Birkhoff normal form for null form wave equations

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Recent trends in nonlinear evolution equations
CIRM - Luminy
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Outline

Nonlinear wave equation

Transformation theory

Hamiltonians and Hamiltonian vector fields

Kernel estimates

Energy estimates

A progress report, with Amanda French (Haverford) and
Chi-Ru Yang (McMaster University and the Fields Institute)

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contrast the behavior of two ODEs

- Quadratic case

$$\dot{z} = z^2 \ , \qquad z(0) = \varepsilon$$
$$z(t) = \frac{\varepsilon}{1 - \varepsilon t} \ , \qquad T_\varepsilon = \frac{1}{\varepsilon}$$

- Cubic case

$$\dot{w} = w^3 \ , \qquad w(0) = \varepsilon$$
$$w(t) = \sqrt{\frac{\varepsilon^2}{1 - 2\varepsilon^2 t}} \ , \qquad T_\varepsilon = \frac{1}{2\varepsilon^2}$$

- The general time of existence does not change when these ODE are replaced by

$$\dot{z} = i\omega z + z^2 + h^{(3)}(z) \ , \qquad \dot{w} = i\omega w + w^3 + k^{(4)}(w)$$

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nonlinear wave equations

- nonlinear wave equations on $\mathbb{R}^n$

$$\partial_t^2 u = \Delta u + N(\partial_t u, \nabla u, \partial_t^2 u, \nabla^2 u) \tag{1}$$

where $N(v) = \mathcal{O}(|v|^{m-1})$. The Cauchy problem

$$u(0,x) = g(x)\,, \qquad \partial_t u(0,x) = h(x)$$

- A basic question in PDEs is the time of existence $T = T_R$ of solutions, for data (g,h) with

$$\|(g,h)\|_Z \leq R$$

for Z an appropriate Sobolev space

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existence time estimates

- It is clear that T_R depends upon the order m of the nonlinearity

Theorem (S. Klainerman, L. Hörmander, J. Shatah (1980s), ... others)

Suppose that

$$\frac{1}{2}(n-1)(m-2) > 1$$

then for Cauchy data $(g,h) \in Z$*, for* R *sufficiently small,* $T_R = +\infty$.

The result reflects a balance of nonlinear effects and the dispersion (decay rate) of solutions in $\mathbb{R}^n$.

- Theorem (decay rates for the linear wave equation)

Homogeneous solutions of the linear wave equation satisfy

$$|u(t,x)|_{L^\infty} \leq \frac{C}{|t|^{\frac{1}{2}(n-1)}}$$

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- Suppose that $m = 3$ (the minimum) then to satisfy the hypotheses one needs

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Suppose that $m \geq 4$, then

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- borderline cases

Theorem (long time existence)

For $n = 2$ and $m = 4$ (respectively $n = 3$ and $m = 3$) and $\|(g,h)\|_Z \leq R$ sufficiently small, then

$$T_R > \exp(C/R^2)\,, \quad \text{respectively} \quad T_R > \exp(C/R)$$

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examples from physics

- compressible fluid dynamics: $m = 3$
- Einstein's equations in general relativity: $m = 3$
- nonlinear Klein - Gordon equation: $m = 4$
  and the time decay is better in this case

Transformation theory

- There is great interest in transforming a problem with $m = 3$ onto one with $m = 4$. There are several results based on this idea

Theorem (Klainerman (1988), Shatah (1989), Pusateri & Shatah (2012))

For nonlinearities which satisfy a null condition then

$T_R = +\infty$ *for dimension* $n = 3$
$T_R = \exp(C/R^2)$ *for dimension* $n = 2$

The idea in this theorem is to change variables $\tau : u \to v$ in order to eliminate the quadratic terms in the equation
It seems hard however to make repeated transformations with the methods of the above articles

- Our course of action is to introduce methods of Hamiltonian systems, and in particular canonical transformations, for this problem

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Lagrangians

- Physically interesting cases are those (systems of) wave equations (1) arising from a Lagrangian $\delta A = 0$, where the action functional is

$$A(u(t,\cdot)) = \int_0^T L(\partial_t u, \nabla u)\, dt$$

  The Lagrangian functional for the wave equation

$$L(u_t, \nabla u) = \int_{\mathbb{R}^n} \tfrac{1}{2}\big((u_t)^2 - |\nabla u|^2\big) + P^{(m)}(u_t, \nabla u)\, dx$$

