Malliavin Calculus of Bismut type for an operator of order four on a Lie group.

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1 Introduction

This talk enters in the philosophy of the beautifull probabilistic index theory of Bismut. With path integrals, we see the formulas which are simpler to check by usinf the theory of parabolic equations when we have them.

Let L be a generator on a space of functions f on a Riemannian manifold M which generates a semi-group P_t:

$$\frac{\partial}{\partial t}P_t f = -LP_t f \tag{1}$$

A natural question is to know if the semi-group has an heat-kernel

$$P_t f(x) = \int_M f(y) p_t(x,y) dy \qquad (2)$$

where dy is the Riemannian measur (which is unique modulo a multiplicative constant).

There are 3 approaches to solve this problem:

-)The microlocal analysis which uses the Fourier transform as a tool.

-)The Harmonic analysis, which uses functional inequalities.

-)The Malliavin Calculus.

The Malliavin Calculus deals only for Markov semigroups: $P_t f \geq 0$ if $f \geq 0$. The others approaches work for a broader class of generator. The object of this work is to fullfill this gap.

Malliavin did a breakdown in stochastic analysis. There a lot of preliminary versions of the Malliavin Calculus, motivated by mathematical physics. See works of

Hida , Albeverio-Hoegh-Krohn, Berezanskii for instance. Malliavin approach to hypoellipticity problem uses an heavy apparatus of differential operations in infinite dimension. Bismut has simplified this approach. In Bismut's approach, only convenient stochastic differential equations appear, therefore convenient semi-groups. This allows us to translate Bismut's approach . We have done some reviews and applications to heat-kernel estimates.

Stochastic analysis works for the whole process. Formulas of stochastic analysis were interpreted by ourself in semi-group theory but they are only valid for the semi-groups. This remark allows us to extend a lot formulas of stochastic analysis. We have done several works for Wentzell-Freidlin estimates for non Markovian semi-group.

In a first stepextend the Malliavin Calculus of Bismut type for a four order operator. In such a case, the semi-group associated do not preserves the positivity.

Let G be a compact Lie group of dimension m which can be seen as a subgroup of the special orthogonal group. Let e_i be an orthonormal basis of its Lie algebra endowed with its biinvariant Euclidean structure. If g is a matrix, e_i can be seen either as a matrix in the tangent space of e, the unit element of G or as $e_i g$ a vector at the tangent space of g. We consider the generator

$$L = \sum_{i=1}^{m} e_i^4 \tag{3}$$

It is an elliptic operator which generates by elliptic theory a semi-group on $C_b(G)$, the space of continuous functions on G endowed with the uniform norm.

Theorem 1 *If $t > 0$, P_t has an heat-kernel*

$$P_t f(g) = \int_G p_t(g, g') dg' \tag{4}$$

where dg' is the Haar measure on G.

If this theorem is rather standard in Analysis, it enters in our general programm to extend stochastic analysis tools in the general theory of parabolic equation.

In the paper where we have translated Malliavin Calculus of Bismut type for diffusion in semi-group theory, we have given the formulas on diffusion processes which lead to the translation of the Malliavin Calculus of Bismut type for diffusion. Here, it is not rigorously possible, but it should possible to repeat in an heuristic way these considerations.

We consider a "process" $b_{i,t}$ on $\mathbb{R}^m$ associated to the driving operator $\sum \frac{\partial^4}{\partial x_i^4} = L^b$ on $\mathbb{R}^m$ such that formally

$$\int_{E[0,\infty[,\mathbb{R}^m]} f(b_t)\text{"}d\mu\text{"} = P_t^b f(0) \qquad (5)$$

where P_t^b is the semi-group associated to L^b and "$d\mu$" the " law" of b_t on a convenient space $E[0,\infty[,\mathbb{R}^m]$ on trajectories of b_t. Of course, these considerations are purely formal, because mainly P_t^f does not preserves positivity.

