Dispersive estimates for the Schrödinger equation on 2-step stratified Lie groups

Hajer Bahouri

CNRS, Créteil University

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Aim

Dispersive estimates on 2-step stratified Lie groups, for the linear Schrödinger equation

The propagator behaves like

- a wave operator on a space of dimension p

- and like a Schrödinger operator on a space of dimension k

This unusual behavior makes the analysis of the solutions tricky and gives rise to uncommon dispersive estimates

Examples

- Heisenberg groups

- bi-Heisenberg groups

- H-type groups

- Diamond groups

- Tensor products of such groups,...

Quantum mechanics, crystal theory, complex analysis, number theory, CR manifolds, ...

Dispersive estimates

Crucial role in the study of evolution nonlinear PDEs

Decay estimate for the L^∞ norm of the solution at time t in terms of

- some negative power of t

- and the L^1 norm of the data

Combined with an abstract functional analysis argument TT^*-argument

Strichartz estimates

space-time Lebesgue norms of the solutions

- gain of some derivative

- and a decrease effect at infinity

Strichartz estimates in the Euclidean space

$\mathbb{R}^d$: constant coefficients

- Brenner, Pecher, Segal, Strichartz,... 1975

- Ginibre-Velo 1995

- Keel-Tao 1998

$\mathbb{R}^d$: variable coefficients

- Kapitanski 1990

- Smith 1998

- Bahouri-Chemin, Klainerman-Rodnianski, Smith, Szeftel,Tataru,...1999

Some generalizations

In domains of $\mathbb{R}^d$

Burq-Lebeau-Planchon, Ivanovici-Lebeau-Planchon, Smith-Sogge,...

In non euclidean settings

- Real hyperbolic spaces : Anker, Banica, Pierfelice, Tataru,...

- Compact and nocompact manifols : Anton, Banica-Duyckaerts, Blair-Smith-Sogge, Burq-Gérard-Tzvetkov, Herr-Tataru-Tzvetkov,...

Others

Frank-Lewin-Lieb-Seiringer, Moyua-Vargas-Vega,..

Dispersive estimates in $\mathbb{R}^d$ frame

Linear Schrödinger equation

$$(S)\begin{cases} (i\partial_t - \Delta)\, u & = & 0 \\ u_{|t=0} & = & u_0 \in L^1(\mathbb{R}^d). \end{cases}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{\|u_0\|_{L^1(\mathbb{R}^d)}}{|t|^{\frac{d}{2}}}\,, \quad t \neq 0\,.$$

Linear wave equation

$$(W)\begin{cases} \left(\partial_t^2 - \Delta\right) u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) \text{ spectrally localized in a ring.} \end{cases}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{\frac{d-1}{2}}} \left(\|u_0\|_{L^1(\mathbb{R}^d)} + \|u_1\|_{L^1(\mathbb{R}^d)}\right)\,, \quad t \neq 0\,.$$

Optimality

Other equations : transport equations,...

Strategy of proof

Schrödinger equation

$$u(t,\cdot) = \frac{1}{(-2i\pi t)^{\frac{d}{2}}} \, \mathrm{e}^{-i\frac{|\cdot|^2}{4t}} \star u_0 \,, \qquad t \neq 0$$

Young inequalities

Wave equation

$$u(t,\cdot) = u^+(t,\cdot) + u^-(t,\cdot)\,,$$

where

$$u^\pm(t,x) = (2\pi)^{-d} \int \mathrm{e}^{ix\cdot\xi}\, \mathrm{e}^{\pm it|\xi|} \widehat{\gamma^\pm}(\xi)\, d\xi$$

with

$$\widehat{\gamma^\pm}(\xi) = \frac{1}{2}\Big(\widehat{u_0}(\xi) \pm \frac{1}{i|\xi|}\widehat{u_1}(\xi)\Big)\,.$$

