Indomitable rho-invariants

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- talk based on joint work with Thomas Schick (and work of many others).

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•⟶Such a natural lift is called a rho-class.

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- Also, in this case: $K_*(C^*(M)^\Gamma) \simeq K_*(C^*_r\Gamma)$
- these groups behave functorially. So, if $\tilde{u} : M \to E\Gamma$ is a Γ-equiv. classifying map then we can use $\tilde{u}_*$ to map to the universal HR sequence:

$$\cdots \to K_*(D^*_\Gamma) \to K_*(B\Gamma) \xrightarrow{\delta} K_{*+1}(C^*_r\Gamma) \to \cdots$$

where $D^*_\Gamma := D^*(E\Gamma)^\Gamma$. It turns out that δ is the assembly map.

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- let $n := \dim M$ be even; for the Dirac bundle $E = E^+ \oplus E^-$
- $[D] := [U^* \chi(D)_+] \in K_1(D^*(M)^\Gamma / C^*(M)^\Gamma) = K_0(M/\Gamma)$, with U a suitable (local) unitary operator $L^2(M, E^+) \to L^2(M, E^-)$.

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- $\mathrm{Ind}(D) = \delta[D] \in K_n(C^*(M)^\Gamma)$, $n = \dim M$, is the index class.
- (it coincides with all possible definitions (e.g. : Mishchenko-Fomenko, Connes-Skandalis, etc))

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- in the past these numeric invariants proved to be extremely useful. Let M be the universal cover of X, $\Gamma = \pi_1(X)$, $\mathcal{R}^+(X) \neq \emptyset$:

Theorem

(P-Schick, 2007): if X has dimension $4k+3$ and $\pi_1(X)$ is not torsion-free then there are infinitely many metrics of PSC that are non-bordant (and thus non-concordant and thus non-pathconnected); they are distinguished by the Cheeger-Gromov rho invariant $\rho^{\mathrm{CG}}(g)$.
In fact $|\pi_0(\mathcal{R}^+(X))/\mathrm{Diffeo}(X)| = \infty$.

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Here is the definition: $\rho_\Gamma[Y, u : Y \to B\Gamma, g_Y] := \rho_\Gamma(g)$ where g is the PSC metric induced by g_Y on $M := u^* E\Gamma$.

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Similarly, one can consider the set $P(X)$ of concordance classes of PSC metrics on X; it has a group structure and $\rho : P(X) \to K_{\dim M+1}(D^*(\widetilde{X})^\Gamma)$ defines a group homormorphism.

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- We have a Γ-covering $\tilde{Z}$ on $Z = X \sqcup (-V)$, $\tilde{Z} = f^*\tilde{V} \sqcup (-\tilde{V})$
- Classic: the associated signature index class $\mathrm{Ind}(D)$ is equal to zero in $K_*(C^*(\tilde{Z})^\Gamma)$ (Mishchenko, Kasparov, ..., Hilsum-Skandalis).

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Definition

Let $X \xrightarrow{f} V$ be a homotopy equivalence. We define $\rho(f) := \tilde{\phi}_*(\rho(D + C_f)) \in K_{*+1}(D^*(\tilde{V})^\Gamma)$ and $\rho_\Gamma(f) := u_*(\rho(f))$

This is a variant of a fundamental definition due to Higson and Roe.

Fundamental properties of $\rho(f)$ and $\rho_\Gamma(f)$

One can appeal to Higson-Roe and Benameur-Roy and see that from $\rho_\Gamma(f)$ we can extract the classic Cheeger-Gromov invariant.

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Crucial in above theorem is the use of (part of) the surgery exact sequence in topology.
In this direction we have:

Theorem

Let $\mathcal{S}(V)$ be the structure set of V. Then there are well defined maps

$$\rho : \mathcal{S}(V) \to K_{*+1}(D^*(\tilde{V})^\Gamma) \text{ and } \rho_\Gamma : \mathcal{S}(V) \to K_{*+1}(D^*_\Gamma)$$

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Elements in $\mathcal{S}(V)$ are equivalence classes $[X \xrightarrow{f} V]$ with f an orientation preserving homotopy equivalence.

$(X_1 \xrightarrow{f_1} V) \sim (X_2 \xrightarrow{f_2} V)$ if there is an h-cobordism X between X_1 and X_2 and a map $F\colon X \to V \times [0,1]$ such that $F|_{X_1} = f_1$ and $F|_{X_2} = f_2$.

