Asymptotic structure and rigidity of free product von Neumann algebras

Cyril HOUDAYER*
(joint works with Yoshimichi Ueda)
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Université Paris-Sud, Orsay

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I. Preliminary background

Terminology

- All the von Neumann algebras we consider are assumed to be **σ-finite**, that is, they possess a faithful normal state.
- An inclusion of von Neumann algebras $P \subset M$ is said to be **with expectation** if there exists a faithful normal conditional expectation $\mathrm{E}_P : M \to P$.
- A von Neumann algebra is **diffuse** if it has no minimal projection.
- A von Neumann algebra $M \subset \mathbf{B}(\mathrm{L}^2(M))$ is **amenable** if there exists a norm one projection $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$.
- By Connes's celebrated theorem (1976), for any von Neumann algebra M with separable predual, we have that

  M is **amenable** if and only if M is **hyperfinite**

What is... a free product von Neumann algebra?

Let I be any nonempty index set and $(M_i, \varphi_i)_{i \in I}$ any family of von Neumann algebras endowed with any faithful normal states.

Definition (Voiculescu 1985)

Up to state-preserving isomorphism, there exists a unique von Neumann algebra M endowed with a distinguished faithful normal state φ which satisfies the following properties:

- For every $i \in I$, $M_i \subset M$ and $\varphi|_{M_i} = \varphi_i$
- $\bigvee_{i \in I} M_i = M$
- The subalgebras $(M_i)_{i \in I}$ are **$*$-free** wrt φ inside M:

$$\varphi(x_1 \cdots x_n) = 0 \quad \forall n \geq 1, \forall i_1 \neq \cdots \neq i_n, \forall x_j \in \ker(\varphi_{i_j})$$

We call M the **free product von Neumann algebra** and write

$$(M, \varphi) = *_{i \in I}(M_i, \varphi_i)$$

Examples of free product von Neumann algebras

Let Γ be any discrete group. Define

$$\lambda_\Gamma = \Gamma \to \mathcal{U}(\ell^2(\Gamma)) \quad \textbf{left regular representation}$$

$$\mathrm{L}(\Gamma) = \lambda_\Gamma(\Gamma)'' \quad \textbf{group von Neumann algebra}$$

$$\tau_\Gamma = \langle \cdot \, \delta_{1_\Gamma}, \delta_{1_\Gamma} \rangle_{\ell^2(\Gamma)} \quad \textbf{canonical trace}$$

Example

Let I be any nonempty index set and $(\Gamma_i)_{i \in I}$ and family of discrete groups. Denote by $\Gamma = *_{i \in I} \Gamma_i$ the free product group. Then

$$(\mathrm{L}(\Gamma), \tau_\Gamma) \cong *_{i \in I} (\mathrm{L}(\Gamma_i), \tau_{\Gamma_i})$$

In particular, when $I = \{1, 2\}$ and $\Gamma_1 = \Gamma_2 = \mathbf{Z}$, we have

$$(\mathrm{L}(\mathbf{F}_2), \tau_{\mathbf{F}_2}) \cong (\mathrm{L}(\mathbf{Z}), \tau_{\mathbf{Z}}) * (\mathrm{L}(\mathbf{Z}), \tau_{\mathbf{Z}})$$

Connes–Takesaki's flow of weights

Let M be any σ-finite type III factor and $\varphi \in M_*$ any faithful state. Denote by

- $\sigma^\varphi : \mathbf{R} \curvearrowright M$ Connes's **modular automorphism group**
- $\mathrm{c}_\varphi(M) := M \rtimes_{\sigma^\varphi} \mathbf{R}$ Connes's **continuous core** of M, which is a type II_∞ von Neumann algebra
- $\theta^\varphi : \mathbf{R} \curvearrowright \mathrm{c}_\varphi(M)$ Takesaki's **dual trace-scaling action**.

By Connes–Radon–Nikodym cocycle theorem, the triplet $(\sigma^\varphi, \mathrm{c}_\varphi(M), \theta^\varphi)$ does not depend on the choice of the state φ.