Stationary phase theorem

Key point : estimate for $t > 0$ in $L^\infty(\mathbb{R}^d)$ of

$$K^{\pm}(t,x) = \int e^{ix\cdot\xi}\, e^{\pm it|\xi|}\varphi(\xi)\, d\xi\,,$$

with φ supported in a ring $\mathcal{C}$ (spectrally localization of the datas in a ring).

$$K^{\pm}(t,tx) = K_1^{\pm}(t,tx) + K_2^{\pm}(t,tx)\,,$$

where

$$K_2^{\pm}(t,tx) = \int_{\mathcal{C}_x} e^{it(x\cdot\xi\pm|\xi|)}\,\varphi(\xi)\, d\xi$$

with

$$\mathcal{C}_x = \Big\{\xi \in \mathcal{C}; \Big|x \pm \frac{\xi}{|\xi|}\Big| \leq \frac{1}{2}\Big\}.$$

Note that if $\mathcal{C}_x$ is not empty, then $x \neq 0$.

Integrating by parts with respect to

$$\mathcal{L} = \frac{\mathrm{Id} - i\big(x \pm \frac{\xi}{|\xi|}\big)\cdot\nabla_\xi}{1 + t\Big|x \pm \frac{\xi}{|\xi|}\Big|^2}$$

which satisfies

$$\mathcal{L}\mathrm{e}^{it(x\cdot\xi \pm |\xi|)} = \mathrm{e}^{it(x\cdot\xi \pm |\xi|)}\,,$$

we infer that

$$\Big|K_2^{\pm}(t,tx)\Big| \leq \int_{\mathcal{C}_x} \Big|({}^t\mathcal{L})^N \varphi(\xi)\Big|\,d\xi \lesssim \int_{\mathcal{C}_x} \frac{1}{\Big(1 + t\Big|x \pm \frac{\xi}{|\xi|}\Big|^2\Big)^N}\,d\xi\,.$$

Change of variables

We write for any $\xi \in \mathcal{C}_x$ the following orthogonal decomposition :

$$\xi = \zeta_1 + \zeta' \quad \text{with} \quad \zeta_1 = \Big(\xi\Big|\frac{x}{|x|}\Big)\frac{x}{|x|} \quad \text{and} \quad \zeta' = \xi - \Big(\xi\Big|\frac{x}{|x|}\Big)\frac{x}{|x|}.$$

Thus

$$\left|x \pm \frac{\xi}{|\xi|}\right| \geq \frac{|\zeta'|}{|\xi|}$$

and

$$\left|K_2^{\pm}(t,tx)\right| \lesssim \int_{\mathcal{C}} \frac{1}{(1+t|\zeta'|^2)^N}\, d\zeta' d\zeta_1 \lesssim \frac{1}{t^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} \frac{1}{(1+|\eta|^2)^N}\, d\eta$$

N large enough

$$\|K_2^{\pm}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{\frac{d-1}{2}}}\,, \quad t \neq 0\,.$$

Along the same lines, integrating by parts with respect to

$$\tilde{\mathcal{L}} = \frac{(x \pm \frac{\xi}{|\xi|}) \cdot \nabla_\xi}{t\left|x \pm \frac{\xi}{|\xi|}\right|^2}$$

ensures that

$$\|K_1^{\pm}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^N}\,, \quad \forall N \quad t \neq 0\,.$$

Dispersive estimates on stratified Lie groups

Heisenberg groups $\mathbb{H}^d$ ($\mathbb{R}^{2d+1}$)

- Sublaplacian : Bahouri-Gérard-Xu 2000

- Full laplacian : Furioli-Melzi-Veneruso 2007

H-type groups ($\mathbb{R}^{2d+p}$)

Del Hierro 2005

Heisenberg and H-type groups are particular cases of 2-step stratified Lie groups respectively of center of dimension 1 and p.

