KK-theory for fundamental C*-algebras of graphs of C*-algebras

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CIRM, November 5, 2015

Some History

Pimsner-Voiculescu's exact sequence

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- 1982 Cross product by free group $\mathbb{F}_n$

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- 1986 Anderson-Paschke HNN groups
- 1986 Pimsner KK-theory of cross product by groups acting on trees

Representations of free products

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1. Representation in the Calkin algebra (Voiculescu absorption theorem for extensions).
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Hypothesis

1. Nuclearity.
2. Existence of conditional expectations.

Cuntz Theorem

A_1 and A_2 unital C*-algebras with one dimensional representations t_1 and t_2.

Theorem (Cuntz 1982)

$A_1 \underset{\mathbb{C}}{*} A_2$ is KK-equivalent to $A_1 \underset{\mathbb{C}}{\oplus} A_2$

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Path π_t of representations of $A = A_1 \underset{\mathbb{C}}{*} A_2$ between $Id_A \oplus t_1 * t_2$ and $Id_{A_1} * t_2 \oplus t_1 * Id_{A_2}$

$$\pi_t(a_1) = \begin{pmatrix} a_1 & 0 \\ 0 & t_1(a_1) \end{pmatrix}$$

$$\pi_t(a_2) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & t_2(a_2) \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ +\sin t & \cos t \end{pmatrix}$$

Graphs of C*-algebras

Definition

A finite graph $\mathcal{G}$ is a collection of vertices v and edges e together with source and range maps s and r. To each edge e we will always have an opposite edge $\bar{e}$ such that source and range are interchanged.

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Definition (Graph of C*-algebras)

Given a finite graph $\mathcal{G}$ we associate to each vertex v a unital C*-algebra A_v and to each edge e a unital C*-algebra $B_e = B_{\bar{e}}$ such that there exists unital injective morphisms s_e and r_e from B_e to $A_{s(e)}$ and $A_{r(e)}$ respectively. We will also require that there exists ucp maps E_e from A_v to any B_e such that $s(e) = v$ with $E_e \circ s_e = Id_{B_e}$

Fundamental C*-algebras

Definition

Choose a maximal tree Y in the graph $\mathcal{G}$. Then the full fundamental C*-algebra P_f is the universal unital C*-algebras generated by the A_v's and unitaries U_e for any edge e such that

- $U_{\bar{e}} = U_e^*$
- $U_e\, s_e(b)\, U_e^* = r_e(b)$ for all $b \in B_e$
- $U_e = 1$ for all e in the tree Y

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Example

- One geometric edge, two vertices : this is the full amalgamated free product $A_1 \underset{B}{*} A_2$
- One geometric edge , one vertex : this is the HNN extension $(A, s_e(B), \theta = r_e \circ s_e^{-1})$

Vertex representations

Choose a vertex v.
Via GNS for A_v and E_e, one gets a B_e-Hilbert module $K_e = 1_{A_v} B_e \oplus K_e^\circ$.
To any path $(e_1, e_2, \cdots, e_n)$ such that $s(e_1) = v = r(e_n)$ and $r(e_i) = s(e_{i+1})$ one can also associate a A_v -Hilbert module :

$$K_{e_1}^{\epsilon_1} \otimes_{B_{e_1}} \cdots \otimes_{B_{e_{n-1}}} K_{e_n}^{\epsilon_n} \otimes_{B_{e_n}} A_v$$

where ϵ_i is null or $\circ$ according to the rule $\epsilon_i = \circ$ if $e_i = \bar{e}_{i-1}$.
We call H_v the direct sum over all admissible paths of these Hilbert modules with convention that A_v is associated to the empty path.

Theorem

For any v, there exists a natural representation of P_f in $\mathcal{L}_{A_v}(H_v)$

Example (Cross product case)

It is a particular case of an HNN extension where $A = B$ and θ is an isomorphism. Then $K_e = A$ and $K_e^\circ = 0$. Only the paths with no change of direction contribute and the associated Hilbert module is isomorphic to A. Hence $H_v = \ell^2(\mathbb{Z}) \otimes A$ and U_e is represented as the bilateral shift.

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Example (Free product case)

$$H_1 = A_1 \oplus \bigoplus_{1=i_1\neq i_2\neq \dots \neq i_n \neq 1} K_{i_1} \otimes_B K_{i_2}^\circ \otimes_B \cdots \otimes_B K_{i_n}^\circ \otimes_B A_1$$

The action of A_1 is straightforward and the action of A_2 is obtained via the isomorphism

$$H_1 \simeq K_2 \otimes_B \left(B \oplus \bigoplus_{2\neq i_1\neq i_2\neq \dots \neq i_n \neq 1} K_{i_1}^\circ \otimes_B K_{i_2}^\circ \otimes_B \cdots \otimes_B K_{i_n}^\circ \right) \otimes_B A_1$$

In the most degenerate case (E_e are morphisms), $H_1 = A_1$.