  The nonlinear term $P^{(m)}(u_t, \nabla u)$ satisfies smallness conditions. $|P^{(m)}(r)| = \mathcal{O}(|r|^m)$ in variables $r = (u_t, \nabla u)$, $m \geq 2$

- The Legendre transform

$$\delta_{u_t} L = u_t + \partial_{u_t} P^{(m)}(u_t, \nabla u) := p$$

  serves to define $p = p(u_t, \nabla u)$. Its inverse gives

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Lagrangians and Hamiltonians

- Under the Legendre transformation, this realizes the nonlinear wave equation (1) as a Hamiltonian PDE

$$H(u, p) := \langle p, u_t \rangle - L(u_t, \nabla u)$$

evaluated at $u_t = u_t(p, \nabla u)$

- In Darboux coordinates

$$\begin{aligned} \partial_t u &= \delta_p H \\ \partial_t p &= -\delta_u H \end{aligned}$$

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Lagrangians and Hamiltonians

- For nonlinear wave equations, if $L = L^{(2)} + \int P^{(m)}$, with

$$L^{(2)} = \int_{\mathbb{R}^n} \tfrac{1}{2}\big((u_t)^2 - |\nabla u|^2\big)\, dx$$

then $H = H^{(2)} + \int R^{(m)}$, with

$$H^{(2)} = \int_{\mathbb{R}^n} \tfrac{1}{2}\big(p^2 + |\nabla u|^2\big)\, dx$$

Furthermore, $N = N^{(m-1)}$ in (1), of order $m - 1$

Birkhoff normal forms

Restrict our considerations to the $n \geq 3$, with $x \in \mathbb{R}^n$

- Solutions of the linear equations $e^{i\xi\cdot x - \omega(\xi)t}$.
  Frequencies are continuous, given by the dispersion relation for the wave equation $\omega(\xi) = |\xi|$
- Normal form - transform the equations to retain only essential nonlinearities

$$\tau : z = \begin{pmatrix} u \\ p \end{pmatrix} \mapsto z'$$

  in a neighborhood $B_R(0) \subseteq Z$
- Conditions:
  1. The transformation τ is canonical, so the new equations are

$$\partial_t z' = J\,\delta H_+(z')\ , \qquad H_+(z') = H(\tau^{-1}(z'))$$

  2. The new Hamiltonian is

$$H_+(z') = H^{(2)}(z') + \left(Z^{(3)} + \cdots + Z^{(M)}\right) + R_+^{(M+1)}$$

  where each $Z^{(m)}$ retains (at most) only resonant terms

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triad resonances

- This transformation procedure is called the reduction to Birkhoff normal form. It is part of averaging theory for dynamical systems
- When $m = 3$ resonances are known as three wave interactions or resonant triads; those that satisfy the resonance relations

$$\omega(\xi_1) \pm \omega(\xi_2) \pm \omega(\xi_3) = 0 \,, \qquad (2)$$
$$\xi_1 + \xi_2 + \xi_3 = 0$$

- The question in PDEs: mapping properties of the transformation $\tau = \tau^{(3)}$, is it well defined, and on which Banach spaces

  When $x \in \mathbb{R}^n$ then $\xi \in \mathbb{R}^n$ is a continuous variable, and the question of resonance becomes more subtle than for finite dimensional Hamiltonian systems

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Triad resonances for wave equations

- Proposition (three wave interactions)

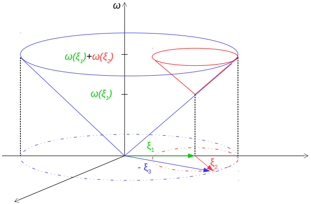
  Resonant triads for the wave equation are colinear

$$\xi_1 : \xi_2 : \xi_3$$

- Proof of Proposition:

  The resonant set is an intersection of light cones

$$LC_{\pm} := \{\Xi := (\xi^0, \xi^1, \dots) \, : \, \xi^0 = \pm\omega(\xi^1, \dots \xi^n)\}$$



