On the L^p Baum-Connes conjecture

G. Kasparov
Vanderbilt University

The right side of the Baum-Connes conjecture is the K-theory of the reduced C^*-algebra $C^*_{red}(G)$ of the group G. This algebra is the completion of the algebra $L^1(G)$ acting on $L^2(G)$ by convolution. If we complete the algebra $L^1(G)$ in the norm of the convolution algebra on $L^p(G)$ we will get the Banach algebra which I will denote $C_r^{*,p}(G)$. The K-theory of this algebra serves as the right side of the L^p-version of the Baum-Connes conjecture. I will discuss the result on this conjecture for a certain class of groups. This is joint work with Guoliang Yu.

A Banach algebra B will be called an L^p-algebra if it isometrically embeds as a closed subalgebra into the algebra $\mathcal{B}(L^p(Z))$ of all bounded linear operators on some space $L^p(Z)$, where Z is a measure space. For such algebra B, the opposite algebra B^{op} embeds into the algebra of operators on the dual space $L^q(Z)$, where $1/p + 1/q = 1$.

For any locally compact group G acting isometrically on an L^p-algebra $B \subset \mathcal{L}(L^p(Z))$ we define the norm on the crossed product algebra $C_c(G, B)$ as the operator norm for the action on $L^p(G \times Z)$ by the formula:

$$(b \cdot l)(t, z)$$

$$= \int_G t^{-1}(b(s)) \cdot l(s^{-1}t, z) ds$$

where b is a compactly supported function on G with values in B and $l \in L^p(G \times Z)$. We define the reduced crossed product algebra $C_r^{*,p}(G, B)$ as the completion of $C_c(G, B)$ in this norm.

In the special case when $B = \mathbf{C}$, the algebra $C_r^{*,p}(G)$ is just the image of the algebra $L^1(G)$ in $\mathcal{L}(L^p(G))$ acting on $L^p(G)$ by convolution. If G is unimodular, the dual to the convolution by an element $f(g) \in L^1(G)$ on $L^p(G)$ is the convolution by $f^*(g)$ on $L^q(G)$, where $f \mapsto f^*$ is the usual involution on $L^1(G)$. Therefore, $C_r^{*,p}(G) \simeq C_r^{*,q}(G)$. By the interpolation theorem, since $C_r^{*,p}(G)$ acts on both $L^p(G)$ and $L^q(G)$, it also acts on $L^2(G)$, so there is a natural homomorphism $C_r^{*,p}(G) \to C_r^*(G)$. The question of surjectivity of the corresponding K-theory map is important for the BC conjecture.

The construction of the left side of the Baum-Connes conjecture and the assembly map requires E-theory for L^p-algebras. This theory has recently been developed by my PhD student Fan Fei Chong. The definition of asymptotic morphisms for Banach algebras goes exactly in the same way as for C^*-algebras. Also the composition of asymptotic morphisms is defined as for C^*-algebras. The first non-trivial point is the construction of an asymptotic morphism corresponding to an extension of Banach algebras: $0 \to J \to A \to B \to 0$.

This requires a quasicentral continuous approximate unit for the ideal J. Also we will assume here that J is an L^p-algebra. In order to construct a quasicentral approximate unit in J, we will assume that the Banach algebra J possesses a countable bounded approximate unit $\{u_i, \ i = 1, 2, ...\}$ such that, if we put $u_0 = 0, b_i = u_i - u_{i-1}$, then for any increasing sequence of indices k_j, the norm $||\sum_j b_{k_j}||$ is bounded by a constant which does not depend on the sequence k_j.

Proposition. Let J be a closed two-sided ideal in a Banach algebra B. Assume that J has an approximate unit which satisfies the above assumptions, and the quotient B/J is separable. Then there is a quasicentral bounded approximate unit in J with respect to B. In the case of a continuous isometric action of a group G on B, the approximate unit can be chosen quasicentral also with respect to G.