Definition (Connes–Takesaki)

The **flow of weights** is the ergodic action $\theta^\varphi : \mathbf{R} \curvearrowright \mathcal{Z}(\mathrm{c}_\varphi(M))$.

Connes's classification of type III factors

Definition (Connes 1972)

Let M be any σ-finite type III factor. We say that M is of type

- III_0 if the flow of weights is properly ergodic.
- III_λ if the flow of weights is $\mathbf{R} \curvearrowright \mathbf{R}/(-\log \lambda)\mathbf{Z}$ $(0 < \lambda < 1)$.
- III_1 if the flow of weights is $\mathbf{R} \curvearrowright \{0\}$.

When M is not of type III_0, Connes's T**-invariant** defined by

$$\mathrm{T}(M) := \{t \in \mathbf{R} : \sigma_t^\varphi \in \mathrm{Inn}(M)\}$$

completely determines the type of M. In that case, we have

- $\mathrm{T}(M) = (-\log \lambda)\mathbf{Z}$ when M is of type III_λ $(0 < \lambda < 1)$
- $\mathrm{T}(M) = \{0\}$ when M is of type III_1

Ultraproduct von Neumann algebras

Let M be any σ-finite von Neumann algebra and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. Define

$$\mathcal{I}_\omega = \{(x_n)_n \in \ell^\infty(M) : x_n \to 0 \ \sigma\text{-}*\text{-strongly as } n \to \omega\}$$
$$\mathcal{M}_\omega = \{(x_n)_n \in \ell^\infty(M) : \lim_{n\to\omega} \|x_n\varphi - \varphi x_n\| = 0, \forall \varphi \in M_*\}$$
$$\mathcal{M}^\omega = \{(x_n)_n \in \ell^\infty(M) : (x_n)_n \mathcal{I}_\omega \subset \mathcal{I}_\omega \text{ and } \mathcal{I}_\omega (x_n)_n \subset \mathcal{I}_\omega\}$$

Then $\mathcal{I}_\omega \subset \mathcal{M}_\omega \subset \mathcal{M}^\omega$ is an inclusion of C*-algebras and $\mathcal{I}_\omega \subset \mathcal{M}^\omega$ is a norm closed ideal.

Definition

1. (Connes 1974) The **asymptotic centralizer** $M_\omega := \mathcal{M}_\omega / \mathcal{I}_\omega$ is a tracial von Neumann algebra.
2. (Ocneanu 1985) The **ultraproduct** $M^\omega := \mathcal{M}^\omega / \mathcal{I}_\omega$ is a σ-finite von Neumann algebra.

The **central sequence algebra** is $M' \cap M^\omega$.

Ultraproduct von Neumann algebras and full factors

We regard $M \subset M^\omega$ as a von Neumann subalgebra and the map

$$\mathrm{E}_\omega : M^\omega \to M : (x_n)^\omega \mapsto \sigma\text{-weak} \lim_{n \to \omega} x_n$$

defines a faithful normal conditional expectation.

We have $M_\omega \subset M' \cap M^\omega$ and moreover $M_\omega = (M' \cap M^\omega)^{\varphi^\omega}$ for every faithful state $\varphi \in M_*$ where $\varphi^\omega = \varphi \circ \mathrm{E}_\omega \in (M^\omega)_*$.

Definition (Connes 1974)

Let M be any factor with separable predual. We say that M is **full** if $M_\omega = \mathbf{C}1$ for some (or any) nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.

Non-type I amenable factors (with separable predual) are not full. By Connes (1974), type III_0 factors are never full.

By Ando–Haagerup (2012), M is full if and only if $M' \cap M^\omega = \mathbf{C}1$ for some (or any) nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.