Hamiltonian flows

- One approach to the transformation $\tau = \tau^{(3)}$ is to construct it as the time $s = 1$ flow of an auxiliary Hamiltonian system

$$\frac{d}{ds} z = J \delta_z K^{(3)}$$

- Define complex symplectic coordinates

$$\begin{aligned} z(x) &= \frac{1}{\sqrt{2}} \Big(\sqrt{|D_x|} u(x) + i \frac{1}{\sqrt{|D_x|}} p(x) \Big) \\ &= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \hat{z}(\xi) e^{ikx} \, d\xi \end{aligned}$$

- In these coordinates, using Plancherel (and dropping 'hat's)

$$H = \int_{\mathbb{R}^n} \omega(\xi) |z(\xi)|^2 + \sum_{m \geq 3} \Big[\sum_{|p|+|q|=m} \iint_{\sum_{\ell=1}^m \xi_\ell = 0} c_{pq}(\xi_1, \dots \xi_m) z^p \bar{z}^q \Big] \, d\vec{\xi}$$

where $z^p \bar{z}^q := \prod_{\ell=1,..p} z(\xi_\ell) \prod_{\ell'=1,..q} \bar{z}(-\xi_{\ell'})$

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null condition

- The Hamiltonian $H^{(3)}$ satisfies the null condition when the interaction coefficients satisfy

$$c_{12}(\xi_1, \xi_2, \xi_3) = 0 \, , \qquad \xi_j \in \mathbb{R}^n$$

for all resonant triads $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{3n}$

This is equivalent to Klainerman's definition (proof given later)

- A particular example is

$$H^{(3)} = \int_{\mathbb{R}^n} p\big(|\nabla u|^2 - p^2\big)\, dx$$

Under Fourier transform, and using complex symplectic coordinates

$$H^{(3)}(z, \bar{z}) = C \iint_{\xi_1+\xi_2+\xi_3=0} \sqrt{\frac{|\xi_1|}{|\xi_2||\xi_3|}} \big(|\xi_2||\xi_3| - \xi_2 \cdot \xi_3\big) \times \big(z_1 z_2 z_3 + z_1 \bar{z}_{-2} \bar{z}_{-3}\big)\, d\xi_1 d\xi_2 + \cdots$$

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cohomological equation

- To eliminate $H^{(3)}$ using a Hamiltonian flow, solve the cohomological equation for $K^{(3)}$

$$\{H^{(2)}, K^{(3)}\} = H^{(3)}$$

  Do this despite the resonant triads (the singularities) of the RHS

  If the Hamiltonian vector field $X^{K^{(3)}}$ has a well defined solution map on an appropriate Banach space Z, this is a good transformation of the nonlinear wave equation

- Solution of the cohomological equation for $K^{(3)}$

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resonant variety

- The first denominator is nonresonant (except at $\xi_1 = \xi_2 = \xi_3 = 0$)
  The second denominator vanishes on the resonant set

$$\begin{aligned} \mathcal{R} \;=\; & \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{3n} : \omega(\xi_1) - \omega(\xi_2) - \omega(\xi_3) = 0\,, \\ & \xi_1 + \xi_2 + \xi_3 = 0\} \end{aligned}$$

- Proposition (null condition)
  The numerator $|\xi_2||\xi_3| - \xi_2 \cdot \xi_3$ in the resonant kernel $K_{2,1}$ vanishes when (ξ_2, ξ_3) are colinear. That is, when

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auxiliary Hamiltonian vector field $X^{K^{(3)}}$

- We seek the transformation as a time $s = 1$ flow of the auxiliary Hamiltonian system

