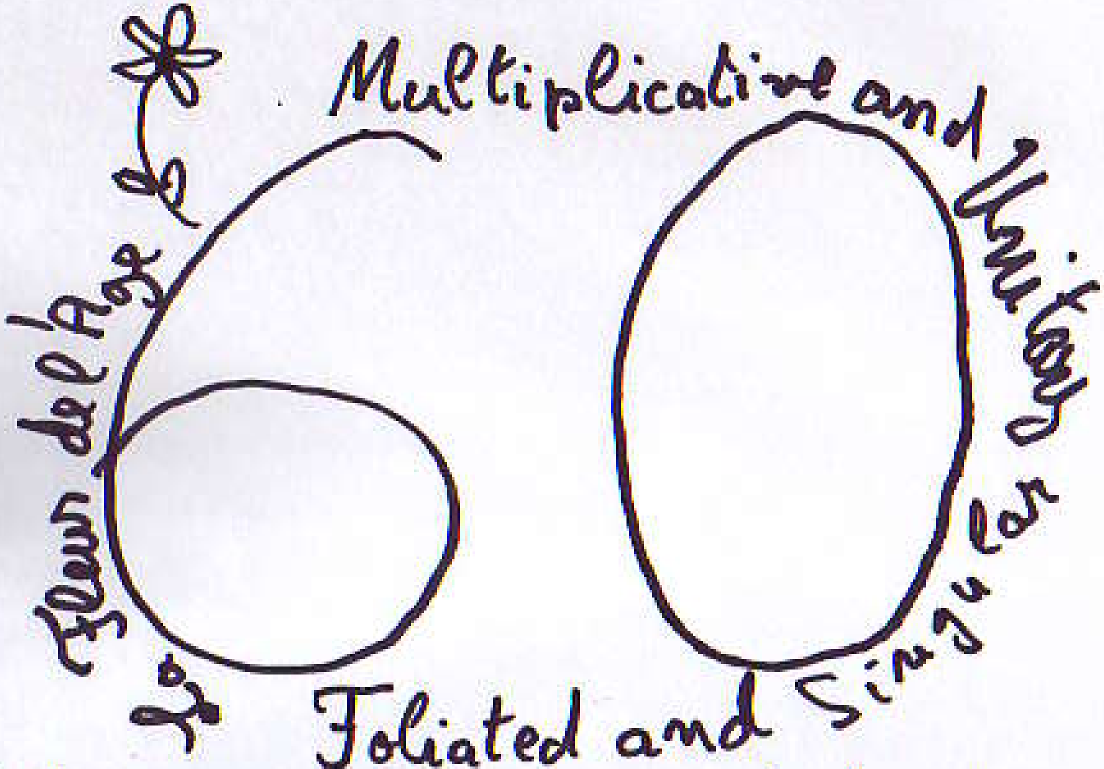
# The Bi-free Extension of Free Probability

Dan-Virgil Voiculescu
UC Berkeley




Happy Birthday
Monsieur Le Professeur
Skandalis !

# An Extension of Free Probability to Systems with two Faces, one of Left Variables and one of Right Variables.

# Free Probability

a probabilistic framework for quantities with the highest degree of noncommutativity.

Noncommutativity a "nuisance" however simplifications occur at the Highest Degree of Noncommutativity

Free Probability =

Noncommutative Probability

+ Free Independence

(Modification of Def. of Independence)

<u>Random variables</u> are quantum mechanical quantities i.e. operators on Hilbert space. $1 \in \mathcal{A}$ algebra of operators on $\mathcal{H}$ Hilbert space, $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ expectation

$$\varphi(a) = \langle a\xi, \xi \rangle, \quad \|\xi\| = 1$$

<u>Free Independence</u> $1 \in \mathcal{B}, \mathcal{C} \subset \mathcal{A}$ subalgebras

$$\varphi(\cdots\cdots (b_j - \varphi(b_j)1)(c_j - \varphi(c_j)1)(b_{j+1} - \varphi(b_{j+1})1)\cdots) = 0$$

$b_{\dots} \in \mathcal{B}$, $c_{\dots} \in \mathcal{C}$

<u>Distribution</u>

Collection of moments of $(a_i)_{i \in I} \subset \mathcal{A}$

$\varphi(a_{i_1} \cdots a_{i_p})$ in general.

1 variable $a = a^*$

$$\mu_a(\cdot) = \varphi(\underbrace{E(\cdot, a)}_{\text{spectral measure}})$$

$\mu_a$ probability measure on $\mathbb{R}$.

