\infty$. If dimension $n \geq 5$ we have

$$\lambda(t) \sim ct^{-\frac{1}{n-4}}.$$

• The nonradial case? Existence of such solutions? In the radial case existence of a nonradial globally defined solution follows from non-continuation results for radial solutions (Galaktionov and Vazquez 1997, Ni, Sacks, Tavantzis 1984).

What about the nonradial case?

Connection with Green's function: $G(x,y)$

$$-\Delta_x G(x,y) = c_n \delta(x-y) \quad \text{in } \Omega, \quad G(x,y) = 0, \quad x \in \partial\Omega.$$

$H(x,y)$ the regular part of $G(x,y)$ namely the solution of the problem

$$-\Delta_x H(x,y) = 0 \quad \text{in } \Omega, \quad H(x,y) = \Gamma(x-y) \quad \text{for all} \quad x \in \partial\Omega.$$

$$G(x,y) = \Gamma(x-y) - H(x,y).$$

where Γ is the fundamental solution

$$\Gamma(x) = \frac{\alpha_n}{|x|^{n-2}},$$

For Problem (1) we establish the existence of a **globally defined solution** with bubbling phenomena as $t \to +\infty$.
Let $q_1, \ldots, q_k$ be given distinct points in Ω.

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_1,q_2) & H(q_2,q_2) & -G(q_2,q_3)\cdots & -G(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1,q_k) & \cdots & -G(q_{k-1},q_k) & H(q_k,q_k) \end{bmatrix}$$

Our result:
A global solution of (1) with its k bubbling points q_j exists if the matrix $\mathcal{G}(q)$ is positive definite.

We can always find k points where $\mathcal{G}(q)$ is positive definite thanks to: $H(x,x) \to +\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$

Theorem (C. Cortázar, M. del Pino, M. Musso)

Assume $n \geq 5$, $\mathcal{G}(q_1, \cdots, q_k)$ is positive definite. Then there exist functions

$x_j(t) \to q_j$ and $0 < \lambda_j(t) \to 0$, as $t \to +\infty$, $j = 1, \dots, k$, and a solution of (1) of the form

$$u(x,t) = \sum_{j=1}^{k} U_{\lambda_j(t), x_j(t)}(x) - \lambda_j^{\frac{n-2}{2}} H(x, q_j) \; + \; l.o.t,$$

as $t \to +\infty$ and for certain positive numbers a_j

$$\lambda_j(t) \sim a_j t^{-\frac{1}{n-4}}.$$

The set of initial conditions around $u(x,0)$ that lead to k-bubbling in infinite time is a codimension k manifold of functions.

Given k points $q_1, \ldots, q_k \in \mathbb{R}^n$ We want to find a solution $u(x,t)$ of (1) with

$$u(x,t) \approx \sum_{j=1}^{k} U_{\lambda_j(t), x_j(t)}(x)$$

where $x_j(t) \to q_j$ and $\lambda_j(t) \to 0$ as $t \to \infty$ for each $j = 1, \ldots, k$.

Ansatz at main order: for a certain fixed positive function $\mu_0(t) \to 0$ and positive constants $b_1, \ldots, b_k$ we have that

$$\lambda_j(t) = b_j \mu_0(t), \quad x_j(t) = q_j$$

Away from the concentration points q_j

$$u_t \approx \Delta u + \sum_{j=1}^{k} U_{\lambda_j,q}(x)^p$$

and

$$\int_\Omega U_{\lambda_j,q}(x)^p dx \approx \lambda_j^{\frac{n-2}{2}} a_n, \quad a_n := \int_{\mathbb{R}^n} U(y)^p dy,$$

Hence, away from the points q_j

$$u_t \approx \Delta u + c_n \mu_0^{\frac{n-2}{2}} \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty).$$

where δ_q is the Dirac mass at the point q.

Letting $u = \mu_0^{\frac{n-2}{2}} v$ we get

$$v_t \approx \Delta v - \frac{n-2}{2}\mu_0^{-1}\dot{\mu}_0 v + \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0, \infty).$$

We assume, as it will be a priori satisfied that $\mu_0^{-1}\dot{\mu}_0 \to 0$, which is the case for instance if $\mu_0 \sim t^{-a}$.