Suppose that formally we could solve the "linear stochastic differential equattion" starting from g

$$dg_t = \sum_{i=1}^{m} e_i g_t db_{i,t} \tag{6}$$

such that

$$P_t f(g) = \int_{E[0,\infty[,\mathbb{R}^m]} f(g_t)\text{"}d\mu\text{"} \tag{7}$$

Malliavin's theorem state that in order hypothesis the map $b_. \to g_t$ is in some "generalized sense a submersion", and that the Malliavin matrix generlized in this context the classical notion of Gram matrix. Namely, if we perturb $db_{i,t}$ into $db_{i,t} + \lambda h_{i,t}dt$, we get formally

$$d_t \frac{\partial}{\partial \lambda} g_t^0 = \sum_1^m e_i \frac{\partial}{\partial \lambda} g_t^0 db_{i,t} + \sum_1^m e_i g_t h_{i,t} dt \tag{8}$$

which can be solved by using the method of variation of constant as

$$\frac{\partial}{\partial \lambda} g_t = g_t \sum_0^m \int_0^t g_s^{-1} e_i g_s dh_{i,s} \tag{9}$$

(We refer to the review [9] in the case of the Brownian motion where these considerations are rigorously performed).

2 The theorem of Malliavin for an operator of order four

We consider the elliptic operator on $G \times \mathbb{R}$

$$\sum_i^m e_i^4 + \sum h_{i,t} e_i \frac{\partial}{\partial u} + \frac{\partial^4}{\partial u^4} = \tilde{L}_t^h \tag{10}$$

It generates by perturbation theory a semi-group (but not a contraction semi-group) on $L^2(G \times \mathbb{R})(dg \otimes du)$ and even a semi-group on $C_b(G \times \mathbb{R})$ by ellipticity.

Theorem 2 (Elementary integration by parts formula). *We have if f is smooth with compact support*

$$\int_0^t P_{t-s} \sum h_{s,i} e_i P_s[f] ds = \tilde{P}_t^h[uf](.,0) \tag{11}$$

Proof: Let us begin by formal computations. Since $\frac{\partial}{\partial u}$ commutes with $\tilde{L}_t^h$, we get:

$$\tilde{P}_t^h[uf](g,u_0) = P_t[f](g)u_0 + \tilde{P}_t[uf](g,0) \tag{12}$$

such that

$$\frac{\partial}{\partial t}\tilde{P}_t^h[uf](g,u_0) = -L\tilde{P}_t^h[uf](g,0) - \sum h_{i,t} e_i P_t[f](x) \tag{13}$$

It is the same parabolic equation with second member as the equation satisfied by the lefthand side of (11).

In order to finish proof, it remain to show that the enlarged semi group can act on the test function $f(x)u$. For that we use the classical Davies gauge transform instead of the Burkholer-Davies-Gundy inequality which work only for Markovian semi-groups.. We consider $g(u) = c(|u|^{2r}+1)$ with a big integer r. The main remark is that

$$g(u)^{-1}\frac{\partial}{\partial u}(g(u).) = \frac{\partial}{\partial u} + C(u) \qquad (14)$$

where C is smooth bounded. Therefore $g(u)^{--1}\tilde{L}_t^h(g(u).)$ spanns a semi-group on $L^2(G \times \mathbb{R})(dg \otimes du)$ which is equal to $g(.)^{-1}\tilde{P}_t^h g(.)$. The result arises because $\frac{u}{g(u)}f(x)$ belongs to $L^2(G \times \mathbb{R})(dg \otimes du)$. $\diamondsuit$

Let $V = G \times M_m$. M_m is the space of symmetric matrices on $LieG$. $(x, v) \in V$. v is called the Malliavin matrix. We consider

$$\hat{X}_0 = (0, \sum < g^{-1}e_i, . >^2) \qquad (15)$$

We consider the Malliavin generator

$$\hat{L} = \sum e_i^4 - \hat{X}_0 \qquad (16)$$

Theorem 3 *$\hat{L}$ spanns a semi-group $\hat{P}_t$ called the Malliavin semi-group on $C_b(M)$.*

Proof: $\hat{L}$ is not a perturbation of an elliptic operator on V, because $\sum e_i^4$ is not an elliptic operator on V. But we can apply the Itô-Stratonovitch formula for general semi-group o to conclude. We consider the flow ϕ_t generated by $\hat{X}_0$ on V. We put $\Phi_t[f](x,u) = f(x, \phi_t v)$. ϕ_t is an isometry of M. Therefore $\Phi_t e_i \Phi_t^{-1} = e_{i,t}$ is without divergence on M and $\sum e_{i,t}^4$ generates a contraction semi-group on $L^2(M)(dg \otimes dv)$. Let us put

$$C(g) = \sum < g^{-1} e_i, . >^2 \qquad (17)$$

If $\hat{f}(g,v)$ is a shape $f^n(u,v) \exp -g(v)$ where f^n is a polynomial in v and g a positive quadratic form in v, the following Volterra expansion converges:

$$\hat{P}_t[\hat{f}](g_0, v_0) = \hat{f}(u_0, v_0) + \\ \sum \int_{0<s_1<..<s_n<t} P_{s_1}[C(.)P_{s_2-s_1} \\ [C(.)...[P_{t-s_n}[D_v^n f(.,.)](g_0, v_0)ds_1..ds_n \tag{18}$$