Type classification and structure of free products

Let $I = \{1, 2\}$. For every $i \in \{1, 2\}$, let (M_i, φ_i) be any von Neumann algebra endowed with any faithful normal state. Assume that $\dim_{\mathbf{C}} M_1 \geq 2$ and $\dim_{\mathbf{C}} M_2 \geq 3$. Write

$$(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$$

Theorem

1. (Ueda 2011) $M = M_c \oplus M_d$ *where* M_c *is a diffuse factor and* $M_d = 0$ *or* M_d *is a discrete von Neumann algebra. Moreover*

$$(M_c)' \cap (M_c)^\omega = \mathbf{C}1 \quad \text{and} \quad \mathrm{T}(M_c) = \ker(\sigma^{\varphi_1}) \cap \ker(\sigma^{\varphi_2})$$

2. (Chifan–H 2008) M_c *is a* **prime** *factor in the sense that whenever* $M = Q_1 \mathbin{\overline{\otimes}} Q_2$, Q_1 *or* Q_2 *is a type* I *factor.*
3. (Boutonnet–H–Raum 2012) M *has no* **Cartan subalgebra**.

Previous results in the tracial case by Ozawa (2004), Peterson (2006) (item 2), Voiculescu (1995), Ozawa–Popa (2007), Ioana (2012) (item 3).

II. Asymptotic structure of free product von Neumann algebras

Popa's maximal amenability result in $\mathrm{L}(\mathbf{F}_2)$

It is well known that any II_1 factor N contains a copy of the hyperfinite II_1 factor R. Kadison asked the following question.

Problem (Kadison 1967)

Let N be any II_1 factor and $x = x^* \in N$ any selfadjoint element. Does there always exist an intermediate hyperfinite subfactor $R \subset N$ such that $x \in R$?

Popa answered negatively the above question by showing the following pioneering result.

Theorem (Popa 1983)

Write $\mathbf{F}_2 = \langle a, b \rangle$. The generator subalgebra $A := \{\lambda_{\mathbf{F}_2}(a)\}''$ is **maximal amenable** *inside $M := \mathrm{L}(\mathbf{F}_2)$, that is, whenever Q is amenable and $A \subset Q \subset M$ then $A = Q$.*

Popa's result motivates our work on the asymptotic structure of free product von Neumann algebras.

Maximal amenability in free products

Let $I = \{1, 2\}$. For every $i \in \{1, 2\}$, let (M_i, φ_i) be any von Neumann algebra endowed with any faithful normal state. Write

$$(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$$

Theorem (H–Ueda 2015)

Let $Q \subset M$ be any amenable von Neumann subalgebra with expectation such that $Q \cap M_1$ is diffuse and with expectation. Then $Q \subset M_1$.

In particular, if M_1 is diffuse and amenable, then M_1 is maximal amenable (with expectation) inside M.

Our result generalizes and strengthens Popa's result (1983) in the case when $M = \mathrm{L}(\mathbf{F}_2)$ with $M_1 = M_2 = \mathrm{L}(\mathbf{Z})$.

It moreover completely settles the question of maximal amenability of the inclusion $M_1 \subset M$ in arbitrary free products.

Asymptotic orthogonality property (AOP)

Our proof uses Popa's **asymptotic orthogonality property (AOP)** strategy and relies on the following key technical result.

Let $P \subset M_1$ be any diffuse von Neumann subalgebra with expectation. Let $\psi \in M_*$ be any faithful state such that

$$\sigma_t^\psi(P) = P \quad \text{and} \quad \sigma_t^\psi(M_1) = M_1 \quad \forall t \in \mathbf{R}$$

Observe that ψ need not be equal to the free product state φ.

Lemma (H–Ueda 2015)

For all $x \in P' \cap M^\omega$ and all $y \in M \ominus M_1$, we have

$$\|yx - xy\|_{\psi^\omega}^2 = \|y(x - \mathrm{E}_{M_1^\omega}(x))\|_{\psi^\omega}^2 + \|(\mathrm{E}_{M_1^\omega}(x) - x)y\|_{\psi^\omega}^2 + \|y\mathrm{E}_{M_1^\omega}(x) - \mathrm{E}_{M_1^\omega}(x)y\|_{\psi^\omega}^2$$

The above lemma generalizes Popa's AOP result inside $\mathrm{L}(\mathbf{F}_2)$.