Hence

$$v_t \approx \Delta v + a_n \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0, \infty),$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Thus as $t \to +\infty$ we get

$$\boxed{v(x,t) \approx a_n \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} G(x, q_j)}$$

where $G(x, y)$ is the Green function of the domain. If $H(x, y)$ denotes regular part, hence:

$$\boxed{u(x,t) \approx \sum_{j=1}^{k} \lambda_j^{\frac{n-2}{2}} \left[\frac{\alpha_n}{|x - q_j|^{n-2}} - H(x, q_j) \right].}$$

Thus a better approximation is

$$u_0(x,t) := \sum_{j=1}^{k} U_{\lambda_j,\xi_j}(x) - \lambda_j^{\frac{n-2}{2}} H(x,q_j).$$

Let

$$S(u) := -u_t + \Delta_x u + u^p.$$

The **error of approximation** is $S(u_0)$. We get after some computation near q_j

$$\lambda_j^{\frac{n+2}{2}} S(u_0)(x,t) = E_0(y,t) + l.o.t.$$

where $y = \frac{x-q_j}{\lambda_j}$ and

$$E_0(y,t) \;=\; \lambda_j \dot{\lambda}_j \left[y \cdot \nabla U(y) + \frac{n-2}{2} U(y) \right]$$

$$+\, pU(y)^{p-1}\left[-\lambda_j^{n-2} H(q_j,q_j) + \sum_{i\neq j} (\lambda_i \lambda_j)^{\frac{n-2}{2}} G(q_i,q_j) \right],$$

We look for a solution

$$u(x,t) = u_0(x,t) + \lambda_j^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi_j}{\lambda_j}, t\right)$$

At main order $S(u_0 + \tilde{\phi}) = 0$ means

$$\lambda_j^2 \phi_t \approx \Delta_y \phi + pU(y)^{p-1}\phi + \lambda_j^{\frac{n+2}{2}} S(u_0),$$

so that at main order we expect

$$\boxed{L_0(\phi) := \Delta_y \phi + pU(y)^{p-1}\phi + E_0(y,t) = 0 \quad \text{in } \mathbb{R}^n.}$$

This equation is solvable for space decaying ϕ if and only if $\int_{\mathbb{R}^n} E_0 Z dy = 0$ for all bounded solution of $L_0(Z) = 0$, which all consist of linear combinations of the functions

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \dots, n, \quad Z_{n+1}(y) := \frac{n-2}{2} U(y) + y \cdot \nabla U(y),$$

We get in particular the necessary condition for the existence of ϕ_0,

$$0 = \int_{\mathbb{R}^n} E_0(y,t)\, Z_{n+1}(y)\, dy \ =$$

$$c_1 \Big[\lambda_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} (\lambda_i \lambda_j)^{\frac{n-2}{2}} G(q_i, q_j) \Big] \ + \ c_2\, \lambda_j \dot{\lambda}_j \,,$$

$$c_1 = -p \int_{\mathbb{R}^n} U^{p-1} Z_{n+1} = (n-p) \int_{\mathbb{R}^n} U^p, \quad c_2 = \int_{\mathbb{R}^n} |Z_{n+1}|^2.$$

We observe that $c_2 < +\infty$ thanks to $n \geq 5$.

Since $\lambda_j = b_j \mu_0$, we get that for all j

$$\mu_0(t)^{n-3}[b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{\frac{n-2}{2}} G(q_i, q_j)] + c_2 c_1^{-1} b_j^2 \dot{\mu}_0(t) = 0$$

$$\dot{\mu}_0(t) = -a\mu_0(t)^{n-3},$$

for some positive constant a, which yields

$$\boxed{\mu_0(t) \;=\; \gamma t^{-\frac{1}{n-4}},}$$

and for suitable γ (chosen taken into account scaling invariance)

$$\boxed{b_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}-1} G(q_i, q_j) = \frac{2b_j}{n-2} \quad \text{for all} \quad j = 1, \ldots, k}$$

This system is solvable if the matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_1,q_2) & H(q_2,q_2) & -G(q_2,q_3)\cdots & -G(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1,q_k) & \cdots & -G(q_{k-1},q_k) & H(q_k,q_k) \end{bmatrix}$$

is positive definite. Indeed, it is equivalent to $\nabla_b I(b) = 0$ where

$$I(b) := \sum_{j=1}^{k} b_j^{n-2} H(q_j,q_j) - \sum_{i\neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}} G(q_i,q_j) - \sum_{j=1}^{k} b_j^2.$$

which has a positive minimizer.