Free Parallel to Classical Probability roughly extends quite unexpectedly far

Basic Classical Prob

---

Free Prob

---

parallel seems to evolve to a You Name It situation for Free Analogues

# Free Prob for Pairs of Faces

## Left and Right Variables

$(\mathcal{A}, \varphi)$ $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$, $\varphi(1) = 1$

$1 \in \mathcal{B} \subset \mathcal{A} \supset \mathcal{C} \ni 1$

Left Face Right Face

$((z_i)_{i \in I}, (z_j)_{j \in J}) \subset \mathcal{A}$

left variables right variables

two-faced pair $(a, b) \subset \mathcal{A}$

bi-partite $[a, b] = 0$


Janus 2 Faces
Past & Future Transition

# Free Product of (pre)-Hilbert Spaces with specified State Vectors

$(\mathcal{H}_\iota, \xi_\iota)$, $\xi_\iota \in \mathcal{H}_\iota$, $\|\xi_\iota\| = 1$, $\mathring{\mathcal{H}}_\iota = \mathcal{H}_\iota \ominus \mathbb{C}\xi_\iota$

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigotimes_{\iota_1 \neq \cdots \neq \iota_n} \mathring{\mathcal{H}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{\iota_n}$$

algebraic, no completions

$$(\mathcal{H}, \xi) = \mathop{\ast}_{\iota \in I} (\mathcal{H}_\iota, \xi_\iota)$$

$$\varphi_\xi : \underbrace{\mathcal{L}(\mathcal{H})}_{\text{all linear operators}} \longrightarrow \mathbb{C}, \quad \varphi_\xi(T) = \langle T\xi, \xi \rangle$$

# Left and Right Factorizations

$$V_i : \mathcal{H}_i \otimes \left( \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigotimes_{i \neq i_1 \neq \dots \neq i_n} \mathring{\mathcal{H}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{H}}_{i_n} \right) \to \mathcal{H}$$

$$W_i : \left( \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigotimes_{i_1 \neq \dots \neq i_n \neq i} \mathring{\mathcal{H}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{H}}_{i_n} \right) \otimes \mathcal{H}_i \to \mathcal{H}$$

$$T \in \mathcal{L}(\mathcal{H}_i)$$

$$\lambda_i(T) = V_i (T \otimes I) V_i^{-1} \in \mathcal{L}(\mathcal{H})$$

$$\rho_i(T) = W_i (I \otimes T) W_i^{-1} \in \mathcal{L}(\mathcal{H})$$

$$[\lambda_i(T), \rho_j(S)] = \delta_{ij} \cdot [T, S] \oplus 0$$

# Bi - Freeness

2 two-faced systems in $(\mathcal{A}, \varphi)$

$((b'_i)_{i\in I'}, (c'_j)_{j\in J'})$ and $((b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ bi-free

if $\exists$ $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$

$((T'_i)_{i\in I'}, (S'_j)_{j\in J'}) \subset \mathcal{L}(\mathcal{H}_1)$, $((T''_i)_{i\in I''}, (S''_j)_{j\in J''}) \subset \mathcal{L}(\mathcal{H}_2)$

distribution $((b'_i)_{i\in I'}, (c'_j)_{j\in J'}, (b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ in $(\mathcal{A}, \varphi)$

$\parallel$

distribution $((\lambda_1(T'_i))_{i\in I'}, (\rho_1(S'_j))_{j\in J'}, (\lambda_2(T''_i))_{i\in I''}, (\rho_2(S''_j))_{j\in J''})$

in $(\mathcal{L}(\mathcal{H}), \varphi_\xi)$, $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$

Bi-freeness has the right properties to serve as a noncommutative independence relation for a new type of systems of non-commutative random variables (2-faced).