$$\frac{d}{ds} z = i\delta_{\bar{z}(x)} K^{(3)} := X^{K^{(3)}}(z, \bar{z})$$

  The flow map $\psi_s(z)$ gives rise to $\tau^{(3)}(z) := \psi_{s=1}(z)$

  The question is whether the flow map exists

- The Hamiltonian vector field

$$\begin{aligned} X^{K^{(3)}}(z,\bar{z}) \quad := \quad & iC \int_{\xi_1+\xi_2+\xi=0} \Big[\sqrt{\frac{|\xi_1|}{|\xi_2||\xi|}} \Big(\frac{|\xi_2||\xi| - \xi_2 \cdot \xi}{|\xi_1| + |\xi_2| + |\xi|} \Big) \bar{z}_{-1} \bar{z}_{-2} \\ & + \sqrt{\frac{|\xi|}{|\xi_1||\xi_2|}} \Big(\frac{|\xi_1||\xi_2| - \xi_1 \cdot \xi_2}{|\xi_1| + |\xi_2| + |\xi|} \Big) \bar{z}_{-1} \bar{z}_{-2} \\ & + 2\sqrt{\frac{|\xi_1|}{|\xi_2||\xi|}} \Big(\frac{|\xi_2||\xi| - \xi_2 \cdot \xi}{|\xi_1| - |\xi_2| - |\xi|} \Big) z_1 \bar{z}_{-2} \Big] \, d\xi_1 + \cdots \end{aligned}$$

  is not Lipschitz on (any reasonable) Banach spaces

auxiliary Hamiltonian vector field $X^{K^{(3)}}$

- We seek the transformation as a time $s = 1$ flow of the auxiliary Hamiltonian system

$$\frac{d}{ds} z = i\delta_{\bar{z}(x)} K^{(3)} := X^{K^{(3)}}(z, \bar{z})$$

  The flow map $\psi_s(z)$ gives rise to $\tau^{(3)}(z) := \psi_{s=1}(z)$

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kernel estimates

- Change variables $w(\xi) := \sqrt{|\xi|}z(\xi)$ so that $\|w\|_{H^s}$ give the standard Sobolev energies for (u,p)

  We are led to study the resonant homogeneous kernels

$$(*) = k(\xi_1, \xi_2, \xi) := \frac{1}{|\xi_2|}\left(\frac{|\xi_2||\xi| - \xi_2 \cdot \xi}{|\xi_1| - |\xi_2| - |\xi|}\right)$$

- Lemma
  Estimates of $(*)$ in conic neghborhoods

$$(*) = \frac{|\xi|}{|\xi_2|}\chi_{|\xi_2| \leq |\xi|/10} + C(\xi_1, \xi_2, \xi)$$

where $C(\xi_1, \xi_2, \xi)$ is a bounded symbol.

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Lie algebras of invariant operators

- Angular momentum operators $\Omega_{j\ell} = x_j \partial_{x_\ell} - x_\ell \partial_{x_j}$
  Dilation operators $\Lambda = \sum_{k=1}^n x_k \partial_{x_k}$

- Under Fourier transform

$$\Omega_{j\ell} = \Omega_{j\ell}(X) \mapsto \xi_j \partial_{\xi_\ell} - \xi_\ell \partial_{\xi_j} = \Omega_{jl}(\xi)$$
$$\Lambda(x) \mapsto -\Lambda(\xi) - nI$$

- The operators Ω and Λ obey the Leibnitz rule with respect to the integral operators

$$X^{(K)}(u, v) = \int_{\xi+\xi_1+\xi_2=0} k(\xi, \xi_1, \xi_2) u(\xi_1) v(\xi_2)\, d\xi_1$$

  Namely

$$\Omega X^{(K)}(u, v) = X^{(K)}(\Omega u, v) + X^{(K)}(u, \Omega v)$$

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Energy estimates for $X^{K^{(3)}}$

- Work in the invariant norm Sobolev spaces

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\begin{aligned}
Z^{\bar{s}} &:= \{ w \,:\, \Lambda^{\beta}\Omega^{\alpha}\partial_x^{\sigma}\hat{w} \in L^2(\mathbb{R}^n) \,,\ |\alpha| + |\beta| + |\sigma| \le \bar{s} \} \\
&= \{ w \,:\, \Lambda_{\xi}^{\beta}\Omega_{\xi}^{\alpha}\langle\xi\rangle^{\sigma} w \in L^2(\mathbb{R}_{\xi}^n) \,,\ |\alpha| + |\beta| + |\sigma| \le \bar{s} \}
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- Energy estimates: For $n \ge 3$ solutions of $\partial_s z = X^{K^{(3)}}(z, \bar{z})$ satisfy

$$
\begin{aligned}
\frac{d}{ds}\|z(s,\cdot)\|_{\bar{s}}^2 &= 2\mathrm{re}\langle z, X^{K^{(3)}}(z,\bar{z})\rangle_{\bar{s}} \\
&\le C\|z\|_{\bar{s}}^3
\end{aligned}
$$