Basic facts about the Heisenberg group

We see $\mathbb{H}^d = \mathbb{R}^{2d+1}$ as

$$\mathbb{H}^d = T^\star \mathbb{R}^d \times \mathbb{R}\,, w = (Y, s) = (y, \eta, s) \in \mathbb{H}^d$$

The non commutative law of product is

$$\begin{aligned} w \cdot w' &= \Big(y + y', \eta + \eta', s + s' + 2\sigma(Y, Y')\Big) \quad \text{with} \\ \sigma(Y, Y') &= \langle \eta, y' \rangle - \langle \eta', y \rangle \quad \text{the symplectic form.} \end{aligned}$$

A vector field X is said left invariant if

$$(Xf) \circ \tau_h = X(f \circ \tau_h)$$

where τ_h is the left translate on $\mathbb{H}^d$ defined by $\tau_h(f)(g) = f(h \cdot g)$, which by elementary differential calculus ensures that

$$\forall w \in \mathbb{H}^d,\ \mathcal{X}(w) = D\tau_w \mathcal{X}(0).$$

This implies that the left invariant vector fields are generated by

$$X_j = \partial_{y_j} + 2\eta_j \partial_s\,, \ \Xi_j = \partial_{\eta_j} - 2y_j \partial_s \quad \text{and} \quad \partial_s.$$

Several objects are linked to this group : Haar measure, Convolution product with Young's inequalities, Sobolev spaces, Littlewood-Paley theory,...

Equations involved a sublaplacian

Sublaplacian

$$\Delta_{\mathbb{H}^d} = \sum_{j=1}^{d} (X_j^2 + Y_j^2)$$

Linear Schrödinger equation on $\mathbb{H}^d$

$$\begin{cases} \left(i\partial_t - \Delta_{\mathbb{H}^d}\right) u &= 0 \\ u_{|t=0} &= u_0 \end{cases}$$

Linear wave equation on $\mathbb{H}^d$

$$\begin{cases} \left(\partial_t^2 - \Delta_{\mathbb{H}^d}\right) u &= 0 \\ (u, \partial_t u)_{|t=0} &= (u_0, u_1) \end{cases}$$

Known results

Heisenberg groups $\mathbb{H}^d$ $(\mathbb{R}^{2d+1})$

- No dispersion for the Schrödinger equation

- Dispersive estimates for the wave equation with an optimal rate of decay of order $|t|^{-\frac{1}{2}}$

H-type groups $(\mathbb{R}^{2d+p})$: multidimensional version of $\mathbb{H}^d$

- Dispersive estimates for the Schrödinger equation with an optimal rate of decay of order $|t|^{-\frac{p-1}{2}}$

- Dispersive estimates for the wave equation with an optimal rate of decay of order $|t|^{-\frac{p}{2}}$

For the Heisenberg and H-type groups

- Only the dimension of the center intervenes.

- Compared with the $\mathbb{R}^d$ framework, there is an exchange in the rates of decay between the wave and the Schrödinger equations.

Optimality.

For general 2-step stratified Lie groups

The optimal rate of decay is not always in accordance with the dimension of the center as it is the case for Heisenberg and H-type groups.

Idea of proof for $\mathbb{H}^d$

Similar to the $\mathbb{R}^d$ case but much more technical

- stationary phase theorem

- an explicit representation of the solution coming from Fourier analysis

Fourier transform on $\mathbb{H}^d$

- the set of *characters* of $\mathbb{H}^d$ is isomorphic to $\mathbb{R}^{2d}$

- defined using irreducible unitary representations of $\mathbb{H}^d$

- family of bounded operators

- inversion and Fourier-Plancherel formulas

- the Fourier transform exchanges convolution and composition

- no formula for the product

Fourier transform on $\mathbb{H}^d$ and sublaplacian

Fourier transform

For $f \in L^1(\mathbb{H}^d)$, we define

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{H}^d} f(w) u_w^\lambda dw,$$

where u^λ called the Schrödinger representation is a continuous group homomorphism from $\mathbb{H}^d$ to $GL(L^2(\mathbb{R}^d))$ and λ belongs to the dual of the center of $\mathbb{H}^d$, which implies that

$$\|\mathcal{F}^{\mathbb{H}}(f)(\lambda)\|_{\mathcal{L}(L^2)} \leq \|f\|_{L^1(\mathbb{H}^d)}.$$