$((b'_i)_{i\in I'}, (c'_j)_{j\in J'})$ and $((b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ bi-free

$\Longrightarrow$ $(b'_i)_{i\in I'}$ and $(b''_i)_{i\in I''}$ freely indep

$(c'_j)_{j\in J'}$ and $(c''_j)_{j\in J''}$ freely indep

$(b'_i)_{i\in I'}$ and $(c''_j)_{j\in J''}$ "classically" indep

$(c'_j)_{j\in J'}$ and $(b''_i)_{i\in I''}$ "classically" indep

# Bi-Free Gaussians

(distributionally, the limits of bi-free central limit processes)

$\mathcal{H}$ Hilbert space

$\mathcal{T}(\mathcal{H}) = \mathbb{C}1 \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n}$ full Fock space

$\ell(h)\xi = h \otimes \xi$ left creation

$r(h)\xi = \xi \otimes h$ right creation

$\mathcal{B}(\mathcal{T}(\mathcal{H}))$, $\varphi(\cdot) = \langle \cdot 1, 1 \rangle$

$h, h^* : I \amalg J \to \mathcal{H}$, ($I, J$ finite)

$a_\iota = \ell(h(\iota)) + \ell^*(h^*(\iota))$, $\iota \in I$

$b_j = r(h(j)) + r^*(h^*(j))$, $j \in J$

$((a_\iota)_{\iota \in I}, (b_j)_{j \in J})$ in $(\mathcal{B}(\mathcal{T}(\mathcal{H})), \varphi)$

Gaussian system

Not bi-partite in general

$[a_\iota, b_j] = (\langle h(j), h^*(\iota)\rangle - \langle h(\iota), h^*(j)\rangle) P$

$P = \langle \cdot 1, 1 \rangle 1$ .

# Combinatorics of Bi-freeness

First step:

Mastnak - Nica

combinatorics of double ended queus.

Charlesworth - Nelson - Skoufranis

Noncrossing Partitions $NC(n)$

Free Prob

Bi-noncrossing Partitions

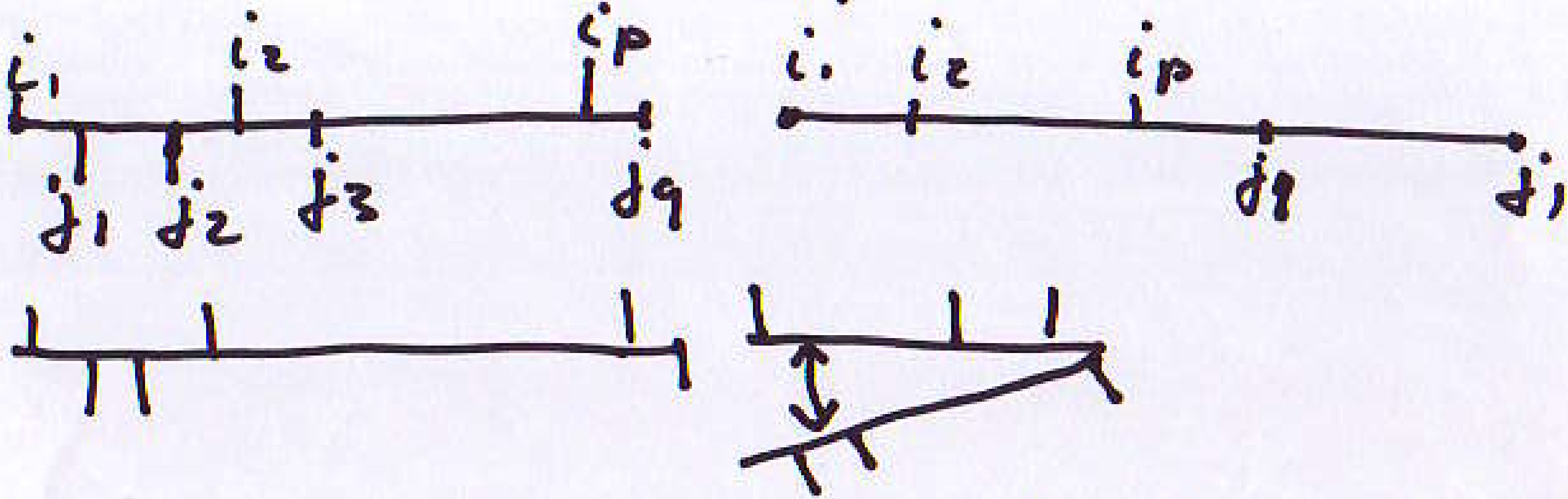
$BNC(\chi)$

$\chi : \{1, \dots, n\} \longrightarrow \{L, R\}$

$\chi^{-1}(L) = \{i_1 < \dots < i_p\}, \ \chi^{-1}(R) = \{j_1 < \dots < j_q\}$

$s_\chi : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$

$$s_\chi(k) = \begin{cases} i_k & 1 \le k \le p \\ j_{n+1-k} & p < k \le n \end{cases}$$