A well-known fact is that the eigenvalue problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N)$$

has exactly one negative, simple eigenvalue λ_0, with a positive, radially symmetric eigenfunction Z_0, which decays like

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|}\,|y|} \quad \text{as } |y| \to \infty.$$

At main order we have

$$\lambda_j^2 \phi_t = L_0(\phi) + E_0(y,t)$$

Let $e(t) := \int_{\mathbb{R}^n} \phi(y,t) Z(y)\, dy$. Then, integrating the equation, using that $\lambda_j(t)^2 \approx \gamma t^{-\frac{2}{n-2}}$ we get

$$\gamma t^{-\frac{2}{n-2}} \dot{e}(t) - \lambda_0 e(t) = f(t) := \int_{\mathbb{R}^n} E_0(y,t) Z_0(y)\, dy.$$

Hence, for some $a > 0$,

$$e(t) = \exp(a t^{\frac{n-2}{n-4}}) \left(e(0) + \int_0^t s^{\frac{2}{n-4}} f(s) \exp(-a s^{\frac{n-2}{n-4}}) ds \right).$$

The only way in which $e(t)$ does not grow exponentially in time is for the specific value of

$$e(0) = \int_{\mathbb{R}^n} \phi(y,0) Z_0(y)\, dy = -\int_0^\infty s^{\frac{2}{n-4}} f(s) \exp(-a s^{\frac{n-2}{n-4}}) ds$$

Therefore the (small) initial condition required for the remainder ϕ should lie on a certain manifold locally described as a translation of the hyperplane orthogonal to $Z_0(y)$. Since we have k of these hyperplanes, these constraints define a *codimension* k *manifold* of initial conditions.

• Similar phenomena holds true for $n = 4$ and $n = 3$ with

$$\mu(t) \sim \begin{cases} e^{-a\sqrt{t}} & \text{if} \quad n = 4 \\ e^{-at} & \text{if} \quad n = 3 \end{cases}$$

• In entire space $\mathbb{R}^3$ we can find (del Pino, Musso, Wei) solutions with a positive single blowing-up bubble as $t \to +\infty$ for $\gamma > 1$ and

$$u_t = \Delta u + u^5 \quad \text{in } \mathbb{R}^3, \qquad \lim_{|x|\to\infty} |x|^{-\gamma} u(x,0) > 0.$$

$$\mu(t) \sim \begin{cases} \frac{1}{t^{\gamma-1}} & \text{if} \quad 1 < \gamma < 2 \\ \frac{\log^2 t}{t} & \text{if} \quad \gamma = 2 \\ \frac{1}{t} & \text{if} \quad \gamma > 2. \end{cases}$$

Formal asymptotics previously derived by Fila and King, 2011.

• **Multiple bubbling at a single point?** The solutions constructed have *simple bubbling*: no "bubble on top of bubble".

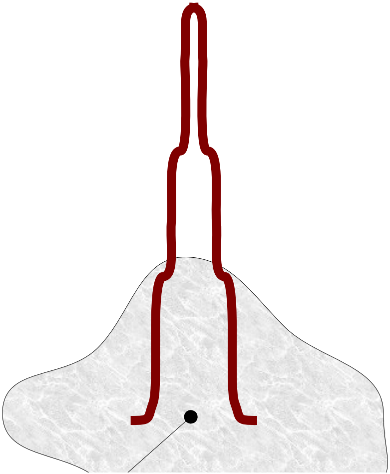
A fact: (del Pino, Dolbeault and Musso, JDE 2003 in a ball, F.Pacard and R.Jing JFA 2005 general domain). The slightly supercritical problem

$$\Delta u + \lambda u + u^{\frac{n+2}{n-2}+\varepsilon} = 0, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has solutions with multiple bubbling at a single point when $0 < \varepsilon \to 0$:

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \mu_k^\varepsilon \ll \mu_{k-1}^\varepsilon \ll \cdots \ll \mu_1^\varepsilon.$$

The analogue of this in the parabolic setting?



$$u_\varepsilon(x) \sim \sum_{j=1}^{3} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x|^2} \right)^{\frac{n-2}{2}}$$

A result for a related problem (Yamabe flow in $\mathbb{R}^n$: conformal evolution of metrics by scalar curvature

Ancient solutions with bubbling as $t \to -\infty$

P.Daskalpoulos, M.D., N. Sesum, Crelle 2016:

$$(u^{\frac{n+2}{n-2}})_t = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \times (-\infty, 0]$$

There exists a radially symmetric solution with the profile

$$u(x,t) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \mu_k \ll \mu_{k-1} \ll \cdots \ll \mu_1.$$

More precisely, as $t \to -\infty$

$$\mu_j(t) \sim |t|^{-b_j(j-\frac{k+1}{2})}, \quad j = 1, \dots, k.$$

Finite time bubbling for the planar harmonic map flow into S^2

$$u_t = \Delta u + |\nabla u|^2 u, \quad |u| = 1 \quad \text{in } \Omega \times (0, T) \qquad (2)$$

$$u = \varphi \quad \text{on } \partial\Omega \times (0, T), \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

Substantial literature is present (since 90s) on the analysis of this ow and its bubbling phenomena (Among them K.C. Chang, W. Ding, A. Freire, F.H. Lin, H. Matano, M. Struwe, G. Tian, P. Topping, R. Ye). **Blowing-up occurs in the form of scalings of steady states**

Our building block is the 1-corrotational harmonic map,

$$U(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

The 1-corrotational symmetric ansatz

$$u(r,\theta,t) = \begin{pmatrix} e^{i\theta}\sin v(r,t) \\ \cos v(r,t) \end{pmatrix},$$

The equation reduces just to

$$v_t = v_{rr} + \frac{v_r}{r} + \frac{\sin(2v)}{2r^2}.$$

$$v(r,t) = w(r) := 2\arctan(r)$$

is the stationary 1-corrotational harmonic map.