If $f \in \mathcal{S}(\mathbb{H}^d) = \mathcal{S}(\mathbb{R}^{2d+1})$, then we have the inversion formula (series)

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} (u_{w^{-1}}^\lambda \mathcal{F}(f)(\lambda) H_{\alpha,\lambda}, H_{\alpha,\lambda})_{L^2(\mathbb{R}^d)} |\lambda|^d d\lambda,$$

where $H_{\alpha,\lambda}(x) = |\lambda|^{\frac{d}{4}} H_\alpha(|\lambda|^{\frac{1}{2}} x)$, with $(H_\alpha)_{\alpha \in \mathbb{N}^d}$ the Hermite functions.

Fourier transform and sublaplacian

In the case of $\mathbb{R}^d$, we have

$$\mathcal{F}_{\mathbb{R}^d}(-\Delta u)(\xi) = |\xi|^2 \widehat{u}(\xi).$$

In the case of $\mathbb{H}^d$, we have

$$\mathcal{F}(-\Delta_{\mathbb{H}^d} u)(\lambda) = 4\mathcal{F}(u)(\lambda) \circ \Delta^{\lambda}_{\text{osc}},$$

with $\Delta^{\lambda}_{\text{osc}}\phi(x) = \Delta\phi(x) - \lambda^2|x|^2\phi(x)$, which gives the spectral representation of $\Delta_{\mathbb{H}^d}$.

Since $\Delta^{\lambda}_{\text{osc}} H_{\alpha,\lambda} = |\lambda|(2|\alpha| + d)H_{\alpha,\lambda}$, we get

$$\mathcal{F}(-\Delta_{\mathbb{H}^d} u)(\lambda) H_{\alpha,\lambda} = 4|\lambda|(2|\alpha|+d)\,\mathcal{F}(f)(\lambda) H_{\alpha,\lambda}.$$

Intuitively, $|\lambda|^{\frac{1}{2}}$ plays the role of the frequency.

Representation of the solution of the Wave equation

Under spectral localization of the Cauchy data

The solution takes the form :

$$u(t,\cdot) = K^{+}(t,\cdot) \star \gamma^{+} + K^{-}(t,\cdot) \star \gamma^{-},$$

with $\gamma^{\pm} = \frac{1}{2}(u_0 \pm i(-\Delta_{\mathbb{H}^d})^{-\frac{1}{2}} u_1)$ and

$$K^{\pm}(t,\cdot) = c_d \sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} (u^{\lambda}_{w^{-1}} \mathrm{e}^{\pm it\sqrt{4|\lambda|(2|\alpha|+d)}} \theta\Big(4|\lambda|(2|\alpha|+d)\Big) H_{\alpha,\lambda}, H_{\alpha,\lambda})_{L^2} |\lambda|^d d\lambda,$$

with θ a smooth radial function compactly supported in a ring $\mathcal{C}$.

Spectral localization of data : $\gamma = \chi\Big(-\Delta_{\mathbb{H}^d}\Big)\gamma\,,\ \chi \in \mathcal{D}$.

$$\mathcal{F}(\gamma)(\lambda) H_{\alpha,\lambda} = \chi\Big(4|\lambda|(2|\alpha|+d)\Big)\, \mathcal{F}(\gamma)(\lambda) H_{\alpha,\lambda}$$

Young's inequalities : L^∞-norm of $K_\pm(t,\cdot)$

Difficulty : series

Ingredients

- accurate stationary phase result
- spectral localization $|\lambda| \sim \frac{1}{|\alpha|}$
- properties of Hermite functions

H-type groups : same arguments

- $\lambda \in \mathbb{R}^p \setminus \{0\}$
- Isotropy (with respect to the dual variable of the center)

General case of 2 step-stratified Lie groups G

Dispersion with an optimal rate of decay of order $|t|^{-\frac{p-1}{2}}$?

where p is the dimension of the center of G

The answer is no in general

Examples of 2 step-stratified Lie groups with center of dimension n

but non dispersion :

tensoriel product of Heisenberg groups.