We will be mainly interested in E-theory on the category of L^p-algebras. The ideal of compact operators that will be used for stabilization is $\mathcal{K}(L^p(Z))$ for some measure space Z. The definition of E-theory goes as in the C^*-category, including equivariant E-theory. Certainly, there are some technical points which are different from the L^2 case, but I will not discuss this.

We define the descent homomorphism

$$j^G : E^G(A, B) \to E(L^1(G, A), L^1(G, B))$$

using the approach of Guentner-Higson-Trout. Combining it with the homomorphism $L^1(G, B) \to C_r^{*,p}(G, B)$, we obtain the map

$$j^G : E^G(A, B) \to E(L^1(G, A), C_r^{*,p}(G, B)).$$

Let $\mathcal{E}G$ be the classifying space for proper actions of G. The left-hand side of the Baum-Connes conjecture (with coefficients in an L^p-algebra B) is the inductive limit $RE_*^G(\mathcal{E}G, B) = \varinjlim E_*^G(C_0(X), B)$ over all G-proper, G-compact, locally compact spaces X.

In particular, the left-hand side for the L^p Baum-Connes conjecture with $B = \mathbf{C}$ is the same as for the conventional Baum-Connes conjecture.

The assembly map

$$\mu_p : RE_*^G(\mathcal{E}G, B) \to$$

$$K_*(C_r^{*,p}(G, B))$$

is defined as follows. We take a non-negative function $c \in C_c(X)$ satisfying $\int_G c(g^{-1}x)dg = 1$ and define the idempotent element $[c] \in C_c(G, C_0(X))$ by

$$[c](g, x) = c(x)^{1/2} c(g^{-1}x)^{1/2}.$$

For any $a \in E_*^G(C_0(X), B)$, we first apply the homomorphism j^G to get an element $j^G(a) \in E(L^1(G, C_0(X)), C_r^{*,p}(G, B))$, and then take the E-theory product of $j^G(a)$ with the idempotent $[c]$.

The L^p Baum-Connes conjecture with coefficients in B is the statement that the map μ_p is bijective.

When $B = \mathbf{C}$, it is clear from the above remark on the map $K_*(C^{*,p}(G)) \to K_*(C^*(G))$ that the assembly map for the L^2 BC conjecture factors through the assembly map for the L^p BC conjecture.

A little bit of Fantasy

At least in the case when the Bost conjecture is true (i.e. when $RE_*^G(\mathcal{E}G, \mathbf{C}) \to K_*(L^1(G))$ is an isomorphism), it seems reasonable to believe that the groups $K_*(C^{*,q}(G))$ increase from $q = 1$ (the case of $K_*(L^1(G))$) to $q = 2$ (the case of $K_*(C^*(G))$). It may happen that the tool to detect the jumps in the groups $K_*(C^{*,q}(G))$ is a certain modified version of the property (RD) adapted to the L^p case.

Main Result. *The L^p-version of the Baum-Connes conjecture with coefficients in any L^p-algebra is true for any discrete group G which admits an affine-isometric, metrically proper action on the space $X = l^p(Z)$, where Z is a countable discrete set equipped with the atomic measure, so that the linear part of this action is induced by an action of G on Z.*

We will use the method which has previously been used in the L^2 case. To simplify notation, I will explain the case without coefficients. The main idea is to construct a Banach algebra $\mathcal{A}(X)$ and two asymptotic morphisms $\mathcal{S} \to \mathcal{A}(X)$ (Bott element) and $\mathcal{S} \otimes \mathcal{A}(X) \to \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{K}$ (Dirac element) which are inverses of each other. Here the algebra $\mathcal{S}$ is a certain $\mathbf{Z}_2$-graded Banach algebra which has the same K-theory as $C_0(\mathbf{R})$, graded by even and odd functions, and $\mathcal{K}$ is the algebra of compact operators on some L^p-space.