Results on existence of blow-up

• Chang-Ding-Ye (1991) in the 1-corotational case, when $\Omega = B_1(0)$, for suitable boundary and initial conditions we have

$$\boxed{v(r,t) \sim w\left(\frac{r}{\lambda(t)}\right)}$$

with $\lambda(t) \to 0$ as $t \to T < +\infty$

• Topping: in the general case $\lambda(t) = o(T-t)^{\frac{1}{2}}$ which means blow-up is always "type II". For certain targets this estimate is nearly optimal.

• Angenent, Hulshof and Matano: In the 1-corrotational case the rate is $\lambda(t) = o(T-t)$.

• Van der Berg, Hulshof, King: Formal analysis. Blow-up in the 1-co-rotational radially symmetric case should typically be

$$\lambda(t) \sim \frac{\kappa(T-t)}{\log^2(T-t)}.$$

• Raphael and Schweyer (2013) constructed a 1-corrotational solution with this bubbling rate for the Cauchy problem in entire space. They prove stability of this bubbling **within the radial 1-corrotational class**.

• The existence and stability issues without radial symmetry is largely open. It has indeed been conjectured by some authors that the bubbling phenomenon is **not** stable once radial symmetry is perturbed.

Theorem (J. Davila, M. del Pino, J. Wei)

Let $q \in \Omega$ be arbitrary and $T > 0$ small. There exist an initial condition $u_0(x)$ and boundary condition $\varphi(x)$ such that the solution $u(x,t)$ to problem (2) blows-up at time T at the point q in the form

$$u(x,t) \approx Q(t) U_{\lambda(t), x_0(t)}(x)$$

with $x_0(t) \to q$ and $\lambda(t) \to 0$, and $Q(0) = Id$ and the limit $Q(T)$ exists, and

$$\lambda(t) \sim \frac{\kappa(T-t)}{\log^2(T-t)}.$$

This blow-up is **stable** in the sense that the same holds for any small perturbation (into S^2) of the initial and boundary condition, at a point q_1 and time T_1 close to q and T

• Also, a similar result can be proven with blow up at k given points of the domain. In this case, the phenomenon is codimension $k-1$-stable.

• One can construct solutions with reverse bubbling, that continue naturally after blow up. With that continuation, the k-blow-up phenomenon is stable, of course with blow-up taking place at different times.

For Keller-Segel, our main result is existence and stability of the **critical mass solution.**

$$u_t = \Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u), \quad u > 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty) \qquad (3)$$

Assuming that $u(x,0) \in L^1(\mathbb{R}^2)$, the following is known:

• If $\int_{\mathbb{R}^2} u(x,0)dx > 8\pi$ then finite-time blow-up always takes place. Bubbling behavior in the radial case with exact rates when mass is close to 8π have been built by Raphael and Schweyer.

• It $\int_{\mathbb{R}^2} u(x,0)dx < 8\pi$ then the solution is global, it goes to zero uniformly as $t \to \infty$ with a self-similar profile (Dobeault and Perthame)

• The solution when $\int_{\mathbb{R}^2} u(x,0)dx = 8\pi$ is globally defined in time. If the second moment of the initial condition is finite, namely $\int_{\mathbb{R}^2} |x|^2 u(x,0)dx < +\infty$, then the solution blows-up in infinite time, with a bubbling behavior (of unknown rates) (Masmoudi, Carlen-Figalli)

• Formal rates of bubbling when mass equals 8π have been studied by Chavanis and Sire and by Campos.

Our result: (with Davila, Dolbeault, Musso and Wei)
There exists a solution $u(x,t)$ of Keller-Segel with fast-decay initial condition, which blow up in infinite time, with a profile which at main order is

$$u(x,t) \approx \frac{8\lambda(t)^2}{(\lambda(t)^2 + |x|^2)^2}$$

where

$$\lambda(t) \sim \frac{1}{\sqrt{\log t}}.$$

All positive initial conditions (not necessarly radial) with fast decay and mass 8π suitably close to $u(x,0)$ lead to the same phenomenon.

Thanks for your attention