Examples of 2 step-stratified Lie groups with center of dimension 1

dispersion with rate of decay of order $|t|^{-\frac{m}{2}}$, with m an arbitrary integer :

Diamond groups.

Difficulties of the general case

New variable in the parameterization of the irreducible unitary representations of the group

- λ in the dual of the center

- ν in the radical of the canonical skew-symmetric form

The isotropy is no longer provided in the general case

- nor for the action of the sublaplacian

- neither for the irreducible unitary representations of the group

Mixed behavior of the propagator

both like a wave operator and a Schrödinger operator

Optimal decay

The propagator behaves like

- a wave operator with respect to the variable λ dimension p

- a Schrödinger operator with respect to the variable ν dimension k

The expected result would be a dispersion phenomenon with an optimal rate of decay of order

$$|t|^{-\frac{k+p-1}{2}}$$

Result (sharp)

- Assumption on the action of the sublaplacian : maximal rank property

- Strict spectral localization (with respect λ)

Basic facts about 2-step stratified Lie groups

A connected, simply connected nilpotent Lie group G is a step 2 stratified Lie group if its left-invariant Lie algebra $\mathfrak{g}$ is endowed with the vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}\,, \quad \text{with} \quad [\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$ of dimension p.

Let $(V_1, \ldots, V_m)$ be an orthonormal basis of $\mathfrak{v}$, the sublaplacian on G writes

$$\Delta_G = \sum_{j=1}^{m} V_j^2.$$

Identification of G and $\mathfrak{g}$ via :

$$\exp \ : \ \mathfrak{g} \ \to \ G$$

The group law on G provided by the Campbell-Baker-Hausdorff formula is a polynomial map.

Several objects are linked to this group : dilations, Haar measure, Convolution product, Sobolev spaces,...

Fourier transform on 2-step stratified Lie groups

Basic properties

$$\mathcal{F}(f)(\lambda,\nu) = \int_G f(x) u^{\lambda,\nu}_{X(\lambda,x)}\, dx$$

- family of bounded operators defined by means of $(u^{\lambda,\nu}, L^2)$

- inversion and Fourier-Plancherel formulas

- exchange convolution and composition

2 variables in the parameterization

- λ in the dual of the center

- ν in the radical of the canonical skew-symmetric form $B(\lambda)$

$$\forall\, U, V \in \mathfrak{v}\,, \quad B(\lambda)(U,V) = \lambda([U,V])$$

Ciatti-Ricci-Sundari, Corwin-Greenleaf, Müller-Ricci.

Key formula : the action of the Sublaplacien is decoupled

$$\mathcal{F}(-\Delta_G f)(\lambda,\nu) = \mathcal{F}(f)(\lambda,\nu)\left(H(\lambda) + |\nu|^2\right)$$

- $|\nu| = \sqrt{\nu_1^2 + \cdots + \nu_k^2}$

- $H(\lambda)$ is the diagonal operator defined on $L^2(\mathbb{R}^d)$

$$H(\lambda)h_{\alpha,\eta(\lambda)} = \zeta(\alpha,\lambda)\, h_{\alpha,\eta(\lambda)} = \sum_{j=1}^{d}(2\alpha_j+1)\eta_j(\lambda)\, h_{\alpha,\eta(\lambda)}\,,$$

- each $\eta_j(\lambda) > 0$ (smooth on a Zariski-open subset of $\mathfrak{z}^\star$) is homogeneous of degree one in λ