For any finite subset $F \subset Z$ and $V_F = l^p(F)$, we consider $\Lambda^*(V_F)$ as the l^p-space of all finite subsets of F. We construct the inductive limit of the spaces $L^p(\Lambda^*(V_F))$. Namely, if $F_1 \subset F_2$ are finite subsets of Z and $\alpha > 0$, let $f_{F_2 - F_1}$ be the scalar function on $V_{F_2 - F_1}$ defined by:

$$f_{F_2 - F_1}(v)$$

$$= \exp(-\sum_{k=1}^{n} |x_k|^p / (p\sqrt{p-1}\alpha)),$$

where $F_2 - F_1 = \{z_k | \; k = 1, \cdots, n\}$, $v = \sum_{k=1}^{n} x_k \delta_{z_k}$.

The isometric embedding:

$$L^p(\Lambda^*(V_{F_1})) \to L^p(\Lambda^*(V_{F_2}))$$

is given by: $\xi \to \xi \otimes (c_n f_{F_2-F_1})$, where the constant $c_n(p,\alpha)$ is chosen in such a way that $c_n f_{F_2-F_1}$ has L^p-norm 1.

For any $\alpha > 0$, we define the Banach space $\mathcal{B}_\alpha$ as the inductive limit of $L^p(\Lambda^*(V_F))$. We define the Banach algebra $\mathcal{K}_\alpha$ as the algebra of all compact operators on the Banach space $\mathcal{B}_\alpha$.

Recall that the number q satisfies: $q-1=(p-1)^{-1}$. Let F be a finite subset of Z, $\{e_k\}_{k\in F}(=\{\delta_{z_k}\}_{k\in F})$ the standard basis for V_F, and x_k the corresponding coordinate functions. We define the operators $d_{k,\alpha}$ and $d^*_{k,\alpha}$ on $L^p(\Lambda^*(V_F))$ by

$$d_{k,\alpha}=(\alpha\sqrt{p-1}\,\partial/\partial x_k$$

$$+x_k|x_k|^{p-2})\mathrm{ext}\,(e_k),$$

$$d^*_{k,\alpha}=(-\alpha\sqrt{q-1}\,\partial/\partial x_k$$

$$+x_k|x_k|^{p-2})\mathrm{int}\,(e_k),$$

and put $D_{k,\alpha}=d_{k,\alpha}+d^*_{k,\alpha}$.

Note that $D_{k,\alpha}^2 = d_{k,\alpha} d_{k,\alpha}^* + d_{k,\alpha}^* d_{k,\alpha}$, and the operators $d_{k,\alpha} d_{k,\alpha}^*$ and $d_{k,\alpha}^* d_{k,\alpha}$ commute with each other and with all other such operators for different values of k.

Now we will normalize our operators $D_{k,\alpha}$. Let $\beta = 1/2 - q/4$. Since $D_{k,\alpha}^2$ is positive in the L^p sense (accretive), there exists the operator

$$D_{k,\alpha}(D_{k,\alpha}^2)^{-\beta} =$$

$$\frac{\sin \pi\beta}{\pi} \int_0^\infty D_{k,\alpha}(D_{k,\alpha}^2 + t)^{-1} t^{-\beta} dt.$$

The Bott-Dirac operator $D_{\alpha,F}$ is the sum $\sum_{k\in F} \lambda_k D_{k,\alpha}(D_{k,\alpha}^2)^{-\beta}$, where the sequence of increasing positive numbers λ_k is specially chosen.

When α varies in the interval $(0, \infty)$, we get a continuous field of L^p Banach algebras $\mathcal{K}_\alpha$ and Bott-Dirac operators D_α as unbounded multipliers of $\mathcal{K}_\alpha$. Similarly to the L^2 case, when $\alpha \to 0$, we will obtain in the limit the algebra $\mathcal{A}(X)$ and the Bott element for the algebra $\mathcal{A}(X)$. The motivation for this is in the representation theory of the crossed product algebra $C^{*,p}(\mathbf{R}, C_0(\mathbf{R}))$, where the linear action $\mathbf{R}$ on itself depends on the parameter α (the speed).