AOP implies maximal amenability

Theorem (H–Ueda 2015)

Let $M_1 \subset Q \subset M$ be any intermediate amenable von Neumann subalgebra with expectation. If M_1 is diffuse, then $Q = M_1$.

Proof.

Since $M_1 \subset Q \subset M$ is with expectation and M_1 is diffuse, Q is necessarily globally invariant under σ^φ and so we can take $\psi = \varphi$.

Assume by contradiction that $M_1 \neq Q$. We may further assume that Q is a factor. We claim that $Q' \cap M^\omega = Q' \cap M_1^\omega$. Indeed, let $x \in Q' \cap M^\omega$. For any $y \in Q \ominus M_1$, since $yx = xy$, AOP implies that $y(x - \mathrm{E}_{M_1^\omega}(x)) = 0$. Since $(Q \ominus M_1)(x - \mathrm{E}_{M_1^\omega}(x)) = 0$ and since Q is a factor, we obtain $x = \mathrm{E}_{M_1^\omega}(x)$.

Observe that $Q \subset M_1 \vee M_2 \subset M_1^\omega \vee M_2 \cong (M_1^\omega, \varphi_1^\omega) * (M_2, \varphi_2)$ and $Q' \cap M_1^\omega = Q' \cap M^\omega$ is diffuse (since Q is amenable). This implies that $Q \subset M_1^\omega$ and hence $Q \subset M_1$. This is a contradiction. □

Interlude: a characterization of property Gamma

We say that M has **property Gamma** if the central sequence algebra $M' \cap M^\omega$ is diffuse.

Theorem (H–Raum 2014)

Let M be any diffuse (possibly type III) von Neumann algebra with separable predual. The following conditions are equivalent.

1. *M has property Gamma.*
2. *There exists a faithful state $\varphi \in M_*$ such that $M' \cap (M^\varphi)^\omega$ is diffuse.*
3. *There exists a faithful state $\varphi \in M_*$ and a decreasing sequence of diffuse abelian subalgebras $A_n \subset M^\varphi$ such that*

$$M = \bigvee_n ((A_n)' \cap M).$$

Previous result in the tracial case by Popa (2003).

III. Rigidity of free product von Neumann algebras

Kurosh theorem for discrete groups

Recall the classical structure theorem by Kurosh for free products of discrete groups.

Theorem (Kurosh 1934)

Let I be any nonempty set and $(\Gamma_i)_{i \in I}$ any family of discrete groups. Denote by $\Gamma = *_{i \in I} \Gamma_i$ the free product group.

For any subgroup $\Lambda < \Gamma$, there exist a free subgroup $\mathbf{F} < \Gamma$, an index set J, a family $(\Lambda_j)_{j \in J}$ of subgroups of $(\Gamma_i)_{i \in I}$ and a family $(\gamma_j)_{j \in J}$ of elements of Γ such that

$$\Lambda \cong \mathbf{F} * \left(*_{j \in J} \, \gamma_j \Lambda_j \gamma_j^{-1} \right)$$

Kurosh theorem motivates our work on rigidity of free product von Neumann algebras. Even though we cannot prove such a precise statement for free product von Neumann algebras, we can still obtain Kurosh-type rigidity results.

A class of anti-freely decomposable von Neumann algebras

The structure of free product von Neumann algebras suggests that the following class is a natural class of von Neumann algebras that cannot be decomposed as a nontrivial free product.

Definition

We say that a nonamenable factor M (with separable predual) is **anti–freely decomposable** if:

1. Either M is not prime, that is, $M = Q_1 \mathbin{\overline{\otimes}} Q_2$ where Q_1, Q_2 are diffuse factors (e.g. M is McDuff).
2. Or M has property Gamma (e.g. M is of type III_0).
3. Or M has a Cartan subalgebra (or more generally M has a regular amenable finite von Neumann subalgebra with expectation $A \subset M$ such that $A' \cap M = \mathcal{Z}(A)$).