- $h_{\alpha,\eta(\lambda)}(\xi) = \prod_{j=1}^{d}\eta_j(\lambda)^{\frac{1}{4}} h_{\alpha_j}\Big(\eta_j(\lambda)^{\frac{1}{2}}\xi\Big)$, with $(h_n)_{n\in\mathbb{N}}$ the basis of Hermite functions normalized in $L^2(\mathbb{R})$

$$\eta_j(\lambda)?$$

The action of the sublaplacian

- homogeneous of degree 1 with respect to λ (in $\mathbb{R}^p$)

- homogeneous of degree 2 with respect to ν (in $\mathbb{R}^k$)

Mixed behavior for the Schrödinger propagator

This unusual behavior in the case when the radical is not trivial of the Schrödinger propagator makes the analysis of the explicit representation of the solution tricky and gives rise to uncommon dispersive estimates

Main result

Linear Schrödinger equation on G

$$\begin{cases} (i\partial_t - \Delta_G)\, u = 0 \\ u_{|t=0} = u_0\,. \end{cases}$$

The expected rate of decay would be of order

$$|t|^{-\frac{k+p-1}{2}}$$

Rank assumption

$\forall \alpha$ in $\mathbb{N}^d$, the Hessian matrix of the map $\lambda \mapsto \zeta(\alpha, \lambda)$ satisfies

$$\operatorname{rank} D^2_\lambda \zeta(\alpha, \lambda) = p - 1$$

$\zeta(\alpha, \lambda)$ is homogeneous of degree one in λ, thus $D^2_\lambda \zeta(\alpha, \lambda)\lambda = 0$.

Hence the assumption may be understood as a maximal rank property.

Statement of the result

Let G be a step-2 stratified Lie group with center of dimension p and a radical index k. Assume that the rank assumption holds. A constant C exists such that if u_0 belongs to $L^1(G)$ and is strictly spectrally localized in a ring, then the associate solution u to the Schrödinger equation satisfies

$$\|u(t,\cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{\frac{k}{2}}(1+|t|^{\frac{p-1}{2}})}\|u_0\|_{L^1(G)}\,,$$

for all $t \neq 0$ and the result is sharp in time.

Relevance of the maximal rank assumption

$\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2} \cdots \otimes \mathbb{H}^{d_n}$ the tensor product of Heisenberg groups

2 step-stratified Lie group with center of dimension n, but no dispersion

$\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2} \ldots \otimes \mathbb{R}^{m_2+p_n}$ the tensor product of H-type groups

2 step-stratified Lie group with center of dimension $p = p_1 + \cdots + p_n$:

dispersion with an optimal rate of decay of order $|t|^{-\frac{p-n}{2}}$

General result for decomposable groups $G = \otimes_{1 \leq m \leq r} G_m$

dispersion with an optimal rate of decay of order $|t|^{-\frac{k+p-r}{2}}$

Idea of proof

In the spirit of the $\mathbb{H}^d$ case but much more technical

- stationary phase theorem

- an explicit representation of the solution

Inversion formula and action of the sublaplacian

$$u(t,x) = \kappa \int_\lambda \int_\nu \mathrm{tr}\left(u^{\lambda,\nu}_{X(\lambda,x^{-1})} \mathcal{F}(u_0)(\lambda,\nu)\, \Theta(\lambda)\, \mathrm{e}^{it|\nu|^2+itH(\lambda)} \right) |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda\,,$$

where $|\mathrm{Pf}(\lambda)| = \prod_{j=1}^d \eta_j(\lambda)$ is the Pfaffian of $B(\lambda)$ and

$$u^{\lambda,\nu}_X \phi(\xi) = \mathrm{e}^{-i\nu\cdot R - i\lambda\cdot Z + \sum_{1\leq j\leq d} \eta_j(\lambda) Q_j \left(\xi_j + \frac{1}{2} P_j\right)} \phi(P+\xi)\,,$$

for $X = (P,Q,R,Z)$ and $\phi \in L^2(\mathbb{R}^d)$.