Note that in the previous formulas for D_α, all $\lambda_k \to 1$ when $\alpha \to 0$.

Let us consider both the symbol $i\xi_k$ of the operator $\partial/\partial x_k$ and the multiplication operator $x_k|x_k|^{p-2}$ as unbounded multipliers of $C_0(\mathbf{R})$. We get two multiplication operators on the exterior algebra of the complexified space $V_F \otimes \mathbf{C}$: namely,

$$b_k = (i\sqrt{p-1}\xi_k$$

$$+x_k|x_k|^{p-2})\text{ext}\,(e_k),$$

$$b_k^* = (-i\sqrt{q-1}\xi_k$$

$$+x_k|x_k|^{p-2})\text{int}\,(e_k),$$

which are unbounded multipliers of $C_0(V_F \otimes \mathbf{C}) \otimes Cliff(V_F \otimes \mathbf{C})$. We put $B_k = b_k + b_k^*$.

After the appropriate normalization, as for $D_{k,\alpha}$ above, we get the Bott element $B_F = \sum_k B_k (B_k^2)^{-\beta}$ for any finite subset $F \subset Z$, where again $\beta = 1/2 - q/4$.

Now we define the algebra $\mathcal{A}(X)$ formally. Let us denote by $\mathbf{C}_+$ the positive complex half-plane $\{z \in \mathbf{C}|\ \mathrm{Re}\, z > 0\}$. We define the algebra $\mathcal{H}_+$ as the algebra of all bounded holomorphic functions on $\mathbf{C}_+$ which correspond via the conformal isomorphism $\mathbf{C}_+ \to D^2 : z \mapsto (1-w)/(1+w)$ to those holomorphic functions on the open unit ball D^2 in the complex plane which extend continuously to the closed unit ball and vanish at the point -1. The norm on $\mathcal{H}_+$ is the sup-norm. Note that any $f \in \mathcal{H}_+$ vanishes at infinity as $\mathrm{const} \cdot |z|^{-m}$ for some integer $m \geq 1$.

Define the cone $C_{1/2}$ as follows: $C_{1/2} = \{z \in \mathbf{C}_+;\ z^2 \in \mathbf{C}_+\}$. Denote by Cl_1 the complex Clifford algebra generated by the elements $\{1, \varepsilon\}$ with the relation: $\varepsilon^2 = 1$. Define $\mathcal{S}$ as the algebra of all holomorphic functions $f : \mathbf{C}_{1/2} \to Cl_1$ such that $f(z) = f_0(z^2) + \varepsilon z f_1(z^2)$, where $f_0, f_1 \in \mathcal{H}_+$. The norm is given by $||f|| = sup_{z \in \mathbf{C}_{1/2}}(|f_0(z^2)| + (1 + |z|^3)|f_1(z^2)|)$. The algebra $\mathcal{S}$ has natural grading induced by the grading of Cl_1. If we consider the algebra $\mathcal{S}$ as ungraded, it has the same K-theory as $C_0(\mathbf{R})$.

In the construction of the algebra $\mathcal{A}(X)$, we will use an inductive system of finite-dimensional *affine* subspaces of X. We will use only those affine subspaces whose underlying vector space is a coordinate subspace of X, i.e. $V_F = l^p(F)$ for some $F \subset Z$. If $V_1 \subset V_2$ are two affine subspaces whose underlying vector subspaces are V_{F_1} and V_{F_2}, then we will consider $V_{F_2 - F_1}$ as the complementary subspace of V_1 in V_2.