Edge representations

Given an edge e such that $s(e) = v$, we can form $H_e = H_v \otimes_{E_e} B_e$

Theorem

- *For any e, H_e is isomorphic to $H_{\bar{e}}$*
- *there exists a natural representation of P_f in $\mathcal{L}_{B_e}(H_e)$*

Example (Free product case)

$$H_B = B \oplus \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n} K_{i_1}^\circ \otimes_B K_{i_2}^\circ \otimes_B \cdots \otimes_B K_{i_n}^\circ$$

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Theorem

If all the ucp maps E_e give a GNS-faithful representation of A_v on K_e then the image of P_f in all vertex or edge representations are isomorphic. This is the analogue of Voiculescu reduced free product.

Vertex reduced free product

Definition

The quotient of $A_1 \underset{B}{*} A_2$ by the intersection of the kernel of the two vertex representations is the vertex reduced free product $A_1 \overset{v}{\underset{B}{*}} A_2$.

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For groups Γ_1, Γ_2, we get the four possibly distinct C*-algebras :

$$\begin{array}{ccccc} & & C^*_r(\Gamma_1) * C^*_r(\Gamma_2) & & \\ & \nearrow & & \searrow & \\ C^*(\Gamma_1 * \Gamma_2) & & & & C^*_r(\Gamma_1 * \Gamma_2) \\ & \searrow & & \nearrow & \\ & & C^*(\Gamma_1) \overset{v}{*} C^*(\Gamma_2) & & \end{array}$$

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The natural quotient map from $A_1 \underset{B}{*} A_2$ to $A_1 \overset{v}{\underset{B}{*}} A_2$ is **always** *a KK-equivalence.*

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- $C^*(\Gamma_1 * \Gamma_2)$ is KK-equivalent to $C^*(\Gamma_1) \overset{v}{*} C^*(\Gamma_2)$ and $C_r^*(\Gamma_1) * C_r^*(\Gamma_2)$ is KK-equivalent to $C_r^*(\Gamma_1 * \Gamma_2)$

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The proof is Julg-Valette for the Bass-Serre tree except that

- there is no tree
- there is no proper action even in the group case
- the homotopy is not a geometric one but a simple rotation argument

The inverse of the quotient map

There are natural partial isometries $F_k \in \mathcal{L}_{A_k}(H_k, H_B \otimes_B A_k)$ between

$$H_k = A_k \oplus \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n \neq k} K_{i_1}^\circ \otimes_B K_{i_2}^\circ \otimes_B \cdots \otimes_B K_{i_n}^\circ \otimes_B A_k$$

and

$$H_B \otimes_B A_k = A_k \oplus \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n} K_{i_1}^\circ \otimes_B K_{i_2}^\circ \otimes_B \cdots \otimes_B K_{i_n}^\circ \otimes A_k$$

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Theorem

- $F_k^* F_k = Id \mod K(H_k)$
- $F_k F_k^* = P_k \otimes_{A_k} 1$ *with* $P_k \in \mathcal{L}_B H_B$ *and* $P_1 + P_2 = Id \mod K(H_B)$
- $[A_1, F_1] = 0$ *and* $[A_2, F_1] \subset K(H_1, H_B \otimes_B A_1)$

The inverse of the quotient map from $A_f = A_1 \underset{B}{*} A_2$ to $A = A_1 \overset{v}{\underset{B}{*}} A_2$ is the element of $KK^0(A, A_f)$ defined by

- the module $(H_1 \otimes_{A_1} A_f \oplus H_2 \otimes_{A_2} A_f) \oplus H_B \otimes_B A_f$
- the natural induced left action of A on H_1, H_2 and H_B
- the fredholm operator

$$F = \begin{pmatrix} 0 & 0 & F_1^* \otimes_{A_1} 1 \\ 0 & 0 & F_2^* \otimes_{A_2} 1 \\ F_1 \otimes_{A_1} 1 & F_2 \otimes_{A_2} 1 & 0 \end{pmatrix}$$

Homotopy is the rotation present in Cuntz theorem. To prove it is defined on the vertex reduced free product we need

Theorem

The partial radial maps φ_k in $A_1 \overset{v}{\underset{B}{*}} A_2$ that is, for reduced words, the multiplication by r to the power the number of letters in A_k for $0 < r < 1$ is UCP.