$BNC(\chi) = \{ \pi \in \mathcal{P}(n) \mid s_\chi^{-1} \pi \in NC(n) \}$

# Simplest Bi-free Convolutions

$(a,b),(c,d)$ bi-free pair in $(\mathcal{A},\varphi)$

left right left right

$[a,b]=0,\ [c,d]=0$

$$\mu_{a+c,b+d} = \mu_{a,c} \boxplus\boxplus \mu_{b,d}$$

$$\mu_{a+c,bd} = \mu_{a,c} \boxplus\boxtimes \mu_{b,d}$$

$$\mu_{ac,bd} = \mu_{a,c} \boxtimes\boxtimes \mu_{b,d}$$

# One-Variable Free Convolutions in Free Probability

$a, b \in (\mathcal{A}, \varphi)$ free

$\mu_{a+b} = \mu_a \boxplus \mu_b$ additive free convolution

R-transform

$G_a(z) = \sum_{n \geq 0} z^{-n-1} \varphi(a^n) = \varphi((z1 - a)^{-1})$

$G_a(K_a(z)) = z$ (near 0)

$R_a(z) = K_a(z) - z^{-1}$

$R_{a+b}(z) = R_a(z) + R_b(z)$

($R_a$ free R-transform, V (1986

Similar multiplicative free convolution

$a, b \in (A, \varphi)$ free

$\mu_{ab} = \mu_a \boxtimes \mu_b$ multiplicative f.c.

S-transform, $\varphi(a) \neq 0$

$$\psi_a(z) = \sum_{n \geq 1} z^n \varphi(a^n) = \varphi((1-za)^{-1}) - 1$$

$$\chi_a(\psi_a(z)) = z, \quad S_a(z) = \frac{z+1}{z} \chi_a(z)$$

$$S_{\mu_a \boxtimes \mu_b}(z) = S_{\mu_a}(z) S_{\mu_b}(z)$$

($S_a$ free S-transform V (1987))

# Partial bi-free transforms

$$G_{a,b}(z,w) = \varphi\left((z1-a)^{-1}(w1-b)^{-1}\right)$$

$$H_{a,b}(z,w) = \varphi\left((1-za)^{-1}(1-wb)^{-1}\right)$$

$$F_{a,b}(z,w) = \varphi\left((z1-a)^{-1}(1-wb)^{-1}\right)$$

various moment generating functions for two-faced pair $(a,b) \subset A$.

Only two-band moments $\varphi(a^p b^q)$.

Reduced partial transforms

$$\tilde{R}_{a,b}(z,w) = 1 - \frac{zw}{G_{a,b}(K_a(z), K_b(w))}$$

$$\tilde{S}_{a,b}(z,w) = \frac{z+1}{z}\,\frac{w+1}{w}\left(1 - \frac{1+z+w}{H_{a,b}(\chi_a(z), \chi_b(w))}\right)$$

$$\tilde{T}_{a,b}(z,w) = \frac{w+1}{w}\left(1 - \frac{z}{F_{a,b}(K_a(z), \chi_b(w))}\right)$$

Reduced: if $\varphi(a^p b^q) = \varphi(a^p)\varphi(b^q)\ \forall p,q \geq 0$

then $\tilde{R} = 0$, $\tilde{S} = 1$, $\tilde{T} = 1$.

If $(a_1, b_1)$ and $(a_2, b_2)$ bi-free in $(\mathcal{A}, \varphi)$ then:

$$\tilde{\mathcal{R}}_{a_1+a_2,\, b_1+b_2} = \tilde{\mathcal{R}}_{a_1, b_1} + \tilde{\mathcal{R}}_{a_2, b_2}$$

$$\tilde{S}_{a_1 a_2,\, b_1 b_2} = \tilde{S}_{a_1, b_1} \tilde{S}_{a_2, b_2}$$

$$\tilde{T}_{a_1+a_2,\, b_1 b_2} = \tilde{T}_{a_1, b_1} \tilde{T}_{a_2, b_2}$$

Together with $R$ and $S$ in free prob can compute $\boxplus\boxplus$, $\boxtimes\boxtimes$, $\boxplus\boxtimes$ at the level of 2-bands moments.

My work defining the bi-free partial $R$-, $S$- and $T$- transforms is analytic. Instead of using my original proofs for the 1-variable $R$- and $S$-transf. as starting point, found alternative proofs of Uffe Haagerup better suited.

Paul Skoufranis soon found alternative combinatorial proofs for the properties of the partial bi-free transform.