We consider a subdivision t_i^l of $[0, \infty[$ of mesh $1/l$. We put $[t]^l$ the closest time of the subdivision by lower values of t. We consider

$$\hat{P}_t^l[\hat{f}](g_0, v_0) = \hat{f}(g_0, v_0) + \\ \sum \int_{0<s_1<..<s_n<t} P_{s_1}[C(g_{[s_1]^l}) \\ P_{s_2-s_1}..C(g_{[s_n]^l})[[P_{t-s_n}[D_v^n f(.,.](g_0, v_0)ds_1..ds_n \tag{19}$$

In order to understand this formula, we divide the simplex in small simplices of the s_i between t_i^l and t_{i+1}^l. We recognize in the previous expression the quantity

$$\hat{f}(g_0, v_0) + P_{t_1^l}[P_{t_2^l - t_1^l}[...[P_{t-[t]^l}[\hat{f}(., v_0 + C(g_{t_1^l}(t_2^l - t_1^l) + ..C(g_{[t]^l}(t - [t]^l)](g_0, 0) \tag{20}$$

by applying the Volterra expansion to the semi group

$$\hat{P}_t^{g_0}[\hat{f}](g_0', v_0) = P_t[\hat{f}(., v_0 + C(g_0)t)](g_0')] \tag{21}$$

generated by

$$\hat{L}^{g_0} = L - C(g_0)D_v \tag{22}$$

where g_0 is frozen. We use the formula

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{j=1}^n \prod_{i=1}^{j-1} a_i(a_j - b_j) \prod_{i=j+1}^n b_i \tag{23}$$

and the fact that when $t \to 0$

$$|P_t|[|C(g) - C(g_0)|](g_0) \to 0 \tag{24}$$

iin order to conclude if f if the type before that $\hat{P}_t^l[\hat{f}](g_0, v_0)$ converges to $\hat{P}_t[\hat{f}](g_0, v_0)$.

We deduce that

$$|\hat{P}_t[\hat{f}](g_0, v_0)| \leq C\|\hat{f}|_\infty \qquad (25)$$

if $\hat{f}$ belongs to $L^2(M)(dg \otimes dv)$.

$\diamondsuit$

Proposition 4 *For all r, there exists a tensor polynomial L_i^r in the elements g, e_i, g^{-1} and their derivatives such that*

$$D^r P_t[f](g) = P_t[\sum_{i=1}^{l} D^r f L_i^r](g) \qquad (26)$$

Proof: This comes from the fact that

$$P_t[f](g) = P_t[f(.g)](e) \qquad (27)$$

where e is the unit element of G.

$\diamondsuit$

We consider the generator

$$L_\lambda = \sum e_i^4 - \lambda \sum < \phi(g), h_t >^i e_i \qquad (28)$$

It generates by elliptic theory a semigroup on $C_b(G)$ which depends smoothly on λ.

We denote it by P_t^λ. We get

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \lambda}P_t^0 f = -L\frac{\partial}{\partial \lambda}P_t^0 f + \sum < \phi(g), h_t >^i e_i P_t[f] \quad (29)$$

By the previous theorem

$$e_i P_t[f](g) = P_t[< Df, g^{-1}e_i >](g) =$$

$$\int_G < Df(g'), g'g^{-1}e_i g > dP_t(g, g') \quad (30)$$

$\diamondsuit$

Therefore

$$\frac{\partial}{\partial \lambda}P_t^0[f] = \sum \int_0^t P_{t-s}[< \phi(g'), h_s >^i P_s[< Df, g'^{-1}e_i > \quad (31)$$

Let us consider the vector field on $G \times T_eG$

$$\tilde{e}_i = (e_i, 0) \quad (32)$$

which operates on $C_b(G \times T_eG)$. We consider

$$\tilde{X}_0 = (0, \sum g^{-1}e_i < \phi(g), h_t >^i) \quad (33)$$

We consider the operator on $G \times T_eG$

$$\tilde{L} = L - \tilde{X}_0 \quad (34)$$

As the Malliavin operator, it is not the perturbation of an elliptic operator on $G \times T_eG$. But we can define as we have defined the Malliavin semi-group $\hat{P}_t$ the semi-group associated $\tilde{P}_t$ by using the Itô-Stratonovitch formula for big order semi-groups].