This is enough to show that the flow map $\psi_s(z)$ exists and is continuous on $B_R(0) \subseteq Z^{\bar{s}}$ for $|s| \le 1$, for small R
In fact $\psi_s(z)$ is smooth on the scale of spaces $Z^{\bar{s}}$

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\|\partial_z \psi_s(z) - I\|_{\bar{s}-1} < C\|z\|_{\bar{s}-1}
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transformed Hamiltonian

- This transformation $z' = \tau^{(3)}(z) = \psi_{s=1}(z)$ has achieved a canonical change of variables of the nonlinear wave equation to one with a new Hamiltonian

$$\begin{aligned} H_+(z') &= H^{(2)}(z') + R^{(4)} \\ &= H^{(2)}(z') + \left(H^{(4)} - \tfrac{1}{2}\{K^{(3)}, \{K^{(3)}, H^{(2)}\}\}\right) + \dots \end{aligned}$$

  Now $m = 4$ and we have the improved existence theory

  Namely if $n \geq 3$ then $T_R = +\infty$.

- In principle, this operation can be repeated. This is particularly important for case $n = 2$ (Alinhac, Hoshige, Delort)

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global existence via an energy estimate

- The standard argument for existence theory for the nonlinear wave equation (1) uses the invariant norm Sobolev estimate

$$|(u(t,\cdot),p(t,\cdot)|_{L^\infty} \le \frac{C}{|t|^{\frac{1}{2}(n-1)}} \|z\|_{\bar{s}}$$

with $\bar{s} \ge (n+2)/2$

- Then energy estimates for (1) give

$$\begin{aligned}\|z(t,\cdot)\|_{\bar{s}} &\le C\exp\Big(\int_0^t |z(s,\cdot)|_{C^1}^{(m-2)}\,ds\Big)\|z(t,0)\|_{\bar{s}}\\ &\le C\exp\Big(\int_0^t (\|z(s,\cdot)\|_{\bar{s}}/\langle s\rangle^{(n-1)/2})^{(m-2)}\,ds\Big)\|z(t,0)\|_{\bar{s}}\end{aligned}$$

This gives an *a priori* bound for $M_T := \sup_{|t|\le T}\|z(t,\cdot)\|_{\bar{s}}$ which is uniform in $T < +\infty$ if the integral $\int_0^{+\infty}(\langle s\rangle^{(n-1)/2})^{-(m-2)}\,ds$ converges

If the integral grows logarithmically, it gives the lower bounds $T_R \ge \exp(C/R^{m-2})$

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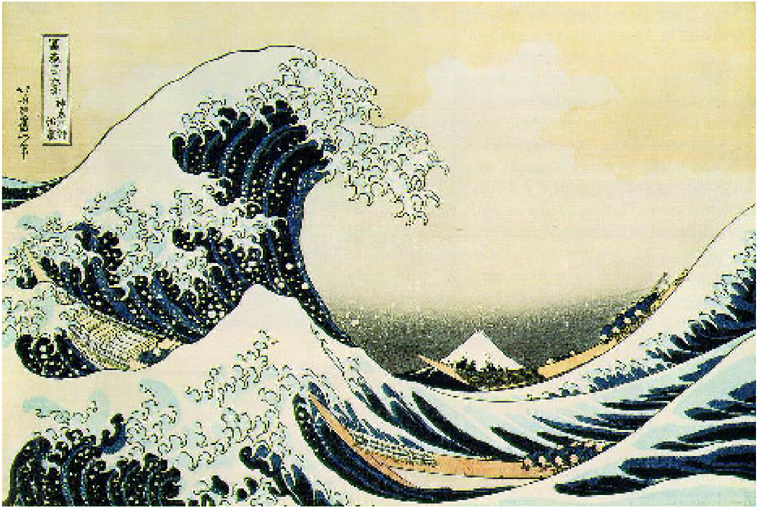

Thank you