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Which is proven via free probability techniques using the isomorphism

$$(A_1 \overset{1}{\underset{B}{*}} A_2) \overset{e}{\underset{A_1}{*}} (A_1 \overset{1}{\underset{B}{*}} C([0,1];B)) \simeq A_1 \overset{1}{\underset{B}{*}} (A_2 \overset{e}{\underset{B}{*}} C([0,1];B))$$

Long exact sequence for fundamental C*-algebras

Choose a set E^+ of oriented edges, then for any separable D, there are long exact sequences for both the full and vertex reduced fundamental C*-algebras (P_f or P_v) analogous to Pimsner exact sequence for group acting on trees.

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Choose a set E^+ of oriented edges, then for any separable D, there are long exact sequences for both the full and vertex reduced fundamental C*-algebras (P_f or P_v) analogous to Pimsner exact sequence for group acting on trees.

Theorem

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+} KK^0(D,B_e) & \xrightarrow{\sum s_e^*-r_e^*} & \bigoplus_{v\in V} KK^0(D,A_v) & \longrightarrow & KK^0(D,P_f) \\
\uparrow & & & & \downarrow \\
KK^1(D,P_f) & \longleftarrow & \bigoplus_{v\in V} KK^0(D,A_v) & \xleftarrow{\sum s_e^*-r_e^*} & \bigoplus_{e\in E^+} KK^0(D,B_e)
\end{array}$$

and

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+} KK^0(B_e,D) & \xleftarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{v\in V} KK^0(A_v,D) & \longleftarrow & KK^0(P_f,D) \\
\downarrow & & & & \uparrow \\
KK^1(P_f,D) & \longrightarrow & \bigoplus_{v\in V} KK^0(A_v,C) & \xrightarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{e\in E^+} KK^0(B_e,D)
\end{array}$$

The boundary map

For any edge e belonging to the graph $\mathcal{G}$, there is a natural element $x_e^{\mathcal{G}} \in KK^1(P_v^{\mathcal{G}}, B_e)$ given by the projection in H_e on the direct sum of Hilbert modules associated to paths $(e_1, ..., e_n)$ with $e_n = e$.

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Theorem

- $x_{\bar{e}} = -x_e$
- *If* $\mathcal{G}_1$ *is a subgraph of* $\mathcal{G}$ *and* π_1 *is the natural morphism from* $P_v^{\mathcal{G}_1}$ *to* $P_v^{\mathcal{G}}$ *then* $[\pi_1] \otimes x_e^{\mathcal{G}} = x_e^{\mathcal{G}_1}$ *if e belong to* $\mathcal{G}_1$ *and* $[\pi_1] \otimes x_e^{\mathcal{G}} = 0$ *otherwise*
- $\sum_e x_e^{\mathcal{G}} \otimes [s_e] = 0$ *in* $KK^1(P_v^{\mathcal{G}}, \oplus_v A_v)$.

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- $\sum_e x_e^{\mathcal{G}} \otimes [s_e] = 0$ *in* $KK^1(P_v^{\mathcal{G}}, \oplus_v A_v)$.
- *The boundary maps in the long exact sequences is given by the Kasparov product with* $x_e^{\mathcal{G}}$ *for the vertex reduced case or with* $[\pi] \otimes x_e^{\mathcal{G}}$ *for the full case where* π *is the natural quotient map* $P_f^{\mathcal{G}} \to P_v^{\mathcal{G}}$

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Let D be the C*-algebra of continuous functions from $]-1,1[$ to $A_1 \underset{B}{*} A_2$ such that $f(]-1,0]) \subset A_1$, $f([0,1[) \subset A_2$ and $f(0) \in B$.

Theorem

The natural map from D to the suspension of $A_1 \underset{B}{*} A_2$ is a KK-equivalence.

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The Kasparov triple for the inverse is
$(H_1 \otimes C_0(]-1,0[) \oplus H_2 \otimes C_0(]0,1[)) \oplus H_B \otimes C_0(]-1,1[)$ and Fredholm operator

$$G(t) = \begin{pmatrix} \cos^- \pi t & 0 & -F_1^* \otimes 1 \sin^- \pi t \\ 0 & -\cos^+ \pi t & F_2^* \otimes 1 \sin^+ \pi t \\ -F_1 \otimes 1 \sin^- \pi t & F_2 \otimes 1 \sin^+ \pi t & Z(t) \end{pmatrix}$$

with $Z(t) = -P_1 \cos^- \pi t + P_2 \, cos^+ \pi t - tP_0$ and $P_1 + P_2 + P_0 = Id_{H_B}$

Corollaries

Theorem

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1. *The full and the vertex reduced fundamental C*-algebras of a graph of C*-algebras are always KK-equivalent.*
2. *Suppose we have a graph of C*- compact quantum groups. If all of the vertex algebras are K-amenable then the fundamental C*-algebra of the graph, which is again a compact quantum groups, is K-amenable*