This allows to show

Proposition 5 *We have if f is smooth with compact support*

$$\frac{\partial}{\partial \lambda} P_t^0[f](g_0) = \tilde{P}_t[< Df, gu >](g_0, 0) \tag{35}$$

Proof: If the Volterra expansion converges, we get

$$\tilde{P}_t[< Df, u >](g, 0) = \sum (-1)^n \int_{0<s_1<..<s_n<t} P_{s_1}^0 \tilde{X}_0..\tilde{X}_0...\tilde{X}_0 P_{t-s_n}[< Df, gu >](g_0, 0) \tag{36}$$

But $u_0 \rightarrow P_{t-s_n}[< Df, gu >](g_0, u_0)$ is linear in u_0. So it remains only the first term in the Volterra expansion.

$\diamondsuit$

Remark: This theorem has to be compared with the remark at the end of the introduction, formula (9).

Theorem 6 (Malliavin) *If the Malliavin condition holds*

$$|\hat{P}_t][v^{-p}](g,0) << \infty \tag{37}$$

for all integer positive integer p, P_t has a smooth density.

Proof: If we consider the semi group Q_t associated to the elliptic operator on $G\times\mathbb{R}$

$$\sum e_i^4 - \sum < \phi(g), h_t >^i e_i \frac{\partial}{\partial u} + \frac{\partial^4}{\partial u^4} \tag{38}$$

we get as in the elementary integration by parts of the beginning of this part

$$\tilde{P}_t[< Df, gu >](g_0,0) = Q_t[fu](g_0,0) \tag{39}$$

In this formula, $< Dfg, u >$ is a scalar. We choose $\hat{X}_0 = \sum < ., g^{-1}e_i >< g^{-1}e_i, . >$.

We get for vector valued semi-groups

$$\hat{P}_t[< Df, gv >](g_0, 0) = Q_t[fu](g_0, 0) \tag{40}$$

If

$$|\hat{P}_t|[|v^{-1}|^p](g_0, 0) < \infty \tag{41}$$

for all positive integers p, we apply the previous integration by parts formulas to the test function $(x, v) \rightarrow f(x)v^{-1}g^{-1}e_j$ in order, following the initial idea of Malliavin that

$$|P_t[D^r f L^r](g_0)| \leq C_r \|f\|_\infty \tag{42}$$

where f is the supremum norm of the test function f where L^r is a tensor polynomial in the e_i.

$\diamondsuit$

3 Inversion of the Malliavin matrix

Theorem 7 *Under the previous elliptic assumptions,*

$$|\hat{P}_t|[\hat{f}]](g_0, 0) < \infty \tag{43}$$

if $t > 0$ *where the test function is the inverse of the Malliavin matrix at power* p

Proof: We remark that

$$|\hat{P}_t^l - \hat{P}_t^{g_0}|[\hat{f}](g_0, v_0) \leq CC(t)t\|\hat{f}\|_\infty \quad (44)$$

where $C(t) \to 0$ when $t \to 0$ if if the test function $\hat{f}$ is positive. We use the Volterra expansion associated to these semi-groups and (23), (24) to conclude. We deduce therefore that

$$|\hat{P}_t - \hat{P}_t^{g_0}|[\hat{f}](g_0, v_0) \leq C(t)t\|\hat{f}\|_\infty \quad (45)$$

We remark that

$$\hat{P}_t^l[\hat{f}](g_0, u) = \hat{P}_{t_1^l}^{g_0}[\hat{P}_{t_2^l - t_1^l}^{g_1}..\hat{P}_{t-t_k^l}^{g_k}[\hat{f}]..](g_0, u_0) \quad (46)$$

where $t_k^l = [t]^l$. We deduce that

$$|\hat{P}_t - \hat{P}_t^l|[\hat{f}](g_0, 0) \leq C(l)\|\hat{f}\|_\infty \quad (47)$$

where $C(l) \to 0$ when $l \to \infty$ if the test function $\hat{f}$ is positive. Moreover $C(g) \geq CI_d$. We have therefore

$$|\hat{P}_t^l|[(v^{-1})^p](g_0, 0) \leq C(p) < \infty \quad (48)$$

for all p where $C(p)$ does not depend on l. Therefore the result.