Reduction to the estimate of the kernel

$$K(t,x) = \int_\lambda \int_\nu \mathrm{tr}\left(u^{\lambda,\nu}_{X(\lambda,x^{-1})} \Theta(\lambda)\, \mathrm{e}^{it|\nu|^2+itH(\lambda)} \right) |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda\,,$$

Separation of the variables λ and ν

- gain of decay rate of order $|t|^{-\frac{k}{2}}$

- reduce to investigate

$$\tilde{K}(t,x) = \sum_{\alpha \in \mathbb{N}^d} \int \mathrm{e}^{it\Phi_\alpha(Z,\lambda)} G_\alpha\Big(P, Q, \eta(\lambda)\Big) |\mathrm{Pf}(\lambda)|\, d\lambda\,,$$

$$\Phi_\alpha(Z,\lambda) = \zeta(\alpha,\lambda) - \lambda \cdot Z\,,$$

and G_α defined from Hermite functions and satisfying good properties. It is related to Wigner transform.

Series

$$I_\alpha(Z) = \int \mathrm{e}^{it\Phi_\alpha(Z,\lambda)} G_\alpha\Big(P, Q, \eta(\lambda)\Big) |\mathrm{Pf}(\lambda)|\, d\lambda$$

Strict spectral localization and homogeneity

$$I_\alpha(Z) = m^{-p-d} \int_{\mathcal{C}} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} \psi_\alpha(\gamma)\, d\gamma\,,$$

where $m = |\alpha|$ and $\mathcal{C}$ a ring.

Key point : investigate the integral near the critical point

$$\tilde{I}_\alpha(Z) = \int_{\mathcal{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} \psi_\alpha(\gamma)\, d\gamma\,,$$

where $\mathcal{C}_\alpha(Z) := \Big\{\gamma \in \mathcal{C};\, |\nabla_\gamma(\Phi_\alpha(Z, \dfrac{\gamma}{m}))| \leq c_0\Big\}$.

Stationary phase theorem

Integration by parts with respect to

$$\mathcal{L}_\alpha = \frac{\mathrm{Id} - i\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))\cdot\nabla_\gamma}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2}$$

which satisfies

$$\mathcal{L}_\alpha \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})} = \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}\,,$$

we get

$$\tilde{I}_\alpha(Z) = \int_{\mathcal{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\frac{\gamma}{m})}({}^t\mathcal{L}_\alpha)^N \psi_\alpha(\gamma)\,d\gamma\,.$$

Classical arguments combined with properties of G_α reduce the problem to the study of

$$\int_{\mathcal{C}_\alpha(Z)} \left(1+t|\nabla_\gamma(\Phi_\alpha(Z,\frac{\gamma}{m}))|^2\right)^{-N} d\gamma\,.$$

Use of the maximal rank assumption to end the proof.

Change of variables

$$\begin{aligned} \mathcal{C}_\alpha(Z) &\longmapsto K \\ \gamma &\longmapsto \tilde{\gamma}\,, \end{aligned}$$

satisfying $|\nabla_\gamma(\Phi_\alpha(Z, \frac{\gamma}{m}))| \geq |\tilde{\gamma}'|$, with $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}')$.

Orthogonal decomposition

$$\frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) = \tilde{\Gamma}_1\, Z_1 + \tilde{\Gamma}' \text{ with } \tilde{\Gamma}_1 = \Big(\frac{1}{m}\nabla_\gamma\zeta(\alpha,\gamma)\Big|\hat{Z}_1\Big) \text{ and } Z_1 = \frac{Z}{|Z|}.$$

We know that if $\mathcal{C}_\alpha(Z)$ is not empty then $Z \neq 0$.

Orthogonal basis $(Z_1, \ldots, Z_p)$

$$\mathcal{C}_\alpha(Z) \ni \gamma \longmapsto \mathcal{H}(\gamma) = (\gamma \cdot Z_1)\, Z_1 + \sum_{k=2}^{p} (\tilde{\Gamma}' \cdot Z_k)\, Z_k = \sum_{k=1}^{p} \tilde{\gamma}'_k\, Z_k\,.$$

$\mathcal{H}$ realizes a diffeomorphism under maximal rank assumption.