# Bi-free Extreme Values

$(A, \varphi)$ v. Neumann algebra with normal state.

$P = P^* = P^2$, $Q = Q^* = Q^2$ in $A$

$P \wedge Q$ projection onto $P\mathcal{H} \cap Q\mathcal{H}$

$P \vee Q$ projection onto $\overline{P\mathcal{H} + Q\mathcal{H}}$

$X = X^*,\ Y = Y^*$ in $(A, \varphi)$

$X \vee Y$ w.r.t. Spectral Order

$$E(X \vee Y; (-\infty, a]) = E(X; (-\infty, a]) \wedge E(X; (-\infty, a])$$

Free max-convolution

$$\mu_{(X_\iota)_{\iota \in I}} \boxed{\vee}\ \mu_{(Y_\iota)_{\iota \in I}} = \mu_{(X_\iota \vee Y_\iota)_{\iota \in I}}$$

$(X_\iota)_{\iota \in I}, (Y_\iota)_{\iota \in I}$ free in $(A, \varphi)$

# Ben-Arous – V.

## Free Extreme Values

1-variable $F_\mu$ distribution function

$$F_\mu(a) = \mu((-\infty, a])$$

$F, G$ distribution functions

$$(F \vee G)(t) = (F(t) + G(t) - 1)_+$$

Classification of free max-stable laws.

# Bi-free Extension (V.)

$$\left( \left( (z_i')_{i\in I}, (z_j')_{j\in J} \right), \left( (z_i'')_{i\in I}, (z_j'')_{j\in J} \right) \right)$$

bi-free, hermitian

$$\left( (z_i' \vee z_i'')_{i\in I}, (z_j' \vee z_j'')_{j\in J} \right)$$

$$\mu_{z'} \boxtimes\!\!\boxtimes \mu_{z''} = \mu_{z' \vee z''}$$

Simplest case $(a,b)$, $[a,b]=0$

$a = a^*$, $b = b^*$.

$\mu$ prob measure on $\mathbb{R}^2$

$F_\mu(s,t) = \mu((-\infty,s]\times(-\infty,t])$

$F, G$ bi-variate distribution functions

$F_j, G_j$ $(j=1,2)$ marginals

$F \boxed{\vee}\boxed{\vee} G$ bi-free max-convolution

$$H = F \boxed{\vee}\boxed{\vee} G$$

$$H_j = \left(F_j + G_j - 1\right)_+ , \quad j = 1, 2$$

$$\frac{H_1(s)\, H_2(t)}{H(s,t)} - 1 =$$

$$= \left(\frac{F_1(s)\, F_2(t)}{F(s,t)} - 1\right) + \left(\frac{G_1(s)\, G_2(t)}{G(s,t)} - 1\right)$$

if $F(s,t) > 0$, $G(s,t) > 0$, $H_1(s) > 0$, $H_2(t) > 0$

and $H(s,t) = 0$ otherwise

# References


V. Free Prob. for Pairs of Faces I
CMP 332 (2014), 955–980

Free Prob. for Pairs of Faces II :
2 Variables Bi-free Partial R-transform
arXiv: 1308.2035

Free Prob. for Pairs of Faces III :
2 Variables Bi-free Partial
S- and T- transforms
arXiv: 1504.03765

V. Free Prob. for Pairs of Faces IV:
Bi-free Extremes in the Plane
arXiv: 1505.05020

Mastnak – Nica Double-ended queues and joint moments of left-right canonical operators on full Fock space
arXiv: 1312.0269

Charlesworth-Nelson-Skoufranis
On two-faced families of non-commutative random variables
arXiv: 1403.4907

Combinatorics of bi-freeness with amalgamation
arXiv: 1408.3251

Skoufranis A combinatorial approach to Voiculescu's bi-free partial transforms
arXiv: 1504.06005

Skoufranis Independences and partial R-transforms in bi-free probability
arXiv: 1410.4265

Some Bi-Matrix models for Bi-Free Limit Distributions
arXiv: 1506.01725

On Operator-Valued Bi-Free Distributions
arXiv: 1510.03896

Gu - Huang - Mingo An analogue of the Levy - Hinčin theorem for bi-free infinitely divisible distributions.
arXiv: 1501.05369

Gao Two-faced families of non-commutative random variables having bi-free infinitely divisible distributions arXiv: 1507.08270

Dykema - Na Principal functions for bi-free central limit distributions
arXiv: 1510.03328