$\diamondsuit$

Proof of theorem1 This comes from theorem 6 and theorem 7.

$\diamondsuit$

4 Varadhan estimates

The object of Varadhan type estimates is to know if Wentzel-Freidlin estimates (large deviation estimates) pass to the density of the involved random variable.

The goal of the Malliavin Calculus is to know if the law of a random variable has a density with respect of the Lebesgue measure.

Bismut [1] in his seminal book pointed out the relationship beteween the Malliavin Calculus and large deviation estimates.

The request of Bismut's book was fullfilled by ourself in our Compte-Rendu [2]. See [3], [4] too.

We have translated in semi group theory Bismut's way of the Malliavin Calculus for Markovian semi-groups. See the review [5],[6] for that.

There are others semi-groups, which are not Markovian, and which are not represented by stochastic processes. We have adapted the considerations of [5] and [6], but in such a case the measure theory on a given "path space" involved is not very well understood. See the review [10].

In this framework, we have performed Wentzel-Freidlin estimates in [9], [11], [12] for Non-Markovian semi-groups by using the normalisation of W.K.B. Analysis [14].

Recently, we have adapted Bismut's way of the Malliavin Calculus for an operator of order four on a Lie group [13]. So we have the two main ingredients for Non-

Markovian semi-groups to get Varadhan types estimates:

-)The Malliavin Calculus.

-)Wentzel-Freidlin type estimates.

The object of this communication is to produce such estimates in a simple case.

5 Statement of the main theorem

Let G be a compact Lie group of dimension m endowed with its biinvariant metric with generic element g. Let e_i be an orthonormal basis of its Lie algebra. e_i can be considered as an orthonormal basis of the tangent space in the unity e $T_e(G)$, or as the the vector fields $e_i g$ or as first order differential operator.

We consider the generator

$$L = \sum_{i=1}^{m} e_i^4 \tag{49}$$

It generates a semi-group on $C_b(G)$ the space of continuous function f on G endowed with its uniform norm which satisfy the parabolic equation

$$\frac{\partial}{\partial t}P_t f = -LP_t f \ ; \ P_0 f = f \qquad (50)$$

We consider the generator

$$L^\epsilon = \epsilon^3 L \qquad (51)$$

Normalisation are those classical of W.K.B. asymptotics (See [14]). L^ϵ generates a semigroup P_t^ϵ

By using the Malliavin Calculus of Bismut type for an order four of [13], we get:

$$P_1^\epsilon f(g_0) = \int_G f(g) p_1^\epsilon(g_0, g) dg \qquad (52)$$

where dg is the normalized volume element on G.

Let us introduce the Hamiltonian on $T^*(G)$

$$H(g, \xi) = \sum_{i=1}^m < e_i, \xi >^4 \qquad (53)$$

We consider the Lagrangian on TG:

$$L(g,p)=\sup_{\xi}(<p,\xi>-H(g,\xi)) \quad (54)$$

A simple computation shows that:

$$C|p|^{4/3}\geq L(g,p)\geq C|p|^{4/3} \quad (55)$$

If $t\in[0,1]\rightarrow G$ is a piecewise C^1 curve on G, we introduce the action

$$S(\phi)=\int_0^1 L(\phi_t,\frac{d}{dt}\phi_t)dt \quad (56)$$

and we consider

$$L(g_0,g)=\inf_{\phi_0=g_0;\phi_1=g}S(\phi) \quad (57)$$

Let us recal that $(g_0,g)\rightarrow l(g_0,g)$ is continuous (See [9]).

Theorem 8 *When $\epsilon\rightarrow 0$, we have uniformly on $G\times G$*

$$\overline{\lim}\epsilon Log|p_1^{\epsilon}(g_0,g)|\leq -l(g_0,g) \quad (58)$$

The next section gives the proof of this Varadhan type estimate, which follows losely in this non-markovian context the proof of these estimates in semi-group theory for diffusion in [7], [8].