Strichartz estimates

Derive from dispersive estimates

General 2-step stratified Lie groups

- Strict spectral localization

- Anisotropic framework

Extend Fourier tranform

Work in progress with Jean-Yves Chemin and Raphael Danchin

- Function point of view of the Fourier transform on $\mathbb{H}^d$

- Characterize the range of $\mathcal{S}(\mathbb{H}^d)$

- Extension of Fourier transform on $\mathcal{S}'(\mathbb{H}^d)$

The Fourier transform as a function

Let us denote $\widehat{\mathbb{H}}^d = \mathbb{N}^{2d} \times \mathbb{R}$ and $\widehat{w} = (n, m, \lambda)$ a generic point of $\widehat{\mathbb{H}}^d$.

Definition *For f in $L^1(\mathbb{H}^d)$, we define $\widehat{f}^{\mathbb{H}}$ as the following function*

$$\mathcal{F}_{\mathbb{H}}(f) : \begin{cases} \widehat{\mathbb{H}}^d & \longrightarrow \quad \mathbb{C} \\ \widehat{w} & \longmapsto \quad \begin{cases} \Big(\mathcal{F}^{\mathbb{H}}(f)(\lambda) H_{m,\lambda} | H_{n,\lambda}\Big)_{L^2} & \text{if } \lambda \neq 0, \\ \delta_{n,m} \displaystyle\int_{\mathbb{H}^d} f(w)\, dw & \text{if } \lambda = 0, \end{cases} \end{cases}$$

where $\delta_{n,m} \overset{\text{def}}{=} 0$ if $n \neq m$ and $\delta_{n,n} \overset{\text{def}}{=} 1$.

Main point : endow the space $\widehat{\mathbb{H}}^d$ with a topology which enables us to recover almost all the classical.

Note that the variable $\widehat{w}$ in $\widehat{\mathbb{H}}^d$ plays the same role as the variable ξ in $\mathbb{R}^d$.

- On $\mathbb{N}^{2d} \times (\mathbb{R} \setminus \{0\})$, the topology induced by $\mathbb{R}^{2d+1}$ is suitable.

- On $\mathbb{N}^{2d} \times \{0\}$, the topology induced by $\mathbb{R}^{2d+1}$ is not appropriate.

The essentiel point is to understand the structure of $\widehat{\mathbb{H}}^d$ near $\lambda = 0$

By definition, $\widehat{\mathbb{H}}^d$ on $\lambda = 0$ reduces to two cases :

- The case of points $(n, n, 0)$ where $\mathcal{F}_{\mathbb{H}}(f)$ takes the value $\displaystyle\int_{\mathbb{H}^d} f(w)\, dw.$

- The case of points $(n, m, 0)$ with $n \neq m$, where $\mathcal{F}_{\mathbb{H}}(f)$ is null.

We have considerd a topology which identifies these two types of points.

The wright picture of $\widehat{\mathbb{H}}^d$ is not the collection of parallelle lines in $\mathbb{R}^{2d+1}$ but the union of two disjoints cones consiting in a countable set of lines.

The map

$$\mathcal{F}_{\mathbb{H}} : \mathcal{S}(\mathbb{H}^d) \longrightarrow \mathcal{S}(\widehat{\mathbb{H}}^d)$$

is a bicontinuous isomorphism.

Extend the Fourier transform on $\mathcal{S}'(\mathbb{H}^d)$.

Applications :

- heat kernel belongs to $\mathcal{S}(\mathbb{H}^d)$

- the operator $g(-\Delta_{\mathbb{H}})$ is the convolution operator by a function of $\mathcal{S}(\mathbb{H}^d)$, ...

Not easy to extend this point of view to 2-step stratified Lie groups.