Basic tool: the delocalized Atiyah-Patodi-Singer index theorem

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Theorem

(delocalized APS index theorem, P-Schick 2013) There exists an index class $\mathrm{Ind}(D, C_\partial) \in K_*(C^*(W)^\Gamma)$ *and*

$$\iota_*(\mathrm{Ind}(D, C_\partial)) = j_*(\rho(D_\partial + C_\partial)) \quad \textit{in} \quad K_0(D^*(W)^\Gamma).$$

Here $j\colon D^*(\partial W)^\Gamma \to D^*(W)^\Gamma$ *is induced by the inclusion* $\partial W \hookrightarrow W$ *and* $\iota\colon C^*(W)^\Gamma \to D^*(W)^\Gamma$ *the natural inclusion.*

Surgery sequences

Surgery sequences

This theorem is crucial in establishing that ρ is well defined on $\mathrm{Pos}_n^{\mathrm{spin}}(Z)$ and on $\mathcal{S}(V)$.

The group $\mathrm{Pos}_n^{\mathrm{spin}}(Z)$ with Z compact and $\pi_1(Z) = \Gamma$, fits into the surgery exact sequence of Stephan Stolz:

$$\to \mathrm{Pos}_n^{\mathrm{spin}}(Z) \to \Omega_n^{spin}(Z) \to R_n^{\mathrm{spin}}(Z) \to \mathrm{Pos}_{n-1}^{\mathrm{spin}}(Z) \to$$

with $R_n^{\mathrm{spin}}(Z)$ depending only on Γ.

The structure set $\mathcal{S}(V)$ fits into the surgery sequence in differential topology, due to Browder, Novikov, Sullivan and Wall:

$$\cdots \to L_{n+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathcal{S}(V) \to \mathcal{N}_n(V) \to L_n(\mathbb{Z}\Gamma)$$

Mapping surgery to analysis

Mapping surgery to analysis

Using as a basic tool the delocalised APS index theorem (with boundary operator invertible) we prove with Thomas Schick the following theorem

Theorem

There exists a well defined and commutative diagram

$$\begin{array}{ccccccc}
\Omega^{\mathrm{spin}}_{n+1}(B\Gamma) & \longrightarrow & R^{\mathrm{spin}}_{n+1}(B\Gamma) & \longrightarrow & \mathrm{Pos}^{\mathrm{spin}}_{n}(B\Gamma) & \longrightarrow & \Omega^{\mathrm{spin}}_{n}(B\Gamma) \\
\downarrow \beta & & \downarrow \mathrm{Ind}_\Gamma & & \downarrow \rho_\Gamma & & \downarrow \beta \\
K_{n+1}(B\Gamma) & \longrightarrow & K_{n+1}(C^*_\Gamma) & \longrightarrow & K_{n+1}(D^*_\Gamma) & \longrightarrow & K_n(B\Gamma)
\end{array}$$

More generally, if Z is a compact space with $\pi_1(Z) = \Gamma$ then

$$\begin{array}{ccccccc}
\Omega^{\mathrm{spin}}_{n+1}(Z) & \longrightarrow & R^{\mathrm{spin}}_{n+1}(Z) & \longrightarrow & \mathrm{Pos}^{\mathrm{spin}}_{n}(Z) & \longrightarrow & \Omega^{\mathrm{spin}}_{n}(Z) \\
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K_{n+1}(Z) & \longrightarrow & K_{n+1}(C^*(\tilde{Z})^\Gamma) & \longrightarrow & K_{n+1}(D^*(\tilde{Z})^\Gamma) & \longrightarrow & K_n(Z)
\end{array}$$

Mapping surgery to analysis II

Mapping surgery to analysis II

Using as a basic tool the delocalised APS index theorem and also interesting work of Charlotte Wahl we also prove a new version of the fundamental theorem of Higson and Roe:

Theorem

There are natural (index theoretic) maps $\mathrm{Ind}, \rho, \beta$ *such that the following diagram is commutative*

$$\begin{array}{ccccccc}
L_{n+1}(\mathbb{Z}\Gamma) & \dashrightarrow & \mathcal{S}(V) & \longrightarrow & \mathcal{N}(V) & \longrightarrow & L_n(\mathbb{Z}\Gamma) \\
\downarrow{\scriptstyle \mathrm{Ind}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{Ind}} \\
K_{n+1}(C^*(\tilde{V})^\Gamma) & \longrightarrow & K_{n+1}(D^*(\tilde{V})^\Gamma) & \longrightarrow & K_n(V) & \longrightarrow & K_n(C^*(\tilde{V})^\Gamma)
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Recent results I

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- Using Lipschitz analysis Zenobi extended the "mapping surgery to analysis" to the topological surgery sequence:

$$\cdots \to L_{n+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathcal{S}^{\mathrm{TOP}}(V) \to \mathcal{N}_n^{\mathrm{TOP}}(V) \to L_n(\mathbb{Z}\Gamma)$$

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  where now V is a topological manifold.
- Notice that $\mathcal{S}^{\mathrm{TOP}}(V)$ has a (exotic) group structure:
  Q. : Is $\rho : \mathcal{S}^{\mathrm{TOP}}(V) \to K_{n+1}(D^*(\tilde{V})^\Gamma)$ a group homomorphism ?
  This is a very interesting question.