6 Proof of the main theorem

Let us recall (see [9]) that

$$\overline{\lim}\epsilon Log|P_1^\epsilon|[1_0](g_0) \leq -\inf_{g\in O} l(g_0,g) \tag{59}$$

if O is an open ball. We consider a positive bump function χ with support a ball centered in g with a small radius such that $\inf_{g'\in O} l(g_0,g')$ is close from $l(g_0,g)$. We consider the measure μ^ϵ

$$f \to P_1^\epsilon[f\chi](g_0) \tag{60}$$

According the framework of the Malliavin Calculus, it is enough to show for all tensor polynomial in the e_i L^r of degree r that

$$|\mu^\epsilon[< D^r f, L^r >]| \leq \|f\|_\infty C\epsilon^{-n(r)} \exp[\frac{-l(g_0,g)+\delta}{\epsilon}] \tag{61}$$

where $\|f\|_\infty$ denotes the uniform norm of f and $n(r)$ is a convenient positive number depending of r.

We apply for that the machinery of [13]. Let $V = G \times M_m$. M_m is the space on symmetric matrices on T_eG. $(g, v) \in V$. v is called the Malliavin matrix. We consider

$$\hat{X}_0^\epsilon = (0, \epsilon^{3/2} \sum_{i=1}^m < g^{-1}e_i, . >^2) \tag{62}$$

and we consider the Malliavin generator

$$\hat{L}^\epsilon = L^\epsilon - \hat{X}_0^\epsilon \tag{63}$$

It generates a semi-group $\hat{P}_t^\epsilon$ called the Malliavin semi-group which acts continuously on $C_b(V)$, the space of continuous bounded function on V $\hat{f}$ endowed with the uniform norm.

The proof follows the proof of theorem 6 of [13], the estimate (11) and the following lemma:

Lemma 9 *For all p, there exists a real*

$n(p)$ such that

$$|\hat{P}_1^\epsilon|[|v^{-p}|](g_0, 0) \leq \frac{C}{\epsilon^{n(p)}} \qquad (64)$$

for $\epsilon < 1$

Proof: The proof of this lemma follows closely the proof of Theorem 7 of [13]. We put $v_\epsilon = \epsilon^{-3/2}v$. It is enough to show that

$$|\hat{P}_1^\epsilon|[|v_\epsilon^{-p}|](g_0, 0) \leq C(p) \qquad (65)$$

if $\epsilon < 1$.

We consider the semi-group $\hat{P}_t^{\epsilon, g_0}$ associated to $L^\epsilon - \epsilon^{3/2}C(g_0)$ ($C(g_0)$ frozen) where $C(g) = \sum_{i=1}^m < g^{-1}e_i, >^2$.

We consider a subdivision t_j^l of mesh $1/l$ of the interval $[0, 1]$. We put $[t]^l$ the closest element of t by lower bound. We consider

$$\hat{P}_t^{l,\epsilon}[\hat{f}](g_0, v_0) =$$
$$P^\epsilon_{t_1^l}[P^l_{t_2^l - t_1^l} ...[P^\epsilon_{t-[t]^l}$$
$$[\hat{f}(., v_0 + \epsilon^{3/2}(C(g_{t_1^l}(t_2^l - t_1^l) + ...C(g_{[t]^l}(t - [t]^l))]]](g_0, 0) \qquad (66)$$

We have clearly

$$|\hat{P}_1^{l,\epsilon}|[(v_\epsilon > C_1)^c](g_0, 0) \qquad (67)$$

We have as in [13] Theorem 7, we have

$$|\hat{P}_t^\epsilon - \hat{P}_t^{g_0,\epsilon}|[\hat{f})(g_0, v_0) \leq \epsilon^{3/2} Ct \|\hat{f}\|_\infty \qquad (68)$$

where $C(t) \to 0$ when $t \to 0$ for any positive test function $\hat{f}$. We deduce that

$$|\hat{P}_t^\epsilon - \hat{P}_t^{l,\epsilon}|[\hat{f}](g_0, v_0) \leq C(l)\epsilon^{3/2} \|\hat{f}\|_\infty \qquad (69)$$

where $C(l) \to 0$ when $l \to \infty$. Therefore the result.

$\diamondsuit$

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