We denote by $\mathcal{C}_{\text{anti-free}}$ the class of **anti–freely decomposable** nonamenable factors.

Kurosh-type rigidity for free products

Let I and J be any nonempty (possibly infinite) index sets. Let $(M_i, \varphi_i)_{i \in I}$ and $(N_j, \psi_j)_{j \in J}$ be any families of nonamenable factors in the class $\mathcal{C}_{\text{anti-free}}$ endowed with any faithful normal states.

Theorem (H–Ueda 2015)

Assume that $M = N$. Then $|I| = |J|$ and there exists a unique bijection $\alpha : I \to J$ such that M_i and $N_{\alpha(i)}$ are stably unitarily conjugate inside M for every $i \in I$.

Our Kurosh-type rigidity result is new for infinite index sets and covers arbitrary (possibly type III) von Neumann algebras.

It generalizes previous results in the tracial case by Ozawa (2004), Ioana–Peterson–Popa (2005) and Peterson (2006) and also Asher (2008).

Our proof relies on a spectral gap theorem inside free products, Popa's intertwining techniques for type III von Neumann algebras and results on tracial amalgamated free products by Ioana (2012) and Vaes (2013).

Popa's intertwining techniques: terminology

Let M be any σ-finite von Neumann algebra and $A, B \subset M$ any von Neumann subalgebras with expectation.

Definition

We say that A **embeds with expectation into B inside M** and write $A \preceq_M B$ if there exist projections $e \in A$, $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that:

- The unital inclusion $\theta(eAe) \subset fBf$ is **with expectation** and
- $av = v\theta(a)$ for all $a \in eAe$.

Popa (2001-2003): When M is tracial, TFAE:

1. $A \preceq_M B$.
2. There is no net of unitaries $(w_i)_{i \in I}$ in $\mathcal{U}(A)$ such that $\lim_i \|\mathrm{E}_B(b^* w_i a)\|_2 = 0$ for all $a, b \in M$.

Popa's intertwining techniques: characterization

Theorem (H–Isono 2015)

Let $M \subset \mathbf{B}(\mathrm{L}^2(M))$ be any σ-finite von Neumann algebra in standard form and $A, B \subset M$ any von Neumann subalgebras with expectation. Assume moreover that A is **finite**. *TFAE:*

1. $A \preceq_M B$.
2. *For any σ-weakly dense subset $X \subset M$, there is no net of unitaries $(w_i)_{i \in I}$ in $\mathcal{U}(A)$ such that $\mathrm{E}_B(b^* w_i a) \to 0$ σ-strongly as $i \to \infty$ for all $a, b \in X$.*
3. *Write $B = B_1 \oplus B_2$ with B_1 semifinite and B_2 of type III. Denote by T_M the canonical operator valued weight from $\langle M, B \rangle$ onto M and fix a fns trace Tr on $\langle M, B \rangle J 1_{B_1} J$. There exists a nonzero element $d \in A' \cap \langle M, B \rangle^+$ such that*

$$\mathrm{Tr}(d\, J 1_{B_1} J) < +\infty \quad \text{and} \quad \mathrm{T}_M(d\, J 1_{B_2} J) \in M.$$

Spectral gap theorem in free products

Let I be any nonempty (possibly infinite) index set. Let $(M_i, \varphi_i)_{i \in I}$ be any family of σ-finite von Neumann algebras endowed with any faithful normal states. Write

$$(M, \varphi) = *_{i \in I}(M_i, \varphi_i)$$

Theorem (H–Ueda 2015)

Let $A \subset M$ be any finite von Neumann subalgebra with expectation. Then at least one of the following assertions is true.

- *There exists $i \in I$ such that $A \preceq_M M_i$.*
- *The relative commutant $A' \cap M$ is amenable.*

The proof consists in two steps:

1. Firstly, we prove spectral gap inside semifinite amalgamated free product von Neumann algebras.
2. Secondly, we use Connes's continuous core $\mathrm{c}(M)$ together with Popa's intertwining techniques to descend back to M.

Thank you!