Recent results II

Let M be a space with a free and co-compact action of Γ. In fact, we could just assume properness.
We denote $S_n^\Gamma(M) := K_{n+1}(D^*(M)^\Gamma)$ where $n = \dim M$ is taken mod 2.

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- using the delocalised APS index theorem for perturbed operators Deeley and Goffeng construct an iso. $\lambda : \mathrm{S}_n^{\Gamma,\mathrm{geo}}(M/\Gamma, \mathcal{L}) \to \mathrm{S}_n^\Gamma(M)$.

Recent results III: products

The following theorem is due to many people: Siegel, Xie-Yu, Zeidler, Zenobi.

Theorem

There is an exterior product

$$S^{\Gamma_1}_j(X_1) \otimes K^{\Gamma_2}_\ell(X_2) \xrightarrow{\boxtimes} S^{\Gamma_1 \times \Gamma_2}_{j+\ell}(X_1 \times X_2)$$

If g_1 is of PSC on X_1 and $g_1 \oplus g_2$ is of PSC on the product then

$$\rho(g_1) \boxtimes [D_{X_2}] = \rho(g_1 \oplus g_2).$$

Zenobi also proves that if $Z_1 \xrightarrow{f_1} X_1$ is a homotopy equivalence then

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Zenobi and Zeidler finds sufficient conditions on X_2 ensuring that the external multiplication by $[D_{X_2}]$ is **injective**.

Consequence $\longrightarrow$ rigidity under products: under these additional assumptions if $\rho(g_1) \neq \rho(h_1)$ (so they are "distinct") then $\rho(g_1 \oplus g_2) \neq \rho(h_1 \oplus g_2)$ (so they are again "distinct"). Similarly, if $\rho[Z \xrightarrow{f_1} X_1] \neq \rho[W \xrightarrow{g_1} X_1]$ then under these additional assumptions $\rho[Z \times X_2 \xrightarrow{f_1 \times \mathrm{Id}} X_1 \times X_2] \neq \rho[W \times X_2 \xrightarrow{g_1 \times \mathrm{Id}} X_1 \times X_2]$.

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New: Deeley and Goffeng use these results in order to map from $S_n^{\Gamma,\mathrm{geo}}(M/\Gamma,\mathcal{L})$ to $H_*^{\mathrm{del}}(\mathcal{B}_\Gamma)$

Recent results V: using rho classes

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Theorem

Let $N_{\mathrm{fin}}(\Gamma) = \{d \in \mathbb{N} \mid \exists \gamma \in \Gamma$ such that $\mathrm{ord}(\gamma) = d, \gamma \neq e\}$ Let Γ be strongly finite embeddable in Hilbert space. Then the rank of $\widetilde{P}(X)$ is $\geq N_{\mathrm{fin}}(\Gamma)$. If we only assume Γ is torsion free then the rank of $\widetilde{P}(X)$ is ≥ 1.

Recent results V: using rho classes

Go back to the group $P(X)$ (concordance classes of metrics of PSC). $\mathrm{Diffeo}(M)$ acts on $P(X)$ and we have the quotient group $\widetilde{P}(X)$.

Xie and Yu, building on very interesting work of Weinberger-Yu, and using rho classes in K-theory, prove the following sharpening of the result of P.-Schick. Let $\dim X = 2n+1$ and $\mathcal{R}^+(X) \neq \emptyset$.

Theorem

Let $N_{\mathrm{fin}}(\Gamma) = \{d \in \mathbb{N} \mid \exists \gamma \in \Gamma$ such that $\mathrm{ord}(\gamma) = d, \gamma \neq e\}$ Let Γ be strongly finite embeddable in Hilbert space. Then the rank of $\widetilde{P}(X)$ is $\geq N_{\mathrm{fin}}(\Gamma)$. If we only assume Γ is torsion free then the rank of $\widetilde{P}(X)$ is ≥ 1.

It would be very interesting to use Zenobi's topological rho map, $\rho^{\mathrm{TOP}} : \mathcal{S}^{\mathrm{TOP}}(X) \to K_{\dim X+1}(D^*(\tilde{X})^\Gamma)$ in order to prove a similar result for $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)$. Partial results (already very interesting !) by Weinberger-Yu.