The algebra $\mathcal{A}(X)$ is the inductive limit of the algebras $\mathcal{S}\hat{\otimes}C_0(V\otimes \mathbf{C})\otimes Cliff(V_F\otimes\mathbf{C})$. When $F_1\subset F_2$, the map

$$\mathcal{S}\hat{\otimes}C_0(V_1\otimes\mathbf{C})\otimes Cliff(V_{F_1}\otimes\mathbf{C})$$

$$\to\mathcal{S}\hat{\otimes}C_0(V_2\otimes\mathbf{C})\otimes Cliff(V_{F_2}\otimes\mathbf{C})$$

is given by $f(s)\hat{\otimes}h \mapsto f(s\varepsilon\hat{\otimes}1 + 1\hat{\otimes}B_{F_2-F_1})\hat{\otimes}h$, where $f\in\mathcal{S}$. Here $f(s\varepsilon\hat{\otimes}1 + 1\hat{\otimes}B_{F_2-F_1})$ is defined as follows. Put $N_F=\sum_{k\in F}(B_k^2)^{q/2}$,

$$f(z\varepsilon\hat{\otimes}1+1\hat{\otimes}B_{F_2-F_1})$$

$$=f_0(z^2+N_{F_2-F_1})+$$

$$f_1(z^2+N_{F_2-F_1})(z\varepsilon\hat{\otimes}1+1\hat{\otimes}B_{F_2-F_1}).$$

The group Γ acts on $X \otimes \mathbf{C}$ as follows: the linear part of the action is the diagonal action on $X \times X = X \otimes \mathbf{C}$, and the 1-cocycle acts over the real part of $X \otimes \mathbf{C}$. This induces the action of Γ on the algebra $\mathcal{A}(X)$. The Bott asymptotic morphism $\mathcal{S}_p \to \mathcal{A}(X)$ naturally comes from the inductive limit construction of $\mathcal{A}(X)$.

Important remark: In the definition of the algebra $\mathcal{A}(X)$ and the Bott asymptotic morphism one can replace $\mathcal{S}$ with a larger algebra of all *continuous* functions (instead of holomorphic functions) satisfying the same conditions as in the definition of $\mathcal{S}$.

The Dirac asymptotic morphism comes from the Connes-Higson construction applied to the extension of the L^p-algebras constructed similarly to the L^2 case. The only significant difference here is the construction of the operator $f(D_{\alpha,F})$ for $f \in \mathcal{S}$. For this, we use holomorphic functional calculus.

Note first that for any function $f \in \mathcal{H}_+$, the integral

$$f(D^2_{\alpha,F}) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - D^2_{\alpha,F}}$$

is well defined if the path γ in $\mathbf{C}_+$ is carefully chosen. In fact, the operator $D^2_{\alpha,F}$ generates a holomorphic semigroup. This means that in a certain sector containing the imaginary axis in $\mathbf{C}$, one has $||(z - D^2_{\alpha,F})^{-1}|| \leq \text{const}/|z|$. The path γ must run in this sector. Since $f \in \mathcal{H}_+$, it has asymptotics $|z|^{-m}$ at infinity for some integer $m \geq 1$, which ensures the convergence of the integral.

As above, in the case of the Bott element, this allows us to define $f(z\varepsilon\hat{\otimes}1 + 1\hat{\otimes}D_{\alpha,F_2-F_1})$, for $f \in \mathcal{S}_p$. Using the same method as in the L^2 case, we construct a continuous field of L^p-algebras, which gives an extension

$$0 \to \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{K} \to \mathcal{F} \to \mathcal{A}(X) \to 0.$$

This extension defines the Dirac asymptotic morphism.

Finally, to prove the L^p Baum-Connes conjecture, we need to consider the inductive limit of the following diagrams over $Y \subset \mathcal{E}\Gamma$:

$$
\begin{array}{ccc}
E_*^\Gamma(C_0(Y),\mathcal{S}) & \rightarrow & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{A}(X)) & \rightarrow & K_*(C_r^{*,p}(\Gamma,\mathcal{A}(X)) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{S}) & \rightarrow & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S})
\end{array}
$$

Here Y is an arbitrary Γ-compact, locally compact space. The horizontal arrows are the Baum-Connes maps. The left vertical arrows are given by the product with the Bott and Dirac asymptotic morphisms.

$$\begin{array}{ccc}
E_*^\Gamma(C_0(Y),\mathcal{S}) \to & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{A}(X)) \to & & K_*(C_r^{*,p}(\Gamma,\mathcal{A}(X)) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{S}) \to & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S})
\end{array}$$

The upper right vertical arrow is defined using commutation properties between the Bott element of $\mathcal{A}(X)$ and the Γ-action, as well as the fact that $C_r^{*,p}(\Gamma)\otimes\mathcal{S}$ can be considered as an algebra of continuous functions on a locally compact space with values in $C_r^{*,p}(\Gamma)$, according to the remark above.

$$
\begin{array}{ccc}
E_*^\Gamma(C_0(Y),\mathcal{S}) \rightarrow & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{A}(X)) \rightarrow & & K_*(C_r^{*,p}(\Gamma,\mathcal{A}(X)) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{S}) \rightarrow & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S})
\end{array}
$$

The lower right vertical arrow is more difficult to define, but we need it only on the image of the middle horizontal arrow, where it is easy to define. Also we need it on the image of the upper right vertical arrow, where it can also be defined.

$$\begin{array}{ccc}
E_*^\Gamma(C_0(Y),\mathcal{S}) \to & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{A}(X)) \to & & K_*(C_r^{*,p}(\Gamma,\mathcal{A}(X)) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{S}) \to & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S})
\end{array}$$

Moreover, like in the L^2 case, it is easy to prove using a homotopy of the Γ-action on X to a linear one (by contracting the 1-cocycle to 0), that the composition of the two vertical arrows is the identity (both on the left and right sides).

$$\begin{array}{ccc}
E_*^\Gamma(C_0(Y),\mathcal{S}) \rightarrow & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{A}(X)) \rightarrow & & K_*(C_r^{*,p}(\Gamma,\mathcal{A}(X)) \\
\downarrow & & \downarrow \\
E_*^\Gamma(C_0(Y),\mathcal{S}) \rightarrow & & K_*(C_r^{*,p}(\Gamma)\otimes\mathcal{S})
\end{array}$$

Next, we observe that the middle horizontal arrow is an isomorphism after passing to the inductive limit over $Y \subset \mathcal{E}\Gamma$. This requires a Mayer-Vietoris type reasoning.

Now the injectivity of the upper horizontal arrow follows from the fact that the composition of the two left vertical arrows is the identity, and the middle horizontal arrow is injective.

$$
\begin{array}{ccc}
E^{\Gamma}_{*}(C_0(Y),\mathcal{S}) \rightarrow & & K_{*}(C^{*,p}_{r}(\Gamma)\otimes\mathcal{S}) \\
\downarrow & & \downarrow \\
E^{\Gamma}_{*}(C_0(Y),\mathcal{A}(X)) \rightarrow & & K_{*}(C^{*,p}_{r}(\Gamma,\mathcal{A}(X))) \\
\downarrow & & \downarrow \\
E^{\Gamma}_{*}(C_0(Y),\mathcal{S}) \rightarrow & & K_{*}(C^{*,p}_{r}(\Gamma)\otimes\mathcal{S})
\end{array}
$$

In order to prove that the lower horizontal arrow is surjective, we take an element $a \in K_{*}(C^{*,p}_{r}(\Gamma)\otimes\mathcal{S})$ and apply to it the Bott asymptotic morphism (i.e. the upper right vertical arrow). The resulting element $b \in K_{*}(C^{*,p}_{r}(\Gamma,\mathcal{A}(X)))$ maps into a via the lower right vertical arrow because the composition of the two right vertical arrows is the identity.

Since the middle horizontal arrow is surjective, this ends the proof.